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# Some Remarks on a Class of Ordinary Differential Equations: the Riccati Property [1]

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**Abstract.** Here ordinary differential equations of third and higher order are considered; in particular, a class of equations which can be solved by quadratures is exploited.

Indeed, crucial to obtain our result is the property of the Riccati equation, according to which, given one particular solution, then its general solution can be determined explicitly.

Thus, what we term the "Riccati" Property is introduced to point out that the members of such a class are differential equations which are of a generalized form of Riccati equation. Trivial examples of differential equations which enjoy the Riccati Property are all linear second order ordinary differential equations.

Here some further examples of ordinary differential equations which enjoy the same Property are considered. In particular, on the basis of group invariance requirements, a method to construct ordinary differential equations which enjoy the Riccati Property is given.

Remarkably, it follows that ordinary differential equations enjoying the Riccati Property are related to nonlinear evolution equations which admit a hereditary recursion operator.

Finally, further connections with nonlinear evolution equations are mentioned.

## 1. Introduction

Here the Riccati equation [7][10] is briefly reconsidered recalling its link with linear ordinary differential equations of the second order. Subsequently, third and higher order ordinary differential equations are considered; in particular, a class of equations which can be solved by quadratures is exploited.

Accordingly, the Riccati equation enjoys the property that given one particular solution, its general solution can be determined explicitly. Hence, in [3], the "Riccati" Property is introduced referring to a class of differential equations which share the same property with the Riccati equation [7]. By definition, thus, such a class of differential equations may comprise equations linear and nonlinear of any order. The reason why we termed such a property "Riccati" is that the best known example of an ordinary differential equation which enjoys it is the Riccati equation ([10],

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[7], [9]). Trivial examples of differential equations enjoying the Riccati Property are all linear second order ordinary differential equations. Indeed, the *variation of constants' method* implies that, known one particular solution, the general one can be obtained in explicit form.

The method to construct ordinary differential equations which enjoy the Riccati Property, proposed in [3], is reconsidered.

Subsequently, following the results obtained in [3], some linear ordinary differential equations of third order are proved to enjoy the Riccati property and, in particular, given one solution, other two independent ones are constructed.

Notably, see [3], given a third order nonlinear evolution equation which admits a hereditary recursion operator, an ordinary differential equation enjoying the Riccati Property can be constructed.

The paper closes providing further remarks concerning how ordinary differential equations which enjoy the Riccati Property are related to nonlinear evolution equations [3].

## 2. The "Riccati" Property

In this section some properties of the Riccati equation which turn out to be crucial for our study are briefly recalled [7], [9], [10]. The nonlinear Riccati equation reads:

$$\frac{dw}{dx} = Q(x) - P(x)w - w^2 \quad , \quad Q(x) \not\equiv 0$$

wherein $P(x)$ and $Q(x)$ are given real functions. It is transformed into

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0$$

by the Riccati transformation

$$w = \frac{u'}{u}.$$

Thus, it can be transformed into a second order **linear** ordinary differential equation, and, therefore, whenever one solution of the Riccati equation is known, then, the general one can be obtained by quadratures [7], [9].

However, typically, the general solution of the Riccati equation cannot be obtained by quadratures.

**Definition 2.1:** *An ordinary differential equation is said to possess the* **Riccati Property** *if, given one particular solution, it is possible to find explicitly its general solution by quadratures.*

Trivially, all linear second order ordinary differential equations enjoy the Riccati Property since, given a particular solution of any second order linear ordinary differential equation, the related general solution can be easily constructed on application of the *variation of constants' method.* Moreover, given a second order ordinary differential equation it can be trasformed into a family of nonlinear first order Riccati equations [10] which depend on a parameter: the integration constant.

Further examples of equations which enjoy such a property are represented by the following linear third order ordinary differential equations in the unknown function $s(x)$:

$$s''' + 4us' + 2u's = 0 \tag{2.1}$$

and

$$u'(s'' - \lambda s) - u(s''' - \lambda s') - 4u^3 s' = 0 \tag{2.2}$$

wherein $u$ is an arbitrary function of $x$ and $\lambda$ a real parameter.

Remarkably, the equation (2.1) reduces to an Euler linear equation on letting $u = -\frac{3}{4}x^{-2}$, namely:

$$x^3 s''' - 3xs' - 3s = 0. \tag{2.3}$$

This particular problem can be studied on use of classical techniques (see for instance [9]). However, even in this special case, our result turns out to be helpful since it allows to write immediatly, given any particular solution of (2.3), its general one.

Indeed, since we proved [3] that if $v_1 = s$ is a solution of (2.1), then both:

$$v_2 = sD_{x_0}^{-1}(s^{-1}) \tag{2.4}$$

and

$$v_3 = \frac{1}{2}s\left(D_{x_0}^{-1}(s^{-1})\right)^2 \tag{2.5}$$

wherein

$$D_{x_0}^{-1} = \int_{x_0}^{x} \cdot \; d\xi$$

are again solutions of (2.1), then the construction of the general solution of (2.3) is straightforward. As an example, one solution is readily checked to be $s = x^3$; the other two independent ones can be evaluated on use of (2.4) and (2.5) wherein $x_0 \in \mathbb{R}\setminus\{0\}$. They, in turn, are:

$$v_2 = -\frac{1}{2}x + \frac{1}{2{x_0}^2}x^3 \quad and \quad v_3 = -\frac{1}{4{x_0}^2}x + \frac{1}{8{x_0}^4}x^3 + \frac{1}{8x} \tag{2.6}$$

and, thus, the general solution reads:

$$s(x) = c_1 x^3 + c_2 x + c_3\frac{1}{x} \tag{2.7}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

The latter provides only a trivial example of application of equations enjoying the Riccati Property; more general is the case represented by (2.1), wherein $u$ is an arbitrarily given function of $x$: its general solution can be constructed via our procedure whenever one particular solution is known. The advantages of our procedure, that induces to write, from a single known solution, a set of independent ones, from which the general solution follows, show up clearly in the case of (2.2). Furthermore, it can be remarked that (2.2) does not reduce to an Euler linear equation for any choice of the parameter $\lambda$ and of the function $u$. In particular, we proved [3] that (2.2) on admitting a solution $v_1 = s$, admits also the solutions:

$$v_2 = sD_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\ s^2 -\ s'^2}\right) \tag{2.8}$$

and

$$v_3 = \frac{1}{2}s\left\{D_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\ s^2 -\ s'^2}\right)\right\}^2 - s\left(D_{x_0}^{-1}(s's^{-3})\right)\ . \tag{2.9}$$

### 3. How to construct equations possessing the Riccati Property

Here the method to construct linear equations which enjoy the Riccati Property [3] is retrieved in detail, furthermore links between the examples presented and well known ordinary differential equations are supplied. For sake of simplicity, we restrict ourselves to illustrate the method in the case of linear third order ordinary differential equations.

**Method:**

1) Consider a third order linear equation in the unknown function $s$ such that:
   - the coefficients depend on a given function $u$;
   - if the equation is considered as an equation in the unknown function $u$, instead of $s$, then it is possible to determine explicitly its solution $u$ as a function of $s$ and its derivatives up to second order, say

   $$u = F(c_1, \ldots, c_n, s, s', s'')$$

   where the $c_1, \ldots, c_n$ are suitable constants of integration.
2) In the representation $u = F(c_1, \ldots, c_n, s, s', s'')$, choose the integration constants such that $F$ is invariant under the substitution $s \to \alpha s$, $\alpha \in \mathbb{R}$. Such a choice is possible since the original equation is linear in $s$ and, hence, invariant under this substitution.
3) Then, evaluate the variational derivative of $F$

   $$F'(s)[v] = \frac{\partial}{\partial \epsilon}F(s + \epsilon v, s' + \epsilon v', s'' + \epsilon v'')_{|\epsilon=0} \tag{3.1}$$

   and consider $F'(s)[v] = 0$ as a second order ordinary differential equation in the unknown function $v$. The homogenuity imediatly implies that one solution is given by $v = s$ and a second independent solution

   $$v = G(s) \tag{3.2}$$

   can be computed on application of the variation of constants' method. This $G(s)$ represents also a solution of the original linear problem.
4) Compute, now, the third solution of the original linear problem via the known two solutions by further application of the variation of constants' method.

The method, we just presented, can be applied to the examples we already gave (2.1) and (2.2). Let us, at first, concentrate our attention on (2.1) and consider it as an ordinary differential equation in the unknown function $u$. This, solved with respect to $u$, gives:

$$u \;=\; \frac{1}{4s^2}(\lambda s^2 \;-\; 2ss'' \;+\; s'^2) + \frac{c_1}{s^2} \tag{3.3}$$

where $c_1$ is a constant of integration. According to step 2), let us require the invariance of $u$ under the substitution $s \to \alpha s, \quad \alpha \in \mathbb{R}$. This implies $c_1 = 0$ and, thus, (3.3) reads:

$$u \;=\; \frac{1}{4s^2}(\lambda s^2 \;-\; 2ss'' \;+\; s'^2) \tag{3.4}$$

Then, step 3), we look for a one-parameter group which transforms the $s$-variable so that the corresponding $u$-variable is left unchanged. Let us denote by $\tau$ the group parameter and by $v = s_\tau$ its infinitesimal generator. Thus, on derivation with respect to $\tau$ of (3.4), it follows that $v$ satisfies the linear differential equation:

$$-s^2 v'' + ss'v' + (ss'' - s'^2)v = 0 \tag{3.5}$$

The homogenuity of (3.4) implies that the special one-parameter group $s \to \tau s$ leaves $u$ unchanged. The infinitesimal symmetry generator of this special group is $s$ itself, thus, $v_1 = s$ is a particular solution of (3.5), which, interpreted as a differential equation in the unknown $v$, enjoys the Riccati Property and, therefore, its general solution can be obtained by quadratures. Applying the variation of constants' method, a second solution of (3.5), independent from the previous one, is obtained:

$$v_2 = \frac{1}{2} s \left( D_{x_0}^{-1}(s^{-1}) \right) . \tag{3.6}$$

Now, since any solution $s$ of (3.4) is a solution of (3.3), the group $s \to s(\tau)$, for which the generator is determined, leaves invariant also (3.3). Furthermore, the linearity of (2.1) implies that the related solution space is a linear space; consequently, any infinitesimal generator of a group acting on its solution space is itself a vector which belongs to the same space of solutions. Thus, any solution $v$ of (3.5), whenever $s$ in (3.5) is a solution of (2.1), satisfies (2.1) as well. This implies that (3.6) represents another solution of (2.1).

Finally, given the two independent solutions $v_1 = s$ and (3.6) of the third order equation (2.1), a third solution, independent from both the previous ones, can be obtained by the method of variation of constants. It follows that, if $s$ is a solution of (2.1), then both (2.4) as well as (2.5) are again solutions of (2.1). Equation (2.1), thus, is proved to enjoy the Riccati Property.

The same method allows to prove that also (2.2) enjoys the Riccati Property; indeed, solution of (2.2) with respect to $u$, gives:

$$2u \;=\; \frac{\lambda\; s - \; s''}{\sqrt{\lambda\; s^2 - \; s'^2 + c_1}} \; . \tag{3.7}$$

wherein $c_1 = 0$ is the only choice of such a constant which implies that $u(s)$ is homogeneous with respect to $s$, namely:

$$2u \;=\; \frac{\lambda\; s - \; s_{xx}}{\sqrt{\lambda\; s^2 - \; {s_x}^2}} \; . \tag{3.8}$$

The subsequent analysis of the group transformations under which $u$, expressed by (3.8), is left unchanged, implies that if $s$ is a solution of (2.2) then

$$sD_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\, s^2 - \, s'^2}\right) \tag{3.9}$$

is again a solution of such equation. The third independent solution can be obtained considering (3.9) as an infinitesimal generator of a one-parameter group acting on the solution space of (3.9) itself. Namely, on use of the group

$$s_\tau = sD_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\, s^2 - \, s'^2}\right) \tag{3.10}$$

and of the linearity of the solution space, it follows that also $s_{\tau\tau}$ represents a solution of (2.2). Explicit computation delivers:

$$s_{\tau\tau} = \frac{1}{2}s\left\{D_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\, s^2 - \, s'^2}\right)\right\}^2 - s\left(D_{x_0}^{-1}(s's^{-3})\right) \ . \tag{3.11}$$

Thus, equation (2.2) is proved to admit, further to the solution $v_1 = s$, the solutions $v_2 = s_\tau$ and $v_3 = s_{\tau\tau}$, given, respectively, by (3.10) and (3.11), that is, in turn, (2.8) and (2.9).

## 4. Connections with Nonlinear Evolution Equations

Here, the connection between ordinary differential equations enjoying the Riccati Property and related nonlinear evolution equations is briefly pointed out. In particular, we show how the examples (2.1) and (2.2) are related to nonlinear evolution equations which admit a hereditary recursion operator. On the other hand, we remark the important role which ordinary differential equations enjoying the Riccati Property may have in connection with the Cauchy problem for nonlinear evolution equations.

Most nonlinear evolution equations admitting a hereditary recursion operator [2] can be written in the form

$$u_t = K(u) \quad , \quad K(u) = \Phi(u)u_x \tag{4.1}$$

wherein $\Phi$ represents the hereditary recursion operator ([2], [4]). In particular, the Korteweg-deVries ( KdV ) equation reads:

$$u_t \;=\; u_{xxx} + 6uu_x \tag{4.2}$$

which can be written in the form (4.1) wherein [2] the hereditary recursion operator is the Lenard operator [1], [2], [4]:

$$\Phi(u) \;=\; D^2 + 2DuD^{-1} + 2u \quad ; \quad D^{-1} = \int_{-\infty}^{x} \cdot \; d\xi. \tag{4.3}$$

2 Note, that both $u$ in (4.2) as well as $s$ in (4.4) are $C^\infty$-functions with respect to $x \in \mathbb{R}$ such that they, together with all their partial derivatives with respect to $x$, vanish at $\pm\infty$ faster than any rational function of $x$.

The Korteweg-deVries equation (4.2) is linked [6] to the KdV zero-interacting soliton equation [5]:

$$s^2 s_t \;=\; s^2 s_{xxx} \;-\; 3 s s_x s_{xx} \;+\; \frac{3}{2} {s_x}^3 \tag{4.4}$$

by the Bäcklund transformation:

$$u \;=\; \frac{1}{2}\left(\frac{s_x}{s}\right)_x - \frac{1}{4}\left(\frac{s_x}{s}\right)^2 \;=\; -\sqrt{s}\left(\frac{1}{\sqrt{s}}\right)_{xx} \;. \tag{4.5}$$

This, when we consider $u$ and $s$ as functions of the variable $x$ only, can be read as an ordinary differential equation which enjoys the Riccati property.

Moreover, (2.1) and (2.2) can be obtained, on application of the results in [5], again considering $u$ and $s$ as functions of the variable $x$ only, from the eigenvalue problem

$$\Phi(u) s_x = \lambda s_x$$

related to the hereditary recursion operator of the Korteweg-deVries and modified Korteweg-deVries equation, respectively.

Further aspects of the connection between evolution equations and ordinary differential equations enjoying the Riccati Property have been studied in [3]; therein the Liouville equation, in the following form[3]

$$h_{xt} = e^{2h} \;. \tag{4.6}$$

is considered. Indeed, such equation enjoys many interesting properties and, in particular, its "general" solution ([12], [1]) is well known since there is a Bäcklund transformation between this equation and the wave equation ([11], [8]). However, it is mostly impossible to adapt this "general" solution to boundary conditions arising in applications ([13]). In these cases, the natural connection between the Liouville equation and the Riccati nonlinear ordinary differential equation, established in [3], allows to solve, by quadratures, the characteristic Cauchy problem.

Furthermore, according to [3], equations which enjoy the Riccati Property are naturally connected with a class of equations sharing with the Liouville equation the property that initial value problems can be solved by quadratures: thus, we termed them *generalized Liouville* equations. In [3] the third order problems enjoying the Riccati Property, (2.1) and (2.2), have been shown to be connected with these generalized Liouville equations and a method to obtain further examples of such generalized Liouville equations has been suggested.

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3 Note, however, that usually in applications the relevant form of the Liouville equation is obtained by a linear transform in the complex plane of independent variables.

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