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ANALYTICAL PROPERTIES OF SOLUTIONS TO A SYSTEM OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Certain analytical properties of self-similar solutions to a system of N partial differential equations, which is the generalized Liouville equation, are investigated.

1. We investigate the analytical properties of the solutions to the following autonomous ordinary differential equation system

$$\begin{aligned}
\left(y_0' \cdot y_0^{-1}\right)' &= y_1, \\
\left(y_1' \cdot y_1^{-1}\right)' &= y_2 - 2y_1, \\
\left(y_k' \cdot y_k^{-1}\right)' &= y_{k+1} - 2y_k + y_{k-1}, \qquad k = \overline{2, N-2}, \\
\left(y_{N-1}' \cdot y_{N-1}^{-1}\right)' &= -2y_{N-1} + y_{N-2},
\end{aligned} \tag{1}$$

which is the self-similar reduction of the partial differential equation system [1,2]

$$\begin{aligned}
\left((w_0)_t' \cdot w_0^{-1}\right)_x' &= w_1, \\
\left((w_1)_t' \cdot w_1^{-1}\right)_x' &= w_2 - 2w_1, \\
\left((w_k)_t' \cdot w_k^{-1}\right)_x' &= w_{k+1} - 2w_k + w_{k-1}, \qquad k = \overline{2, N-2}, \\
\left((w_{N-1})_t' \cdot w_{N-1}^{-1}\right)_x' &= -2w_{N-1} + w_{N-2}
\end{aligned} \tag{2}$$

if $w_l(x,t) = \exp(-z)y_l(z)$, $\exp z = \tau$, $\tau = xt$, where $y_l' = \frac{dy_l}{dz}$, $l = \overline{0, N-1}$.

This system (2) is the generalized Liouville equation

$$w_{xt} = \exp w \tag{3}$$

with $N = 2$ or $N = 3$. Actually, if $N = 2$ and if $w_0 = 3w_1$, we obtain Eq. (3), where $w_1 = \exp w$. In the case where $N = 3$ and $w_2 = w_1$, the second and third equations of system (2) can also be considered as Eq. (3) (up to the scaling of x and t).

Equation (3) often appears in different branches of mathematics and physics. For example, it was shown in [3] that a number of nonlinear models can be described by one nonlinear equation, (3): the gravitation theory with a constant scalar, the Born–Infeld massless field, and the relativistic string. It was established in [4, 5] that the description of stationary plasma configuration topology in cross-compatible fields can also be reduced to the investigation of solutions to Eq. (3). Various properties of the solutions to Eq. (3) are described in detail in a number of papers by different authors. For instance, it was shown in [6] that Eq. (3) can be integrated by inverse scattering problem methods. It was proved in [7] that certain completely integrable dynamic systems, which describe nontrivially interacting relativistic particles, correspond to the singular solutions of Eq. (3). Finally, it was proved in [8] that Eq. (3) possesses the Painleve property, which leads to the general solution formula for Eq. (3), presented first in [9].

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2. Differential equation system (1) is a P-type system, if $N = 2$ and $y_0 = 3y_1$, because it can be reduced to the equation

$$y_1 y_1'' - y_1'^2 = y_1^3, \tag{4}$$

which is the Painleve III equation [10]

$$yy'' = y'^2 + \exp z(\alpha y^3 + \beta y) + \exp 2z(\gamma y^4 + \delta); \tag{P$_3$}$$

if $y = \exp(-z)y_1$, $\alpha = 1$, $\beta = \gamma = \delta = 0$. Equation (4) implies

$$\left(y_1' y_1^{-1}\right)^2 = 2y_1 + K, \tag{5}$$

where K is a constant of integration. Equation (5) is integrated by elementary functions. In particular, if $K = 0$, its solution is $y_1 = \left(-z/\sqrt{2} + \widetilde{C}\right)^{-2}$, where $\widetilde{C}$ is an arbitrary constant. Thus, the functions

$$y_0 = 3(-z/\sqrt{2} + \widetilde{C})^{-2}, \qquad y_1 = (-z/\sqrt{2} + \widetilde{C})^{-2} \tag{6}$$

are the solutions of system (1) if $N = 2$.

If $N = 3$ and $y_2 = y_1$, system (1) is reduced to

$$(y_0' \cdot y_0^{-1})' = y_1, \qquad (y_1' \cdot y_1^{-1})' = -y_1. \tag{7}$$

The second equation is (P$_3$) if $y = \exp(-z)y_1$, $\alpha = -1$, and $\beta = \gamma = \delta = 0$.

The first integral of system (5) is

$$y_0' \cdot y_0^{-1} + y_1' \cdot y_1^{-1} = C_1, \tag{8}$$

where C_1 is an arbitrary constant. Equation (8) implies

$$y_0 = y_1^{-1} \cdot \exp(C_1 z + C_2), \tag{9}$$

where C_2 is an arbitrary constant.

The second equation of system (7) is a P-type equation, so (9) implies that system (7) is a P-type system.

3. In the general case, if $N > 2$, system (1) has the first integral

$$(N \log y_0)' + ((N-1)\log y_1)' + \cdots + (\log y_{N-1})' = \widetilde{C_1}, \tag{10}$$

where $\widetilde{C_1}$ is an arbitrary constant. It is easily derived from (10) that

$$\log(y_0^N \cdot y_1^{N-1} \cdots y_{N-1}) = \widetilde{C_1} z + \widetilde{C_2}$$

or

$$y_0 = (y_0^{N-1} \cdot y_2^{N-2} \cdots y_{N-1})^{-1/N} \cdot \exp\left((\widetilde{C_1} z + \widetilde{C_2})/N\right), \tag{11}$$

where $\widetilde{C_2}$ is an integration constant.

If $N = 2 (y_0 \neq 3y_1)$, system (1) has the two-parameter set of solutions with poles

$$y_0 = \frac{6}{(z - z_0)^2} + 3\beta + \sum_{k=1}^{\infty} \alpha_k (z - z_0)^k,$$

$$y_1 = \frac{2}{(z - z_0)^2} + \beta + \sum_{k=1}^{\infty} \beta_k (z - z_0)^k, \tag{12}$$

where z_0, β are arbitrary constants and parameters α_k, β_k are uniquely defined by parameter β. It should be mentioned that if $\beta = 0$ in (12), then $\alpha_k = \beta_k = 0$, $k = 1, 2, \ldots$. Thus, we obtain the one-parameter set of solutions of form (6).

Let us consider the following system of differential equations:

$$\begin{aligned} \left(y_1' \cdot y_1^{-1}\right)' &= y_2 - 2y_1, \\ \left(y_k' \cdot y_k^{-1}\right)' &= y_{k+1} - 2y_k + y_{k-1}, \qquad k = \overline{2, N-2}, \\ \left(y_{N-1}' \cdot y_{N-1}^{-1}\right)' &= -2y_{N-1} + y_{N-2}, \end{aligned} \tag{13}$$

which is system (1) without the first equation. We shall use the approach based on the Kowalewskaya–Painleve test presented in [11] (see review [12]) to prove the following.

Theorem. *System* (13) *has a family of solutions with poles depending on* N *arbitrary constants.*

Proof. Using the methods of [13], it is easy to show that the solutions of (13) with poles, under the assumption of their existence, have the form

$$y_l = \frac{a_l}{t^2} + \frac{b_l}{t} + \sum_{q=2}^{\infty} \alpha_{q-2}^{(l)} t^{q-2}, \qquad l = \overline{1, N-1}, \qquad t = z - z_0, \tag{14}$$

where z_0 is an arbitrary parameter.

Substituting (14) into (13), we obtain the system of linear equations determining the coefficients $a_l \neq 0$, b_l $(l = \overline{1, N-1})$:

$$\begin{aligned} -2a_1 + a_2 &= 2, \\ a_{k-1} - 2a_k + a_{k+1} &= 2, \qquad k = \overline{2, N-2}, \\ -2a_{N-1} + a_{N-2} &= 2, \\ a_1(b_2 - b_1) + 2b_1(a_2 - a_1 - 2) &= 0, \end{aligned} \tag{15}$$

$$\begin{aligned} a_k(b_{k+1} - 2b_k + b_{k-1}) + 2b_k(a_{k+1} - 2a_k + a_{k-1} - 2) &= 0, \qquad k = \overline{2, N-2}, \\ a_{N-1}(b_{N-2} - 2b_{N-1}) + 2b_{N-1}(a_{N-2} - 2a_{N-1} - 2) &= 0. \end{aligned} \tag{16}$$

It is not difficult to verify that the numbers $a_l = -l(N-l)$ $(l = \overline{1, N-1})$ determine the unique solution of system (15). Moreover, system (16) has a zero solution: $b_1 = b_2 = \cdots = b_{N-1} = 0$.

We can find the coefficients $\alpha_{q-2}^{(l)}$, $(q = \overline{2, N}, l = \overline{1, N-1})$ from the $N-1$ systems of linear homogeneous equations

$$\begin{aligned} (s + 2 - 2N)\alpha_{q-2}^{(1)} + (N-1)\alpha_{q-2}^{(2)} &= 0, \\ k(N-k)\alpha_{q-2}^{(k-1)} + [s - 2k(N-k)]\alpha_{q-2}^{(k)} + k(N-k)\alpha_{q-2}^{(k+1)} &= 0, \qquad k = \overline{2, N-2}, \\ (N-1)\alpha_{q-2}^{(N-2)} + (s + 2 - 2N)\alpha_{q-2}^{(N-1)} &= 0, \qquad s = q^2 - q, \qquad q = \overline{2, N}. \end{aligned} \tag{17}$$

Each system (17) has a nontrivial solution if $\det A = 0$, where A is the tridiagonal matrix

$$A = \begin{bmatrix} a_{11} & b_{12} & 0 & \ldots & 0 & 0 & 0 \\ c_{21} & a_{22} & b_{23} & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & c_{N-2,N-3} & a_{N-2,N-2} & b_{N-2,N-1} \\ 0 & 0 & 0 & \ldots & 0 & c_{N-1,N-2} & a_{N-1,N-1} \end{bmatrix}$$

with the nonzero elements $a_{ii} = s - 2i(N-i)$, $i = \overline{1, N-1}$; $b_{jj+1} = j(N-j)$, $j = \overline{1, N-2}$; $c_{k+1,k} = (k+1)(N-k-1)$, $k = \overline{1, N-2}$. Now, we shall show that $\det A = (s-\alpha_1)(s-\alpha_2)\dots(s-\alpha_{N-1})$, $\alpha_m = m(m+1)$, and $m = \overline{1, N-1}$. Adding the first $N-2$ lines to the last line of matrix A, we get the following equality: $\det A = (s-2)\det A_1$, where A_1 is the matrix whose elements equal those of matrix A, except for the elements of the last line. Each element in the last line of matrix A_1 is equal to 1.

Multiplying the first $N-3$ lines of matrix A_1 by the numbers $N-2$, $N-3$, ..., 2, respectively, and the last line by $-2(N-2)$ and adding it to the next to the last line, we obtain the following equality: $\det A = (s-2)(s-6)\det A_2$, where A_2 is the matrix whose elements equal those of matrix A_1, except for the elements of the next to the last line. The elements of this line are equal to $N-2$, $N-3$, ..., 2, 1, 0, respectively.

Multiplying the first $N-4$ lines of matrix A_2 by $\lambda_1 = \frac{(N-2)(N-3)}{2}$, $\lambda_2 = \frac{(N-3)(N-4)}{2}$, ..., $\lambda_{N-4} = 3$ ($\lambda_1 = \lambda_2 + N - 3$, $\lambda_2 = \lambda_3 + N - 4$, ..., $\lambda_{N-5} = \lambda_{N-4} + 3$, $\lambda_{N-4} = 3$), respectively, and the next to the last line by $-3(N-3)$ and adding it to the $N-3$th line, we obtain the following equality: $\det A = (s-2)(s-6)(s-12)\det A_3$, where A_3 is the matrix whose elements equal those of A_2, except for the elements of the third line from the last. Its elements are equal to λ_1, λ_2, ..., λ_{N-4}, 1, 0, 0, respectively. Moreover, the elements of the last three lines of matrix A_3 satisfy the following equalities: $a_{il} = a_{il+1} + b_{i+1l+1}$, $i = N-3, N-2, N-1$; $l = \overline{1, N-1}$. All the elements above the diagonal of the matrix are zeroes.

Following the proposed method, we sequentially transform matrix A_3 $N-4$ times and get the equality: $\det A = (s-2)(s-6)\dots[s-(N-2)(N-1)]\,[s-(N-1)N]\cdot 1$, where

$$1 = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ N-2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \lambda_1 & N-3 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \dots & 3 & 1 & 0 & 0 \\ N-2 & N-3 & N-4 & N-5 & \dots & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \end{vmatrix},$$

Q.E.D.

Each $q^2 - q - m(m+1) = 0$ equation, $m = \overline{1, N-1}$, has only one positive root, $q = m+1$. Thus, equation $\det A = 0$ has exactly $N-1$ positive roots: $q_1 = 2$, $q_2 = 3$, ..., $q_{N-1} = N$.

It is easy to see that $\operatorname{rank} A = N-2$ if $q = 2, 3, \dots, N$. This means that each line of coefficients of the matrix

$$B = \begin{bmatrix} \alpha_0^{(1)} & \alpha_0^{(2)} & \dots & \alpha_0^{(N-1)} \\ \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(N-1)} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{N-2}^{(1)} & \alpha_{N-2}^{(2)} & \dots & \alpha_{N-2}^{(N-1)} \end{bmatrix}$$

has one arbitrary coefficient. The other coefficients of the given line are uniquely defined by the value of that arbitrary coefficient. If $q > N$ in expansion (14), then the coefficients $\alpha_{q-2}^{(1)}$, $\alpha_{q-2}^{(2)}$, ..., α_{q-2}^{N-1} ($q = N+1, N+2, \dots$) are uniquely defined by the elements of matrix B. Thus, every component y_l of expansion (14) depends on N arbitrary constants (taking into account that z_0 is an arbitrary parameter). Q.E.D.

It should be mentioned that if all the elements of matrix B are zeroes, then $\alpha_{q-2}^{(l)} = 0$ ($l = \overline{1, N-1}$, $q = N+1, N+2, \dots$) and thus we obtain the explicit one-parameter solution of system (13): $y_l = -\frac{l(N-l)}{(z-z_0)^2}$, $l = \overline{1, N-1}$.

Therefore, if $N > 2$, system (1) has the explicit solution

$$y_0 = \left[(-1)^{N-1}(N-1)!\right]^{-\frac{1}{N}} (z-z_0)^{(N-1)} \exp\left((\widetilde{C}_1 z + \widetilde{C}_2)/N\right), \qquad y_l = -\frac{l(N-l)}{(z-z_0)^2}$$

and the solution

$$y_0 = (y_1^{N-1} \cdot y_2^{N-2} \cdots y_{N-1})^{-\frac{1}{N}} \exp\left((\widetilde{C}_1 z + \widetilde{C}_2)/N\right),$$
$$y_l = -\frac{l(N-l)}{(z-z_0)^2} + \sum_{q=2}^{\infty} \alpha_{q-2}^{(l)} (z-z_0)^{q-2}, \qquad l = \overline{1, N-1}. \tag{18}$$

Moreover, each formal series in (18) defining y_l depends on N arbitrary constants.

REFERENCES

1. B. Fuchssteiner and M. Lo Schiavo, *Appl. Math. Lett.*, **6**, 97 (1993).
2. B. Fuchssteiner and M. Lo Schiavo, *Manuscr. Math.*, **79**, 27 (1993).
3. B. M. Barbashov, V. V. Nesterenko, and A. M. Cherviakov, *Theor. Math. Phys.*, **40**, 572 (1979).
4. N. N. Komarov, *Yad. Sint.*, **3**, 174 (1963).
5. B. M. Fadeev, I. F. Kvartshava, and N. N. Komarov, *Yad. Sint.*, **5**, 202 (1965).
6. V. A. Andreev, *Theor. Math. Phys.*, **29**, 1027 (1976).
7. A. K. Pogrebkov, *Theor. Math. Phys.*, **45**, 951 (1980).
8. K. M. Tamizhmani and M. Lakshmanan, *J. Math. Phys.*, **27**, 2257 (1986).
9. J. Liouville, *J. Math. Pures Appl.*, **18**, 71 (1853).
10. E. L. Ince, *Ordinary Differential Equations*, Dover, N.Y. (1944).
11. A. P. Veselov and A. B. Shabat, *Funct. Anal. Appl.*, **27**, 10 (1993).
12. M. D. Kruskal and P. A. Clarkson, *Stud. Appl. Math.*, **86**, 87 (1992).
13. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981).