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A new Approach to the Quantum KdV

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Abstract

We present a new formulation for the quantum evolution equation of KdV type. It is shown explicitly that a generalization of the usual recursion operator is possible even when we follow the rules of quantization and assume that the nonlinear field variables do not commute. We also demonstrate that this recursion operator generates in a recursive way an infinite number of Hamiltonians commuting with each other, thus giving a basis for the complete integrability of the quantum mechanical evolution of the field. It is discovered that the reason why the Recursion operator for the quantum KdV was not discovered earlier lies in the fact that this recursion operator is more closely connected to the general theory of the KP than to that of the KdV.

1 Introduction

Nowadays, quantization of integrable evolution equations is one of the important problems in the study of nonlinear systems.

Apart from the recent introduction of *integrable quantum mappings* (see [4]), at present there exist two principal avenues to this intricate problem; yet none of them is totally satisfactory. The first one is that of Faddeev [11], who formulated the Quantum Inverse Problem, based on classical Lax pairs. He constructed some kind of excited states called "string configurations". However, in this formalism it actually is difficult to find a proper definition

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of *complete integrability* of the quantum mechanical nonlinear system. A few years later, the idea of conformal invariance ([21],[5]) was used to quantize KdV-like systems. The main ingredient was the use of operator product expansions. Actually this was suggested from the commutation rules of the Virasoro algebra and the second Hamiltonian structure of the KdV system. In this formalism one is able to deduce the operator form of the equation very elegantly but one is not able to deduce in a recursive way an infinite number of conserved quantities. So, even in this framework a concrete definition of complete integrability is lacking.

Another important development in that direction is the introduction of Integrable Quantum Mapping

Recently a proper formulation of some of these problems, for nonlinear equations whose field variables do not commute, has been put forward in [17] (see also [2]). There the physically relevant problem of the Heisenberg spin chain, when the spin operator is no more a classical variable, has been studied. It was demonstrated how one can construct in such cases, a *mastersymmetry* and hence can speak of recursive determination of integrals of motion (See also [1] for the interpretation of master symmetries in these cases). Here in this communication we have used such an approach for the specific case of KdV equation. We show in the following how one can have a generalization of the second symplectic operator, and hence of the recursion operator, in the non-commuting situation of the nonlinear field variables. So we have an operator generalization of the recursive way of constructing infinitely many Hamiltonians. We also demonstrate that these Hamiltonians do commute with each other. The necessary hamiltonian formulation in case of noncommunative structures we use here (see [17], [2] or [18] for a general presentation of a suitable theory) is closely connected to the theory presented by Magri-Morosi [22], a theory similar to that recently used in the investigation of $2+1$-dimensionsional equations in by Dorfman-Fokas [10]. The present paper is an extended version of the resuts mentioned in [19].

2 Quantization of KdV

We follow here the general concept introduced in [20]. Recall that for the KdV

$$u_t = u_{xxx} + 6uu_x \tag{2.1}$$

the Poisson bracket structure is defined by

$$\{F_1(u), F_2(u)\} := \int_{-\infty}^{+\infty} (\nabla F_1(u))(\nabla F_2(u))_x dx \tag{2.2}$$

where F_1, F_2 are scalar fields and where ∇ denotes the operation of taking the gradient. We shall follow the usual approach that quantum brackets are considered to be operator generalizations of the classical Poisson brackets [9]. First we rewrite (2.2) for the case

$$F_i(u) = \int_{-\infty}^{+\infty} \varphi_i(x)u(x)dx \tag{2.3}$$

where $\varphi_i(x)$ are suitable test functions. For these special fields we find

$$\{F_1, F_2\} = \int_{-\infty}^{+\infty} \varphi_i(x)\varphi_2(x)_x dx \ . \tag{2.4}$$

Now, taking limits such that $\varphi_1(x) \to \delta(\hat{x})$ and $\varphi_2(x) \to \delta(x - \hat{x})$ we obtain

$$\{u(\hat{x}), u(\tilde{x})\} = \delta_{\hat{x}}(\hat{x} - \tilde{x}) \tag{2.5}$$

that the Poisson bracket between field variables at different points is a derivative of the δ-distribution. So quantization of the KdV-field must lead to

$$[u(x), u(\tilde{x})] = i\delta_x(x - \tilde{x}) \tag{2.6}$$

whenever the field $u(x)$ is generalized to be a distribution-valued operator field.

However, serious difficulties have to be overcome in order to make this heuristic approach precise. In order to show that an algebra, fulfilling this relation, exists at all we have to give an interpretation of terms like

$$(u(x)u(\tilde{x}) - u(\tilde{x})u(x))^2$$

which would be equal to $\delta_x(x - \tilde{x})^2$, a quantity not yet defined. So, we first have to make some remarks about distribution multiplication. We follow closely the concept introduced in [13] and [12] (see also [14]).

A distribution $\phi(x)$ is said to be *almost-bounded* if, for every $n \in \mathbb{N}$, its n-th derivative is of the form

$$\phi^{(n)}(x) = b(x) + \triangle(x) \tag{2.7}$$

where b is locally bounded and where $\triangle$ is a distribution with discrete support such that the support has no accumulation point.

The degree $G(\phi)$ of an almost-bounded distribution ϕ, which is not a proper function, is the smallest m such that ϕ is the m-th derivative of a continuous function. For continuous functions f (in the space of almost-bounded distributions) the degree $G(f) = -m$ is the negative of the greatest number m such that $f^{(m)}$ is continuous. So, taking the derivative increases the degree by 1. The δ-distribution $\delta(x)$ has degree - 2.

A fundamental observation is that in the space of almost-bounded distributions, there is a canonical algebraic structure fulfilling associativity, product-rule of differentiaton, translation invariance with respect to x, and having the property that it extends the usual pointwise algebra of functions. The algebra is non-commutative, hence there must be two different algebras (interchange of order of factors). In that algebra the product of two distributions with discrete support vanishes. The two product realizations are given by

$$\phi(x)\tilde{\phi}(x) = \lim_{\epsilon \downarrow 0} \phi(x+\epsilon)\tilde{\phi}(x) \tag{2.8}$$

or

$$\phi(x)\tilde{\phi}(x) = \lim_{\epsilon \downarrow 0} \phi(x)\tilde{\phi}(x+\epsilon) \ . \tag{2.9}$$

To make the following considerations consistent we choose the realization given by, say (2.8).

Now we denote by $u(x)$ a variable in the space of real almost-bounded distributions of degree 3, and we define $\mathcal{F}(x)$ to be the algebra generated, via the operations allowed in the space of complex almost-bounded distributions (like taking derivatives, etc.), by $u(x)$ its translations $u(x+\tilde{x}), \tilde{x} \in \mathbb{R}$ and by the almost-bounded distributions itself. Observe that this is a non-commutative algebra. By $\otimes\mathcal{F}(x)$ we define the algebra of arbitrary tensor products of elements of $\mathcal{F}(x)$. We will realize an algebra fulfilling (2.6) by taking suitable congruence classes in that algebra. Consider the ideal J generated in $\otimes\mathcal{F}(x)$ by the following relations $\simeq$:

$$\phi_1(x) \otimes \phi_2(\tilde{x}) \simeq \phi_1(x)\phi_2(\tilde{x}) \otimes 1 \simeq 1 \otimes \phi_1(x)\phi_2(\tilde{x}) \tag{2.10}$$

$$u(x) \otimes \phi_1(\tilde{x}) \simeq u(x)\phi_1(\tilde{x}) \otimes 1 \simeq 1 \otimes u(x)\phi_1(\tilde{x}) \tag{2.11}$$

$$\phi_1(\tilde{x}) \otimes u(x) \simeq \phi_1(\tilde{x})u(x) \otimes 1 \simeq 1 \otimes \phi_1(\tilde{x})u(x) \tag{2.12}$$

$$u(\hat{x}) \otimes u(\tilde{x}) - u(\tilde{x}) \otimes u(\hat{x}) \simeq i\delta_x(\hat{x} - \tilde{x}) \otimes 1 \tag{2.13}$$

$$A \otimes 1 \simeq 1 \otimes A \simeq A \tag{2.14}$$

where ϕ_1, ϕ_2 are arbitrary elements of the space of almost-bounded distributions, and where $A \in \otimes\mathcal{F}(x)$.

Taking now the quotient

$$QF(x) = \otimes\mathcal{F}(x)/J \tag{2.15}$$

of $Q\mathcal{F}(x)$ with respect to the ideal J we have found our quantum realization (named $QF(x)$, the *quantum fields* generated by $\mathcal{F}(x)$).

One should remark that here the usually subtle point of embedding non-linear terms of the ditribution- and operator-valued field variables is circumvented here by taking quotients only with respect to those algebraic rules which are really needed in building up the rquired structure. Using in this construction the almost-bounded distributions ensures that we do not encounter difficulties with respect to the distribution-valued nature of our quantities.

The operation of taking adjoints in $QF(x)$

$$QF(x) \stackrel{\text{adjoint}}{\longrightarrow} QF(x)^+ \tag{2.16}$$

is derived from the operation

$$(A_1 \otimes A_2 \ldots \otimes A_n)^+ = \bar{A}_n \otimes \bar{A}_{n-1} \otimes \ldots \otimes \bar{A}_1 \tag{2.17}$$

which is compatible with the congruence classes given by J.

In order to indicate the product of A and B in $QF(x)$ we just write AB, and we embed $\mathcal{F}(x)$ into $QF(x)$ by considering the elements of $\mathcal{F}(x)$ as equivalence classes of 1-times contravariant tensors. In this new algebraic structure we then have

$$u(x) \cdot u(\tilde{x}) - u(\tilde{x}) \cdot u(x) = i\delta_x(x - \tilde{x}). \tag{2.18}$$

Since elements of $QF(x)$ may be considered as operators (by multiplication) on $QF(x)$ itself, we have found the required operator representation of the Poisson structure of the KdV. Now, we have the prerequisites to define the time evolution for quantum systems by taking suitable Hamiltonian operators. For example, taking

$$H = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} u_\xi(\xi) u_\xi(\xi) + u(\xi)u(\xi)u(\xi) \right\} d\xi \tag{2.19}$$

and defining the action of a commutator on an integral, as integral (in convolution sense) over the commutator with its integrand, then we find

$$u(x)_t = i[H, u(x)] = u_{xxx}(x) + 3u_x(x)u(x) + 3u(x)u_x(x)\ . \tag{2.20}$$

This equation we consider as the quantum version of the KdV, it indeed leaves the crucial relation (2.18) invaraint. The main problem of this paper is to prove that this equation, indeed, is completely integrable in the usual sense, i.e. that it has infinitely many commuting symmetry groups (or conserved quantities which are in involution).

3 Densities

In order to give a recursive description of the symmetries and the conserved quantities of the evolution (2.20), an alternative representation of its dynamics is introduced. We closely follow the presentation of [17], [20] and [19] (see also [2], or [16] where this construction was first used).

Define the space of *densities* to be the quotient of $QF(x)$, first with respect to

$$\mathcal{L}_1 = \text{linear span } \{AB - BA \mid A, B \in QF(x)\} \tag{3.1}$$

and then with respect to

$$\mathcal{L}_2 = \text{linear span } \{DA - AD \mid A \in QF(x)\} \ , \tag{3.2}$$

where D denotes differentiation with respect to the variable x. The equivalence relation coming from the successive spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ will be denoted by $\equiv$ instead of $=$. The construction of density space is done in such a way that the factorspaces can be understood as the kernel of a trace operation which is chosen in such a way that formal integrals (from $-\infty$ to $+\infty$) over total derivatives vanish. Indeed, it is exactly that kernel, in case our operators are Hilbert-Schmidt operators vanishing rapidly with respect to x at $\pm\infty$.

The factorization with respect to $\mathcal{L}_1$ should not be misunderstood as if a structure were created where the ordering of operators is ignored. This because we only factorize with respect to the structure of *linear space*. This results in the fact that only *cyclic* permutations are allowed; thus generating a structure where exactly those operations are allowed which usually are performed when *traces* are considered. The factorization with respect to $\mathcal{L}_2$ then generates a structure similar to that where integrals over traces are taken in such a way into account that integrals over total derivatives are ignored (thus making the differential operator a self-adjoint one).

Let $\mathbf{A}$ and $\mathbf{B}$ elements in $QF(x)$. Define for all $\mathbf{A}, \mathbf{B} \in QF(x)$ an *inner product* in $QF(x)$ by

$$< \mathbf{AB} >= \text{equivalence class of } \int_{-\infty}^{+\infty} \mathbf{A}(x)\mathbf{B}(x)dx \ . \tag{3.3}$$

Observe that, due to (3.1) the differential operator is antisymmetric with respect to that density-valued inner product. Let $F = F(u)$ be a density depending in some way on the field variable u. Then define its *directional* *derivative* of F in the direction of an element $\mathbf{B}$ of $QF(x)$ by

$$F^{'}[\mathbf{B}] = \frac{\partial}{\partial\epsilon}_{|\epsilon=0}[F(u+\epsilon\mathbf{B})] \ . \tag{3.4}$$

These definitions, and the notion of density, provide as simple result [17] that there is a unique operator ∇, mapping densities into density-valued linear functionals on $QF(x)$ such that

$$F^{'}[\mathbf{B}] =< \nabla F, \mathbf{B} > \text{ for all } \mathbf{B} \in QF(x) \ . \tag{3.5}$$

The quantity ∇F is said to be the *gradient* of F. In case that the densities can be understood as kernels of traces then the gradient defined this way is indeed the classical gradient of the corresponding scalar quantity given by the integral over the trace.

For example, one obtains for the gradient of

$$H = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} u_\xi(\xi) u_\xi(\xi) + u(\xi)u(\xi)u(\xi) \right\} d\xi \tag{3.6}$$

as

$$u_{xx}(x) + 3u(x)u(x) \ . \tag{3.7}$$

The evolution equation (2.20) can now be rewritten as

$$u_t = D\nabla H \tag{3.8}$$

where H is given above in (3.6), and where D denotes the operator of taking the derivative with respect to x. This suggests, to consider this as alternative dynamical formulation of the system (2.20). But now, this is a flow not on all of $QF(x)$ but rather on the manifold given by those $u(x)$ which are realizations of (2.6). However, we have found a hamiltonian system in the classical sense since D is an implectic operator, i.e. the inverse of a symplectic operator.

We proceed as usual and, given an implectic operator Θ, define a Lie algebra structure in the space of densities by

$$\{G, H\}_\Theta =< \nabla G, \Theta \nabla H > \tag{3.9}$$

which fulfills the Jacobi identity on this special manifold. These brackets are called *Poisson brackets.* A density G, which depends explicitly on time, is then *invariant* under the flow

$$u_t = \Theta\nabla H \tag{3.10}$$

if and only if

$$G_t + \{G, H\}_\Theta \equiv 0 \ . \tag{3.11}$$

If G does not depend on the variable t then G is invariant if and only if its Poisson bracket with H vanishes, Therefore, also in this formulation, H is said to be the *hamiltonian* of (3.10). Furthermore, again as usual, the map

$$G \longrightarrow \Theta \nabla G \tag{3.12}$$

is a Lie algebra homomorphism from the Poisson brackets into the vector fields.

Observe now, that we found a new hamiltonian structure, for the given quantum system, which drastically differs from the one we had before in (2.20). The main differences are:

- The manifold under consideration is not anymore the manifold of all selfadjoint elements of $QF(x)$ but rather the manifold of all $u(x)$ fulfilling (2.6). Thus we have reduced the dynamics to a manifold which is considerably smaller.
- The dynamical system now is a truly nonlinear one, whereas in its canonical formulation $A_t = i[H, A], \ A \in QF(x)$ it was a linear one.

This new approach, which completely fits into the classical formulation of hamiltonian systems, now allows to look for other implectic operators which generate the same dynamics. So, we may use the bi-hamiltonian formulation given by that in order to construct the recursion operator in the usual way.

4 The recursion operator of the quantum KdV

As we have seen, identification of equations (2.20) and (3.10) allows that we can follow the usual procedure of soliton theory for constructing higher integrals. However, one should observe that now the algebraic structure where the recursion operator has to be found is far more complex than it usually is the case in soliton theory; this because a subtle realization of of the noncummutativity aspects of the quantum field variables had to be implemented. These noncommutativity aspects, of course, are also more complicated than those one usually encounters when matrix soliton equations are considered. This complication results from the fact that noncommutativity not only arises from the algebraic nature of the space where field variables take their values, but also from the points in x-space where these variables are localized.

We are now able to derive from the second symplectic operator of the KdV a modified form of the second hamiltonian formulation of the quantum kdV (3.7). Denote by u the field variable and introduce

$$L(u)A := uA \tag{4.1}$$

$$R(u)A := Au \tag{4.2}$$

where $a \in QF(x)$. These are the operators of multiplication with u from the left and from the right, respectively. Then set:

$$\Theta = D^3+DL(u)+DR(u)+R(u)D+L(u)D+(L(u)-R(u))D^{-1}(L(u)-R(u)) \tag{4.3}$$

which gives an operator being antisymmetric with respect to the inner product defined in the last section. We claim Θ satisfies, for arbitrary $A, B, C \in QF(x)$, the relation

$$< A, \Theta'[\Theta B]C > + \text{ cyclic permutations } = 0 \; . \tag{4.4}$$

Here $\Theta'[A]$ denotes Frechet derivative in the direction A. Whence we can conclude that Θ is an implectic operator and provides therefore the second hamiltonian formulation of (3.8). The actual verification of equation (4.4) is tedious and elaborate.[1] However, the operator (4.3) is formally, up to a change of sign, the same as the one considered in the general KP-theory presented in [22] or [10]. So, one may apply formally the structural argumants found in these papers, although the operator space considered there is quite different from the one considered here.

Since the operator Θ satisfies the required condition (4.4), we can now consider the conserved quantity:

$$H_0 = \frac{1}{2}\int_{-\infty}^{+\infty} u(\xi)u(\xi)d\xi \tag{4.5}$$

then from the rules laid down in the previous section we get

$$\nabla H = u \tag{4.6}$$

whence, $u_t = \Theta\nabla H_0$ again is the flow (2.20).

So we have now found two classical hamiltonian formulations for the quantum flow (2.20). This allows to apply the usual theory of hereditary

[1] The proof can be found in the appendix of this paper.

operators [15] in order to have a recursive generation of conserved densities and vector fields. We observe that replacing $u(x)$ by $u(x)+\alpha$ in the operator (4.3) preserves the implectic character of that operator (trivial Bäcklund transformation). But this transformation now yields the operator $\Theta + 4\alpha D$. Hence, Θ and D are compatible implectic operators, so

$$\Phi = \Theta D^{-1} \tag{4.7}$$

is hereditary and generates out of the vector field, given by the right side of (2.20), a hierarchy of commuting flows. All these flows then constitute symmetry group generators for the quantum KdV, since that equation is among the members of the hierarchy. On the other hand, we may consider the density (4.5) as a conserved quantity for (2.20), then recursive application of Φ^+ yields other densities, which are conserved. This is an immediate consequence that Φ^+ generates, because of its hereditaryness, a sequence of elements whose Poisson brackets commute.

So, we have indeed given two infinitely many hierarchies of invariants for the quantum KdV, the conserved densities and the symmetry generators.

5 Concluding Remarks

In this final and concluding section we add some comments relevant to the problem. It may be noted that the problem we have treated differs considerably from those discussed by other authors, see for example [3], [6], [7], [8] or [23].

In some work the variable $u(x)$ was realized as a boson field fulfilling the crucial commutation relations (2.6). However, the additional structure implemented by interpretation as a boson field prevented the complete integrability of that quantum version of the KdV, so for higher integrals quantum corrctions were needed.

Actually some basic questions related to integrable quantum nonlinear systems were not even considered in the initial formulations which can be found in the literature. These questions include a proper definition of *complete integrability* in the quantum case, the equivalence of the Hamiltonian formulation and symplectic formulation, and last, the question of involution of the quantum integrals of motion. Furthermore, the important problem of whether an algebraic frame can be found in which arbitrary products of field variables make sense at all, ususally is left open. In general, in nonlinear field theories, where the quantization is given such that the commutator or anticommutator of field variables amounts to delta-distributions, one runs into serious difficulties because nonlinear expresssions in delta-distributions are not well defined. Therefore the present paper presents together with the aspects on complete quantum-integrability also the aspect of showing that indeed such a quantization can be done in a rigorous and coherent way for nonlinear fields. Its essential point consists of a new theory which allows the embedding of certain nonlinear delta-distribution terms in a suitable algebraic structure.

Another aspect is to be pointed out. In general in quantized field theories, because of difficulties with proper i.e. scalar consered quantities, one introduces in certain situations infinite-valued terms and the like. Such an approach sometimes sheds serious doubts on wether or not these methods are rigorous. In the present paper, by working with quotients it is, for one example, shown that these methods are rigorous in so far as they allow for quantities which are not necessary scalar quantities. Our density approach amounts more or less to the same as those theories where nonfinite scalar terms are introduced, it is just another way of considering an extension of the scalars.

The QISM formulation of Faddeev, which is an ingenious theory, studies the spectrum generated by the quantization of a classical inverse problem. This formulation, however, does not consider the proper setup for the definition of a quantum nonlinear equation of motion.

On the other hand, though this problem was the starting point of the formulation by Kupershmidt and Mathieu, still a background regarding the basic structure was absent. Furthermore the recursive generation of quantum integral of motions were impossible in both formulations. So we hope that our point of view will fill up some gaps left out by others, especially the problem of quantizing an infinite number of conserved quantities, together with a formal treatment of their renormalization.

The reason that only one example, namely the KdV, is presented in this paper, is due to the fact that unfortunately the cmputations needed for finding a genuine nonlinear hamiltonian formulation for operator-valued flows, which are of the complexity needed for quantum fields, are considerable. The complexity of these computations is also seen from the fact that the hereditary property of our recursion operator also includes, if other interpretations like those in [10] are chosen, also comprises the complete integrability of fields in severeal space dimensions.

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6 Appendix

We show that (4.4) holds. We consider forms $F(A,B,C)$ being linear in A,B,C. Among these forms we introduce $\simeq$ as *equivalence modulo* those forms having the property that the sum of their cyclic permutations with respect to A,B,C vanishes. And, for further abbreviation, we denote, for example, by $\{...\}_{as(A,B)}$ the antisymmetric sum with respect to A,B of the forms in the bracket $\{...\}$. Similarly, $\{...\}_{sym(A,C)}$ denotes the symmetric sum. When checking the following computation the reader should be aware of the following rules

$$\{F(A,B,C)_{sym(A,B)}\}_{as(A,C)} \simeq 0 \tag{6.1}$$

since the entries of $sym(\ )$ and $as(\ )$ are different. Furthermore we have the obvious identities

$$\{F(A,B,C)_{sym(A,C)}\}_{as(A,C)} = 0 \tag{6.2}$$

$$\{F(A,B,C)\} - \{F(B,C,A)\} \simeq 0\ . \tag{6.3}$$

We have to show

$$< A, \Theta'[\Theta B]C > \simeq 0\ . \tag{6.4}$$

We split up this expression according to the order of differentiation. By $< A, \Theta'[\Theta B]C >_n$ we denote the n-th order contribution. We recall that

$$\Theta B = B''' + (uB)_x + (Bu)_x + B_x u + uB_x + u(D^{-1}(uB - Bu)) - (D^{-1}(uB - Bu))u\ . \tag{6.5}$$

6.1 4-th order terms

Utilizing the definition of the inner product, together with the skewsymmetry od D and integration by parts, we find

$$\begin{aligned} < A, \Theta'[\Theta B]C >_4 &\simeq\ < A, \{B'''C\}_x + \{CB'''\}_x + B'''C_x + CB'''_x > \\ &\simeq -< A_x, B'''C + CB''' > + < A, B'''C_x + C_x B''' > \\ &\simeq -< CA_x, B''' > + < A_x, CB''' > + < C_x A, B''' > + < A, C_x B''' > \\ &\simeq\ < C, A_{xx}B_{xx} > + < A_{xx}, CB_{xx} > - < C_{xx}, AB_{xx} > - < A, C_{xx}B_{xx} >\ . \end{aligned}$$

Now, using the equivalence defined in the densities, we see that in an inner product cyclic permutations for operators are allowed, so the sum of all cyclic permutations of these terms vanish vanish because of rule 6.3.

6.2 2-order terms

We use the notation $\Lambda_+(a)$ for the symmetric left-right application $(\Lambda a + a\Lambda)$ of Λ to a, and by $\Lambda_-(a)$ the antisymmetric left-right application $\Lambda a - a\Lambda$.

We use freely the fact that cyclic permutations are allowed in inner products, that the differential operators D and D^{-1} are skewsymmetric and that integration by parts can be performed. We obtain

$$
\begin{aligned}
< A, \Theta'[\Theta B] C >_2 \;\simeq\;& \left\{< A\{(u_+ B') + (u_+ B)'\}_+ C' >\right\}_{as(A,C)} + < (u_- C) D^{-1} B'''_- A >_{as(A,C)} \\
\simeq\;& \left\{< A(u'B + Bu')C' > + < AC'(u'B + Bu') >\right\}_{as(A,C)} \\
& + \left\{2 < A(uB' + B'u)C' > +2 < AC'(uB' + B'u) >\right\}_{as(A,C)} \\
& + \left\{< (uC - Cu) B''_- A > + < (uC - Cu) D^{-1}(B''_- A') >\right\}_{as(A,C)} \\
\simeq\;& - \left\{< (BC'A)'u > + < (C'AB)'u >\right\}_{as(A,C)} \\
& - \left\{< (BAC')'u > + < (AC'B)'u >\right\}_{as(A,C)} \\
& 2 \;\left\{< (B'C'A)u > + < (C'AB')u >\right\}_{as(A,C)} \\
& 2 \;\left\{< (B'AC')u > + < (AC'B')u >\right\}_{as(A,C)} \\
& + \left\{< CB''_-(A)u > - < B''_-(A)Cu >\right\}_{as(A,C)} \\
& + \left\{- < CD^{-1}(B''_- A')u > + < D^{-1}(B''_- A')Cu >\right\}_{as(A,C)} \\
\simeq\;& - \left\{< (C'AB)'u > + < (BAC')'u >\right\}_{as(A,C)} \\
& + \left\{2 < B'C'Au > +2 < AC'B'u >\right\}_{as(A,C)} \\
& - \left\{< CB''Au > - < CAB''u >\right\}_{as(A,C)} \\
& + \left\{- < B''ACu > +AB''Cu\right\}_{as(A,C)} \\
& + \left\{- < CB'A'u > + < CD^{-1}(B'A'')u > + < CD^{-1}(A'B'')u\right\}_{as(A,C)} \\
& + \left\{- < A'B'Cu > + < D^{-1}(A''B')Cu > + < D^{-1}(B''A')Cu >\right\}_{as(A,C)} \\
\simeq\;& - \left\{< C''BAu > + < C'A'Bu > + < BA'C'u >\right\}_{as(A,C)} \\
& - \left\{< BAC''u > -2 < B'C'Au > -2 < AC'B'u >\right\}_{as(A,C)} \\
& + \left\{< CB''Au > - < CAB''u > - < B''ACu >\right\}_{as(A,C)} \\
& + \left\{< AB''Cu > - < CB'A'u > - < A'B'Cu >\right\}_{as(A,C)} \\
\simeq\;& - \left\{< C''ABu > + < BAC''u >\right\}_{as(A,C)} \\
& + \left\{< CB''Au > + < AB''Cu >\right\}_{as(A,C)} \\
& - \left\{< CAB''u > + < B''ACu >\right\}_{as(A,C)} \\
\simeq\;& 0
\end{aligned}
$$

Let us explain how the cancellations took place. We use the notation $[3, 5, (6.1)]$ if the 3 and the 5-th term cancel because of rule (6.1), and so on. Going from the second to the third equality we used $[1, 4, (6.1)]$ and $[6, 7, (6.1)]$. Going from the third to the fourth equality we used $[10, 11, (6.1)]$ and $[13, 14, (6.1)]$. From the fourth to the fifth equality we had $[2, 5, 12, (6.3)]$ and $[3, 6, 11, (6.3)]$. Then finally $[2, 5, (6.1)]$, $[1, 6, (6.1)]$ and $[3, 4, (6.1)]$.

6.3 0-th order terms

We obtain

$$
\begin{aligned}
< A, \Theta'[\Theta B]C >_0 \simeq & \left\{< A(u_- D^{-1} u_- B)_+ C' >\right\}_{as(A,C)} \\
& + < (A(u_+ D + Du_+)B)_- D^{-1} u_- C > \\
& + < (Au_- D^{-1}((u_+ D + Du_+)B)_- C > \\
\simeq & \left\{< A(u_- D^{-1} u_- B)_+ C' > + < A(u_+ DB + Du_+ B)_- D^{-1} u_- C >\right\}_{as(A,C)} \\
\simeq & \left\{< C' A(u_- D^{-1}(u_- B)) > + < AC'(u_- D^{-1}(u_- B)) >\right\}_{as(A,C)} \\
& + \left\{< A(u_+ B' + (u_+ B)')D^{-1} u_+ C >\right\}_{as(A,C)} \\
& - \left\{< (u_+ B' + (u_- B)')AD^{-1} u_- C >\right\}_{as(A,C)} \\
\simeq & - \left\{< u_-(C'A)D^{-1} u_- B > + < u_-(AC')D^{-1} u_- B >\right\}_{as(A,C)} \\
& + \left\{< Au_+(B')D^{-1} u_- C > - < A' u_+(B)D^{-1} u_- C >\right\}_{as(A,C)} \\
& - \left\{< Au_+ B(u_- C) > + < u_+(B')AD^{-1} u_- C >\right\}_{as(A,C)} \\
& + \left\{< u_+(B)A' D^{-1} u_- C > + < u_+(B)A(u_- C) >\right\}_{as(A,C)} \\
\simeq & \{- < A(uB + Bu)(uC - Cu) > + < (uB - Bu)A(uC - Cu) >\}_{as(A,C)} \\
& - \{< AuBuC > + < ABuuC >\}_{as(A,C)} \\
& + \{< AuBCu > + < ABuCu >\}_{as(A,C)} \\
& + \{< uBAuC > - < uBACu >\}_{as(A,C)} \\
& - \{< BuAuC > - < BuACu >\}_{as(A,C)} \\
& - \left\{< uC'AD^{-1}(u_- B) > - < C'AuD^{-1}(u_- B) >\right\}_{as(A,C)} \\
& - \left\{< uAC'D^{-1}(u_- B) > - < AC'uD^{-1}(u_- B) >\right\}_{as(A,C)} \\
& + \left\{< AuB'D^{-1}(u_- C) > + < AB'uD^{-1}(u_- C) >\right\}_{as(A,C)} \\
& - \left\{< A'uBD^{-1}(u_- C) > + < A'BuD^{-1}(u_- C) >\right\}_{as(A,C)} \\
& - \left\{< uB'AD^{-1}(u_- C) > + < B'uAD^{-1}(u_- C) >\right\}_{as(A,C)} \\
& + \left\{< uBA'D^{-1}(u_- C) > + < BuA'D^{-1}(u_- C) >\right\}_{as(A,C)} \\
\simeq & \ 0
\end{aligned}
$$

In the fifth equality the following terms cancel $[1, 4, (6.3)]$, $[2, 6, (6.1)]$, $[3, 8, (6.1)]$, $[2, 7, (6.3)]$, $[9, 17, (6.1)]$, $[10, 16, (6.3)]$, $[11, 19, (6.3)]$, $[12, 14, (6.1)]$, $[13, 20, (6.1)]$, and $[15, 18, (6.1)]$.

6.4 -2-th order terms

$$
\begin{aligned}
< A, \Theta'[\Theta B] C >_{(-2)} &\simeq \left\{ < A(u_- D^{-1} u - B)_- D^{-1} C > \right\}_{as(A,C)} \\
&\simeq \left\{ < A(u_- D^{-1}(u_- B))_- D^{-1}(u_- C) > \right\}_{as(A,C)} \\
&\simeq \left\{ < A(u D^{-1}(u_- B) - D^{-1}(u_- B) u)_- D^{-1}(u_- C) > \right\}_{as(A,C)} \\
&\simeq \left\{ < A u D^{-1}(u_- B) D^{-1}(u_- C) > - < A D^{-1}(u_- B) u D^{-1}(u_- C) > \right\}_{as(A,C)} \\
&+ \left\{ - < A D^{-1}(u_- C) u D^{-1}(u_- B) > \right\}_{as(A,C)} \\
&+ \left\{ < A D^{-1}(u_- C) D^{-1}(u_- B) u > \right\}_{as(A,C)} \\
&\simeq 0
\end{aligned}
$$

In the last equality the following terms cancel $[1,(6.1)]$, $[2,3,(6.1)]$ and $[4,(6.1)]$.