

# A new integrable system: The interacting soliton of the BO

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## Abstract

The interacting soliton equation of the Benjamin-Ono equation is found to be  $s_t = Hs_{xx} - cs_x - is_xH(s^{-1}s_x) - is^{-1}s_xH(s_x)$ , where  $H$  is the Hilbert transform. It is integrable in the sense that it has infinitely many commuting symmetries and conservation laws in involution. Only solitons with speed specified by the spectral parameter  $c$  emerge from its time evolution. Since it is connected to the BO by the interacting soliton projection solutions can be obtained via derivatives of solutions of the Benjamin-Ono equation, with respect to asymptotic phases.

## 1 Introduction

Surprisingly, in most work on interacting fields a differential geometric invariant definition of the notion *particle* is missing. In [7] for nonlinear Schrödinger equations the notion of particles was investigated in a differential geometric setup. Heuristically we adopted the viewpoint that those parts of a joint field are called particles which have the property that small changes of their states have only negligible effects on the other particles. For a nonlinear equation of motion for a joint wave function, such that its solution  $\psi$  represents an  $N$ -particle system we found:

**Observation:** *The dynamics of the gauge group generators  $i\varphi_1, \dots, i\varphi_N$  of the individual particles is that of a symmetry group generator of the dynamics of the joint wave function. Furthermore the vector fields given by these gauge group generators  $i\varphi_1, \dots, i\varphi_N$  are commuting with each other (in the vector field Lie algebra).*

This observation leads for nonlinear Schrödinger equations to

**Definition 2:** A field  $\psi$  with gauge invariant energy function  $\mathcal{E}(\psi)$  is an  $N$ -particle system if there is a decomposition

$$\psi = \varphi_1 + \cdots + \varphi_N$$

such that the different gauges  $\varphi_k$  are an abelian set of symmetry group generators for the dynamics

$$\psi_t = i \operatorname{grad} \mathcal{E}(\psi)$$

of the wave function  $\psi$ .

The role of the generator of the gauge group is easily generalized to arbitrary hamiltonian situations because its special property comes from the fact that it is obtained by mapping a gradient of a conservation law via the hamiltonian formulation onto a symmetry group generator.

We therefore generalize the situation above to an arbitrary translation invariant situation where a dynamic for a field variable  $u$  is given by

$$u_t = K(u). \tag{1.1}$$

Here  $u = u(x), x \in \mathbb{R}^n$  and  $K(u)$  is a vector field which commutes with the generators of translation, i.e. with the dynamics given by the vector fields  $u_{x_i}, i = 1, \dots, n$ .

**We define:**  $u$  to be an  $N$ -particle solution if there is a decomposition

$$u = u_1 + \cdots + u_N$$

such that the individual translations  $u_{kx_i}, i = 1, \dots, n, k = 1, \dots, N$ , are an abelian set of symmetry group generators for the dynamics around the given solution.

To see that this definition makes sense one should consider a solution which is in a heuristic sense an  $N$ -particle solution. I.e. we assume that asymptotically for large time  $t$  the solution  $u$  decomposes into traveling waves

$$u(x, t) \simeq \sum_{i=1}^N s_i(x + c_i t + q_i) \quad \text{for } t \rightarrow \infty \tag{1.2}$$

where the  $s_i(x + c_i t + q_i)$  are traveling wave solutions of (1.1). The different  $c_i$  are the speeds of the asymptotically emerging waves and the  $q_i \in \mathbb{R}^n$  describe suitable phases. Furthermore, we assume that a small change in the asymptotic data of one of emerging waves has only negligible effects on the other emerging waves. Then taking the gradients with respect to the  $q_i$

$$\sigma_i := \nabla_{q_i} u := \begin{pmatrix} \frac{\partial}{\partial q_1} \\ \vdots \\ \frac{\partial}{\partial q_n} \end{pmatrix} u \tag{1.3}$$

we easily find that

- i) The  $\sigma_i$  are generators of one-parameter symmetry groups around the given solution. This is because they are derivatives with respect to the invariant quantities  $q_i$  such that these derivatives commute with the time derivative, hence they are invariant vector fields.
- ii) The  $\sigma_i$  converge asymptotically towards the emerging traveling waves.
- iii) The vector fields  $\sigma_i$  commute with each other. To see this one observes that this is true asymptotically because a change in asymptotic data of one of the emerging waves has only negligible effects on the others. Now, because the vector fields are invariant with respect to the given flow this must be true in general.
- iv) Finally, we have

$$u_x = \sum_{i=1}^N \sigma_i \quad (1.4)$$

To see this, again we derive this equation for  $t \rightarrow \infty$  by observing that the  $x$ -gradient in (1.2) is equal to the sum of  $q_i$ -gradients, then we conclude that this must hold for all  $t$  (due to the invariance of the fields  $\sigma_i$ ).

Hence the  $s_i$  given by  $s_{i,x} := \sigma_i$  have all properties one reasonably can expect from particles. Therefore we call them *interacting particles* for (1.1).

All this has nothing to do with integrable systems since these considerations carry over to any nonlinear system. However, in the case of integrable nonlinear systems, like the KdV or the sine-Gordon equation, the  $\sigma_i$  have a remarkable additional property. For their dynamics an uncoupled formulation

$$s_{i,t} = G_i(s_i) \quad (1.5)$$

can be found which does not depend on  $u$  but only on  $s_i$ . Hence the dynamics of interacting particles in these cases can be described in terms of self-interaction alone [5]. In these cases the  $\sigma_i$  are called *interacting solitons* and (1.5) is then called the *interacting soliton equation*.

The observations above followed from the fundamental fact that in these cases the  $\sigma_i = s_{i,x}$  were eigenvectors of the recursion operator.

The question arises whether or not this possibility of describing the interacting particles by self-interaction also is true for other integrable systems even when these do not admit recursion operators.

In case of the BO (Benjamin-Ono equation) ([2] and [10]) we are giving in this paper an affirmative answer for this problem. The BO is

$$u_t = Hu_{xx} + 2uu_x \quad (1.6)$$

where  $H$  denotes the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} \int_{+\infty}^{-\infty} \frac{f(\xi)d\xi}{\xi - x} \quad (\text{principal value integration}) .$$

## 2 The method

In order to get an idea how to obtain the dynamics for interacting solitons we briefly review how the interacting soliton equation is obtained when a recursion operator exists [5].

Let  $\Phi(u)$  be a recursion operator for (1.1). Then its eigenvectors

$$\Phi(u)\sigma = \lambda\sigma \quad (2.1)$$

have the same dynamics as the generators of one-parameter symmetry groups:

$$\sigma_t = K'(u)[\sigma] := \frac{\partial}{\partial \epsilon|_{\epsilon=0}} K(u + \epsilon\sigma) . \quad (2.2)$$

The problem is how the dynamics can be uncoupled such that the field  $u$  does not appear anymore in this time evolution. In order to do that we consider (2.1) as a Lie-Bäcklund-transformation between  $u$  and  $\sigma$ . This transformation is unique modulo an additional symmetry. For suitable boundary conditions we solve this relation for  $u$  when  $\sigma$  is given and obtain  $u = U(\sigma) = U(s_x)$ . We insert the relation in (2.2) in order to obtain the interacting soliton equation. The resulting equation is again integrable since, modulo uniqueness up to a symmetry for the given system, it is related to the original integrable equation by a transformation between dependent variables, hence a manifold transformation which does not destroy integrability.

**Example:** For the mKdV (modified Korteweg de Vries equation)

$$u_t = u_{xxx} + 6u_x u^2 =: K(u)$$

the eigenvector equation given by the recursion operator is

$$\lambda s_x = s_{xxx} + 4D u D^{-1} u s_x .$$

This yields

$$u_x u^{-3} (s_{xx} - \lambda s) - u^{-2} (s_{xxx} - \lambda s_x) = 4s_x .$$

Solving the homogeneous equation and applying the method of variation of constants gives

$$u = \frac{\lambda s - s_{xx}}{2(\sqrt{\lambda s^2 - s_x^2} + C)}$$

where  $C = 0$  because of the boundary condition at infinity. Inserting this into

$$s_{xt} = K(u)'[s_x]$$

we obtain the interacting soliton equation for this case

$$s_t = s_{xxx} + \frac{3(\lambda s - s_{xx})^2}{2(\lambda s^2 - s_x^2)} s_x .$$

Now we recall that the eigenvector problem for the recursion operator is in many cases ([4], [6]) equivalent to a nonlinear eigenvector problem given by the auto-Bäcklund transformation (ABT) of the system. Let us briefly review this fact.

Consider an auto-Bäcklund transformation for (1.1)

$$B(u, \bar{u}, \lambda) = 0 . \quad (2.3)$$

That means, for any value of  $\lambda$ , we have that when  $u$  is a solution of (1.1) then  $\bar{u}$  also has to be a solution. For example for the KdV there is the well known auto-Bäcklund transformation

$$B(u, \bar{u}, \lambda) = u + \bar{u} + \lambda D^{-1}(u - \bar{u}) + \frac{1}{2}\{D^{-1}(u - \bar{u})\}^2 = 0 . \quad (2.4)$$

Now we consider the following *nonlinear spectral problem*:

Given a solution  $u$  of (1.1), consider the partial variational derivative  $B_u(u, \bar{u}, \lambda)$  of  $B(u, \bar{u}, \lambda)$  and find those  $\lambda$ 's such that there is some non-zero vector field  $\omega$  and some  $\bar{u}$  on the manifold under consideration such that

$$B_u(u, \bar{u}, \lambda)[\omega] = 0 \quad \text{and} \quad B(u, \bar{u}, \lambda) = 0 . \quad (2.5)$$

That means we try to find those values of  $\lambda$  where the implicit function theorem is not valid. Then in all known cases where a recursion operator exists, this nonlinear spectral problem is equivalent to a linear one, namely the one given by the recursion operator.

Mostly this problem is easily linearized in a purely algorithmic way. Let us see this at one

**Example:** Consider the case of the KdV. Variational derivative of (2.4) with respect to  $u$  yields the operator:

$$B_u = I + (D^{-1}(u - \bar{u}))D^{-1} + \lambda D^{-1} . \quad (2.6)$$

And the spectral problem (2.5) reads as follows

$$0 = \omega + (D^{-1}(u - \bar{u}))D^{-1}\omega + \lambda D^{-1}\omega . \quad (2.7)$$

This is only formally linear since  $\bar{u}$  and  $\omega$  are not independent. Abbreviation  $D^{-1}\omega = v$  allows to write

$$D^{-1}(u - \bar{u}) = -\left(\frac{v_x}{v} + \lambda\right) . \quad (2.8)$$

Writing  $u + \bar{u}$  as  $2u - (u - \bar{u})$  and replacing all terms  $u - \bar{u}$  in (2.4) by (2.8) we obtain

$$2u + \left(\frac{v_x}{v} + \lambda\right)_x + \frac{1}{2}\left(\frac{v_x}{v} + \lambda\right)^2 - \lambda\left(\frac{v_x}{v} + \lambda\right) = 0 \quad (2.9)$$

which certainly is a *nonlinear* eigenvalue equation. By multiplication with  $v^2$  we get

$$2uv^2 + v_{xx}v - \frac{1}{2}v_x v_x = \frac{1}{2}\lambda^2 v^2 . \quad (2.10)$$

If this problem can be linearized there must be operators  $A(v)$  and  $\Psi(u)$  such that  $A(v)v = Cv^2$  and  $A(v)\Psi(u)v$  is equal to the left hand side of (2.10). Comparison of suitable terms yields in an algorithmic way:

$$D^{-1}vD\{v_{xx} + 2uv + 2D^{-1}(uv_x)\} = \lambda^2 D^{-1}vDv . \quad (2.11)$$

Hence  $A(v) = D^{-1}vD$  and  $\Psi(u) = D^2 + 2u + 2D^{-1}uD$ . Going back to  $\omega = v_x$  we see that  $\omega$  is a solution of (2.7) if and only if  $\omega$  is an eigenvector of  $\Phi(u) = D\Psi(u)D^{-1} = D^2 + 2u + 2DuD^{-1}$ , the recursion operator of the KdV.

The foregoing considerations suggest that the results which can be obtained from the recursion operator in principle can be derived from the auto-Bäcklund transformation. Therefore, also for those integrable equations with only an ABT, the interacting soliton equation certainly is hidden in that relation. However, we have to find a different approach:

We consider an equation

$$u_t = K(u) \quad (2.12)$$

such that there is an ABT

$$B(u, \bar{u}, \lambda) = 0. \quad (2.13)$$

For a given solution  $u$  with emerging soliton  $s_i(x + ct + q)$ , having asymptotic speed  $c$  and phase  $q$  at  $t \rightarrow \infty$ , we consider the spectral parameter  $\lambda_0 = \lambda(c)$  such that in the solution  $\bar{u}$  this soliton is annihilated by the ABT. Then clearly, taking the gradient of (2.13), for  $\lambda = \lambda_0$  with respect to  $q$ , we obtain

$$\nabla_q B(u, \bar{u}, \lambda_0) = B_u(u, \bar{u}, \lambda_0)[\nabla_q u] = 0. \quad (2.14)$$

Now we use this equation together with (2.13) in order to express  $u$  and  $\bar{u}$  in terms of  $s_x := \nabla_q u$ . These representations we use to replace  $u$  in the dynamic

$$s_{xt} = K(u)'[s_x] \quad (2.15)$$

for  $s$ . The result is the interacting soliton equation for  $s$ . This equation is integrable. A considerable class of its solutions can be obtained by differentiation of solutions of the original equation with respect to the asymptotic phases.

### 3 Application to the BO

Consider the BO

$$u_t = Hu_{xx} + 2uu_x. \quad (3.1)$$

For convenience we define  $H1 = 0$ . Observe that  $K$  is translation invariant. Equation (3.1) has ([1, p.204])

$$s_c(t, x) = \frac{ic}{c(x - x_0) + c^2t + i} - \frac{ic}{c(x - x_0) + c^2t - i} \quad (3.2)$$

as one-soliton solutions, i.e. solutions of  $cu_x = Hu_{xx} + 2uu_x$ . This is easily seen from

$$iH\left(\frac{1}{x - \alpha}\right) = \begin{cases} +\frac{1}{x - \alpha} & \text{if imaginary part of } \alpha < 0 \\ -\frac{1}{x - \alpha} & \text{if imaginary part of } \alpha > 0 \end{cases} \quad (3.3)$$

A typical two soliton looks like

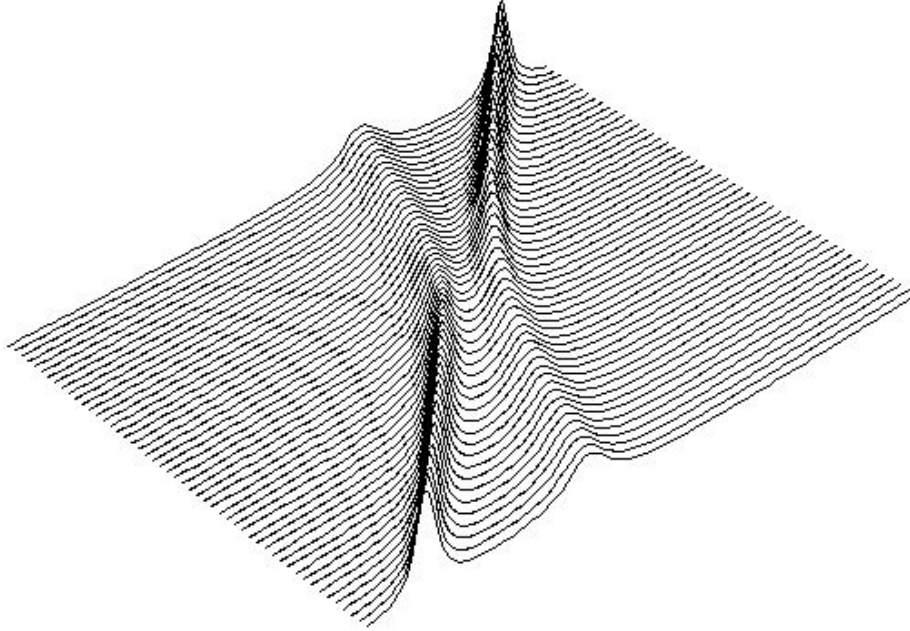


Fig. 1: A two soliton for the BO

Here the line in front is parallel to the  $x$ -axis, the chosen asymptotic speeds are  $c_1 := 0.7$  and  $c_2 := 0.25$  and the  $t$ -slices have been scaled by  $3/2$ .

For the BO we have the well known ABT [9]

$$\exp(iD^{-1}(u - \bar{u})) - 1 - \frac{1}{c}\{iH(u - \bar{u}) + (u + \bar{u})\} = 0 \quad (3.4)$$

which annihilates those solitons in  $u$  with asymptotic speed  $c$ . Equation (2.14) now has the form

$$is \exp(iD^{-1}(u - \bar{u})) - \frac{1}{c}(iHs_x + s_x) = 0 . \quad (3.5)$$

Introducing orthogonal projection operators

$$P_{\pm} := \frac{1}{2}(I \pm iH)$$

we write (3.4) and (3.5) as

$$\exp(iD^{-1}(u - \bar{u})) - 1 + \frac{2}{c}P_-(u - \bar{u}) - \frac{2}{c}u = 0 \quad (3.6)$$

and

$$D^{-1}(u - \bar{u}) = -i \ln \left( -\frac{2i}{cs} P_+ s_x \right) . \quad (3.7)$$

Inserting this last result into (3.6) we get

$$-\frac{2i}{cs}P_+(s_x) - 1 - \frac{2i}{c}P_-\left(\ln\left(\frac{2}{ics}P_+s_x\right)\right)_x = \frac{2u}{c}. \quad (3.8)$$

Observe that the decomposition given by the projection  $P_{\pm}$  is the usual decomposition into functions being analytic in the upper and lower half of the complex plane, respectively. Hence terms like

$$P_-\left(\frac{P_+s_{xx}}{P_+s_x}\right) = 0$$

vanish. Using this we rewrite (3.8) as

$$u = -\frac{c}{2} - \frac{i}{s}P_+(s_x) + iP_-\left(\frac{s_x}{s}\right). \quad (3.9)$$

The dynamics (2.15) in case of the BO is

$$s_{xt} = K(u)'[s_x] = Hs_{xxx} + 2s_xu_x + 2us_{xx}$$

or

$$s_t = Hs_{xx} + 2us_x. \quad (3.10)$$

Eliminating  $u$  by use of (3.9) we find for the interacting soliton equation

$$\begin{aligned} s_t &= Hs_{xx} - cs_x + 2is_xP_-s^{-1}s_x - 2is^{-1}s_xP_+s_x \\ &= Hs_{xx} - cs_x + s_xH(s^{-1}s_x) + s^{-1}s_xH(s_x) \end{aligned} \quad (3.11)$$

This equation is certainly integrable, its invariants can easily be obtained from those of the BO by using (3.9) as a manifold transformation. Observe that the invariants of the BO are easily computed by the usual mastersymmetry approach [3]:

To be concrete, replacing in any conserved scalar field of the BO the variable  $u$  by (3.9) leads to a conserved quantity for (3.11). Replacing in a symmetry generator of the BO the quantity  $u$ , then mapping the result by the inverse of the variational derivative of the right hand side of (3.11) yields the symmetry generators for (3.11).

Solutions are easily found for this equation: Since Fig. 1 represents a two soliton for the BO with asymptotic speeds  $c_1 := 0.7$  and  $c_1 := 0.25$  the corresponding derivatives with respect to the phases lead after one integration to the interacting solitons for the equations with these  $c$ -values. Plots show the expected asymptotic behaviour, and the effects due to the nonlinearity:

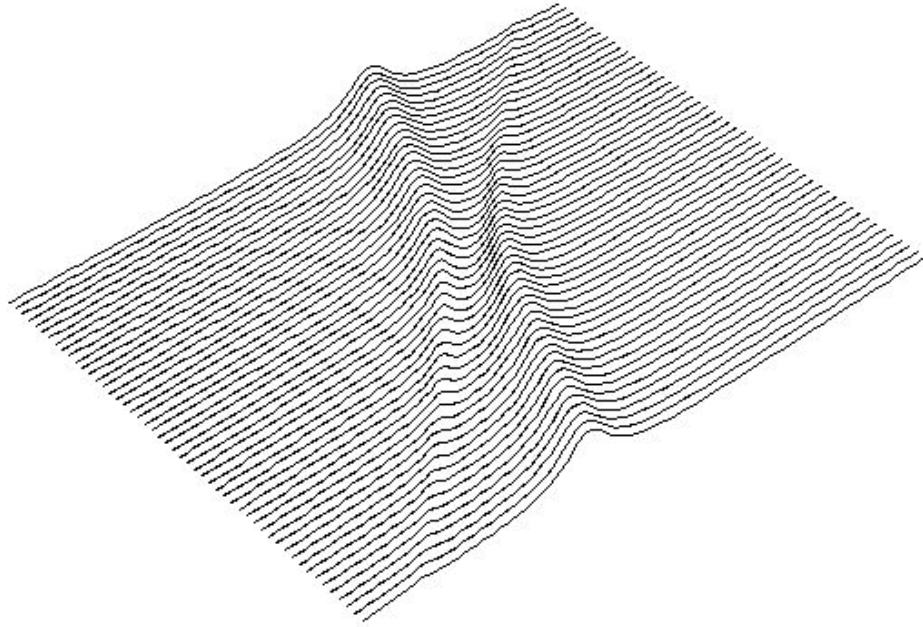


Fig. 2: The slower interacting soliton for the BO,  $c_1 := 0.25$

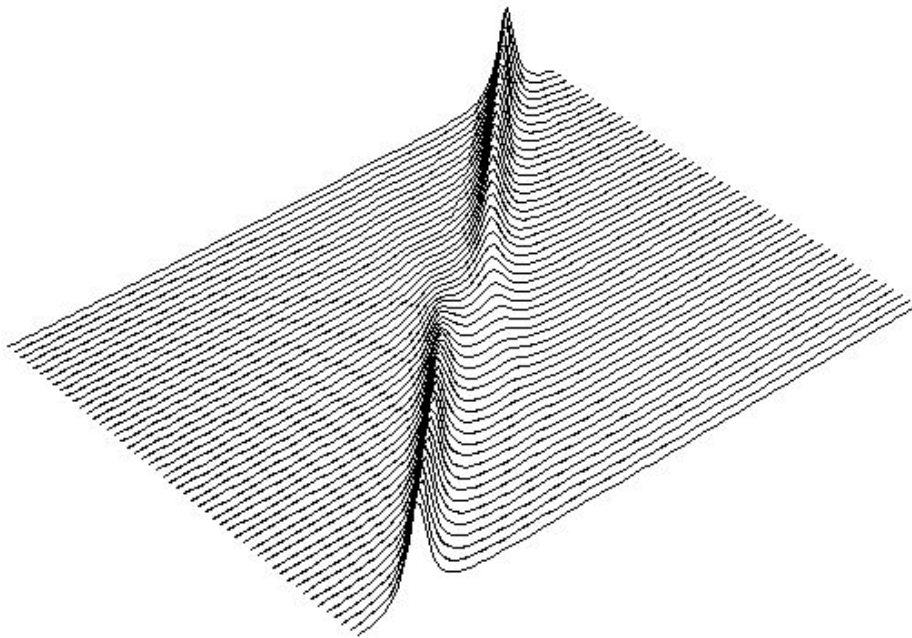


Fig. 3: The faster interacting soliton for the BO,  $c_1 := 0.6$

As known from the general theorie [8] (3.11) the dynamics corresponding to the special hamiltonian is given by the asymptotic speed  $c$ . The quantity  $c$  can be considered as an action variable. We are interested in finding the interaction dynamics for the corresponding angle variable, or rather the vector field obtained from it via the hamiltonian formulation of the system. To find this we denote by  $u(c, q, t)$  the solution of the BO having asymptotically an emerging soliton with speed  $c$  and phase  $q$ . Now we fix some  $t_0$  and we want to produce the solution  $u(c + \delta c, q, t)$  by a change of  $u(c, q, t_0)$  and then taking this as initial condition with respect to an evolution of time  $t - t_0$  under the given dynamics. We observe that  $u(c + \delta c, q, t)$  results out of a change  $c \rightarrow c + \delta c$  for the initial condition at time  $t = 0$  therefore the necessary change for the  $t_0$ -translated initial value problem is  $c \rightarrow c + \delta c$  and  $q \rightarrow q + t_0 \delta c$ . Hence, the quantity

$$\tau_x := \left( \frac{\partial}{\partial c} + t \frac{\partial}{\partial q} \right) u(c, q, t) \quad (3.12)$$

must have the dynamics of a symmetry group generator, i.e. is a solution of

$$\tau_{xt} = K(u)'[\tau_x] . \quad (3.13)$$

Taking the derivative of the ABT with respect to this derivative we obtain

$$i(\tau - i/c) \exp(iD^{-1}(u - \bar{u})) - \frac{1}{c}(iH\tau_x + \tau_x) = 0 . \quad (3.14)$$

Running through a similar computation as before we find

$$u = -\frac{c}{2} - \frac{i}{\tau - i/c} P_+(\tau_x) + iP_-\left(\frac{\tau_x}{\tau - i/c}\right) . \quad (3.15)$$

which inserted in (3.13) yields

$$\tau_t = H\tau_{xx} - c\tau_x + \tau_x H((\tau - i/c)^{-1}\tau_x) + (\tau - i/c)^{-1}\tau_x H(\tau_x) . \quad (3.16)$$

Hence,  $\tilde{s} := \tau - i/c$  is another solution of the interacting soliton equation for the BO, however with different asymptotics. Plots for these quantities, for the parameters considered before, are most easily obtained:

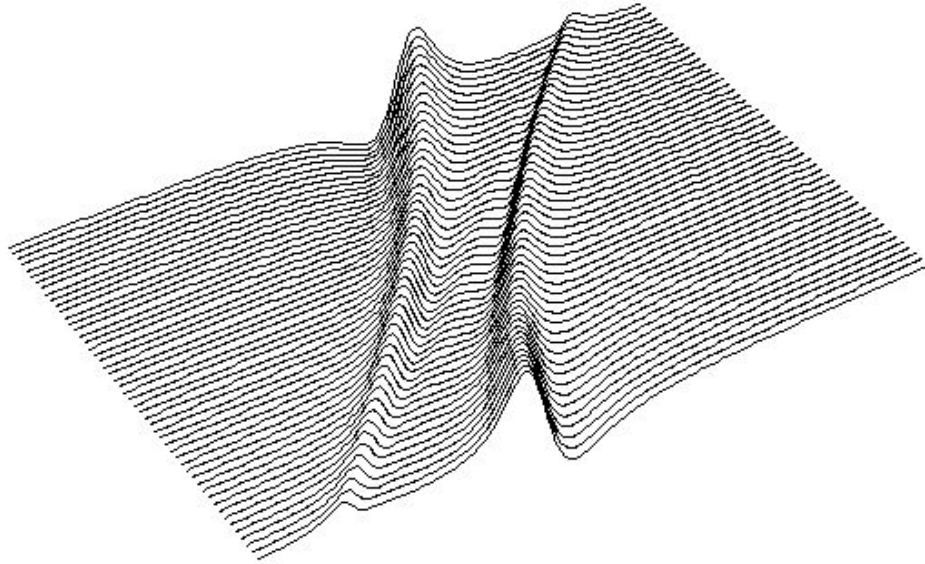


Fig. 4: The slower angle soliton for the BO

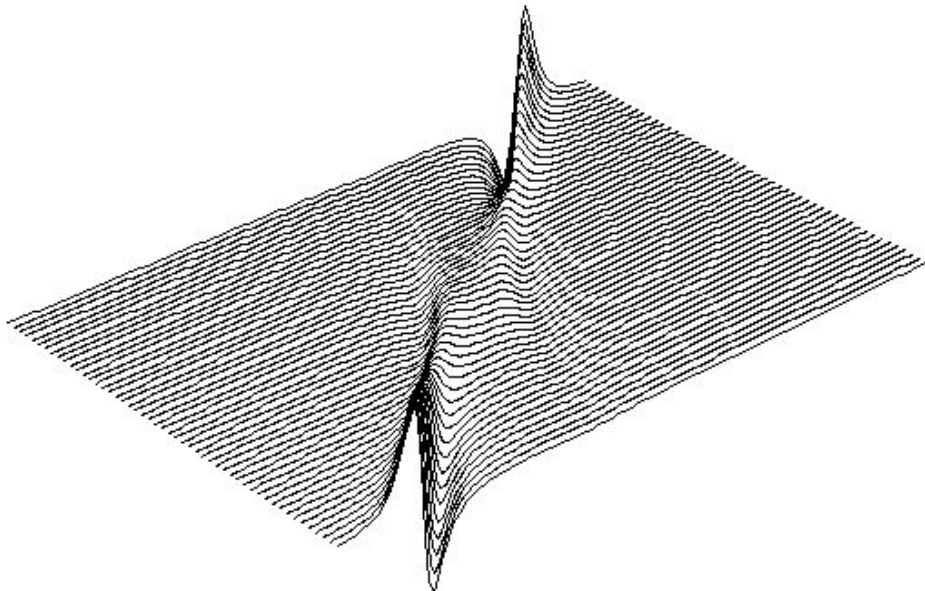


Fig. 5: The faster angle soliton for the BO

Since the derivatives with respect to phases were already plotted in Figs. 2 and 3 here we only plotted the derivatives with respect to the speeds. Therefore the plot represents the derivative of the density of an angle variable of the BO minus  $t \times$  the density of the corresponding angle variable.

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