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# Integrable Nonlinear Evolution Equations with time-dependent Coefficients

Benno Fuchssteiner
University of Paderborn
D 4790 Paderborn
Germany

**Abstract**

We exhibit a simple and straight forward method for generating completely integrable nonlinear evolution equations with time-dependent coefficients. For the equations under consideration, we relate the solutions to those of equations given by vector fields which are independent of time, thus explicit links between equations are obtained. As application of the proposed method we show that the linear superposition, with arbitrary time dependent coefficients, of different members of an integrable hierarchy is again integrable. Furthermore, it turns out that for some integrable equations (like the KdV, the BO or the KP) the resolvent operator of lower order flows can be explicitly obtained from that of any higher order flow. We completely classify (demonstrated for the KdV) those flows which can be generated by Lie homomorphisms coming from first order problems. Many well known equations which can be found in the literature are of that type. As an application of such a first order link we give a direct link from KdV to the cylindrical KdV, and from there to the KP with nontrivial dependence on the second spatial variable.

## 1 Introduction

Very often in the literature (see the extensive literature survey in the application section) modifications of well known integrable systems are found which again turn out to be integrable. These so callled equations with variable coefficients play an important role in applications. They originated, in the case of the KdV, from shallow water problems in water with variable depth, and today they generally turn up where modifications of integrable systems are needed to take inhomogeneous properties of media into account.

In most examples the additional terms are somehow related to the symmetry analysis, or the scaling analysis, of the underlying equations. Usually, the methods for integrability results for these equations are ad-hoc methods. However, looking at the variety of examples one has the impression that there must be a general approach to these equations. Indeed, there is such a general approach. This approach is simple, transparent and straightforward and will be presented in this letter. Consider an evolution equation of the form

$$u_t = K(u) \tag{1.1}$$

where $u$ evolves on some suitable manifold of functions in the independent variable $\vec{x} = (x_1, \ldots, x_n)$. Then, such an equation is said to be integrable if an infinite dimensional symmetry group (represented by its infinitesimal generators) can be found. Here, as in the literature, we abbreviate *infinitesimal generator of a symmetry group* (or semigroup) just by the notion of *symmetry*. Or more precise, the notion of *symmetry* of a given flow stands for a vector field invariant under this flow. In all known cases of integrable equations the symmetry algebra can be extended to a Virasoro algebra of vector fields (i.e. an algebra of symmetries and mastersymmetries or a hereditary algebra, see [1], [2], [3] and [4]). The commutation relations of this Virasoro algebra are

$$[K_n, K_m] = 0 \tag{1.2}$$

$$[\tau_n, K_m] = (m + \rho)K_{n+m} \tag{1.3}$$

$$[\tau_n, \tau_m] = (m - n)\tau_{n+m} \tag{1.4}$$

where $\rho$ is a fixed number, $m$, $n$ run from either $-1$ or 0 to infinity[1] and where the $\tau_n$, $K_m$ are suitable vector fields on the manifold under consideration. One should recall that the commutator can be expressed (in any suitable parametrization of the manifold) as

$$[K, G] := G'(u)[K(u)] - K'(u)[G(u)] \tag{1.5}$$

where $G'[K]$ denotes the variational derivative

$$G'(u)[K(u)] := \frac{\partial}{\partial \epsilon} G(u + \epsilon K(u))_{|\epsilon=0} \tag{1.6}$$

[1]There are also meaningful cases where the $m, n$ run from $-\infty$ to $+\infty$ (see [5]).

in the direction of the vector field $K$. Equation (1.1) is said to be *integrable* if $K(u)$ is in the linear hull of the $K_n$, $n \in \mathbb{N}$ of such a Virasoro algebra.

One can rephrase these observations in a more concrete way by introducing time-dependent symmetries. To see this, consider a one-parameter family of vector fields $K(u, t)$ and define by that a time-dependent flow

$$u_t = K(u, t). \tag{1.7}$$

Then another family of vector fields $G(u, t)$ is said to be a time-dependent symmetry (see [6]) if

$$[K(u, t), G(u, t)] + \frac{\partial G(u, t)}{\partial t} = 0. \tag{1.8}$$

This concept generalizes the concept of symmetries and has the advantage that we may include, from the beginning, equations where the right-hand side depends explicitly on time. Now, we return to equation (1.1), where the flow is given by a time-independent vector field $K(u)$. If we assume that $K(u)$ is a linear combination of the $K_n$ of some Virasoro algebra, then we can make out of the corresponding $\tau_m$ of that Virasoro algebra time-dependent symmetries. A simple computation shows that the

$$\tau_m(u, t) := \tau_m(u) + t[K(u), \tau_m(u)] \tag{1.9}$$

are indeed time-dependent symmetries for (1.1). This is an immediate consequence of the fact that by the commutation relations of the Virasoro algebra, and by use of the Jacobi identity, we obtain

$$[K(u), [K(u), \tau_m(u)]] = 0. \tag{1.10}$$

So, we may say that (1.1) is integrable if it admits a Virasoro algebra of time-dependent symmetries. This definition, which, verbatim, can be carried over to equation (1.7), is the notion of integrability on which this paper is based.

## 2 The Main Observation

Denote the Lie algebra defined in (1.5) as $\mathcal{L}$. The Lie derivative given by some $M \in \mathcal{L}$ on $\mathcal{L}$ we denote by $\Lambda_M$, i.e. for all $K \in \mathcal{L}$

$$\Lambda_M K := [M, K]. \tag{2.1}$$

It is well known that, formally, the exponential of $\Lambda_M$

$$\exp(\Lambda_M) := \sum_{n=0}^{\infty} \frac{\Lambda_M^n}{n!} \tag{2.2}$$

is a Lie algebra isomorphism on $\mathcal{L}$. By "formal" we mean that whenever application of $\Lambda_M$ to $K$ as well as to $G$ converges then

$$\exp(\Lambda_M)[K,G] = [\exp(\Lambda_M)K, \exp(\Lambda_M)G]. \tag{2.3}$$

Again, we introduce one-parameter families $H(u,t)$ of vector fields; these are always assumed to be differentiable in $t$. The derivative with respect to $t$ we denote by $\partial_t$. Our main result is:

**Theorem 2.1:**

(a) *Let $G(u,t)$ be a symmetry for*

$$u_t = K(u,t) \tag{2.4}$$

*and let $H(u,t)$ be a family of time-dependent vector fields. Then*

$$\Gamma(u,t) := \exp(\Lambda_H)G \tag{2.5}$$

*is a symmetry for the equation*

$$u_t = \exp(\Lambda_H)K - \sum_{n=0}^{\infty} \frac{(\Lambda_H)^n}{(n+1)!}\partial_t H. \tag{2.6}$$

*Furthermore, the Lie algebras of symmetry group generators of (2.4) and (2.6) are isomorphic. So, if (2.4) admits a Virasoro algebra as symmetry group generators, then (2.6) also admits such an algebra of symmetry group generators.*

(b) *Let $\sigma$ be a new evolution parameter and consider the equation*

$$U_\sigma = -H(U,t) \tag{2.7}$$

*Whenever $U = U(\vec{x},t,\sigma)$ is a solution of (2.7) such that $u(\vec{x},t) := U(\vec{x},t,\sigma = 0)$ is a solution of (2.4) then $U(\vec{x},t,\sigma = 1)$ is a solution of (2.6).*

Before we prove this statement we illustrate by a simple example how this result works. Later on we shall give examples which are more meaningful.

**Example** 2.2: We claim that, for any $h(t)$, the equation

$$u_t = u_{xxx} + 6\exp(h(t))uu_x - h'(t)u \tag{2.8}$$

admits a Virasoro algebra as symmetries. Indeed, this flow, for example, admits $u_x$ as well as

$$u_{xxxxx} + 20\exp(h(t))u_{xx}u_x + 10\exp(h(t))uu_{xxx} + 30\exp(2h(t))u^2u_x \tag{2.9}$$

as symmetries, and so on.
Furthermore it turns out that $u(x,t)$ is a solution of (2.8) if and only if $u(x,t)\exp(h(t))$ is a solution of the KdV.

This is easily explained, and generalized, by introducing the *u-scaling degree*. Let $m(u)$ be any monomial in $u$, $u_x$, $u_{xx}$, .... Its *u-scaling factor* is its polynomial degree minus 1. For example, the scaling factors of $u^3$, $u^2u_x^2$, $u(u_{xx})^4$ are 2, 3 and 4. By $\exp(\lambda S)$ we denote the operator which acts on a linear sum of monomials by multiplying each if its summands by the exponential of $\lambda \times$ *its u-scaling factor*. So,

$$\exp(hS)(u_{xxx} + 6uu_x) = u_{xxx} + 6\exp(h)uu_x. \tag{2.10}$$

Now, choose in theorem 2.1

$$H(u,t) = h(t)u \tag{2.11}$$

and observe that application of $\exp(\Lambda_H)$ coincides with application of the scaling exponential $\exp(hS)$. Furthermore, observe that the sum

$$\sum_{n=0}^{\infty} \frac{(\Lambda_H)^n}{(n+1)!}\partial_t H \tag{2.12}$$

reduces to its first term ($n=0$) since $H$ and $\partial_t H$ do commute. Therefore application of 2.1 to the KdV-hierarchy yields a hierarchy of time-dependent flows of which (2.8) is the second member. Indeed, the whole Virasoro algebra of symmetries and mastersymmetries of the KdV can be carried over to equation (2.8).
The relation between solutions of (2.8) and the KdV is a consequence of theorem 2.1.b. However in this simple case it is easily checked by a direct computation.

Proof of theorem 2.1:
(a): The basis for the Lie algebra $\mathcal{L}$ was a manifold $\mathcal{M}$ of functions in $\vec{x}$. We now change that viewpoint by considering a manifold consisting of orbits on $\mathcal{M}$, i.e. we consider the field variable $u$ as a function in $\vec{x}$ *and* $t$. By $\mathcal{L}_{ex}$ we denote the vector field Lie algebra with respect to flows on this extended manifold.

Since $\mathcal{L}$ can be considered as a subalgebra of $\mathcal{L}_{ex}$ (see [7]) we denote the Lie algebra in $\mathcal{L}_{ex}$ also by [ , ]. Using this Lie algebra we can now rephrase condition (1.8): A time-dependent vector field $G(u,t)$ is a symmetry group generator of

$$u_t = K(u,t)$$

if and only if

$$[K(u) - u_t, G] = 0 \tag{2.13}$$

in the extended Lie algebra $\mathcal{L}_{ex}$. After this observation the proof of theorem 2.1 is straight forward. We consider, all in the extended Lie algebra, the Lie algebra isomorphism $\exp(\Lambda_H)$. Then application of this isomorphism to $K(u) - u_t$ leads to the right hand side of (2.6), and $G$ is transferred by this to the $\Gamma$ given in (2.5). Using the fact that $\exp(\Lambda_H)$ is an isomorphism, and taking the first nontrivial symmetry of the KdV, we see from (2.10) that indeed (2.9) must be a symmetry for (2.6). Again, the isomorphy of the Virasoro algebra of the symmetry group generators of (2.6) and (2.4) follows from the isomorphy property of $\exp(\Lambda_H)$.
(b): Consider the manifold of functions $U$ in $\vec{x}, t$ and $\sigma$ and on that the solution manifold $\mathcal{M}_1$ of $R(U) = 0$, where

$$R(U) := \exp(\sigma\Delta_{H(U,t)})(U_t - K(U,t)) \ . \tag{2.14}$$

Observe that its fibers $\sigma = 0$ and $\sigma = 1$ are the solution manifolds of (2.4) and (2.6), respectively. Taking the total $\sigma$-derivative of $R(U)$ on the whole manifold we find

$$R'[H] - H'[R] + R'[U_\sigma] = 0$$

or, since $R(U) = 0$ on $\mathcal{M}_1$,

$$R'[H + U_\sigma] = 0 \ . \tag{2.15}$$

Hence, $U_\sigma = -H$ leaves $\mathcal{M}_1$ invariant., and obviously this flow transports from the fiber $\sigma = 0$ to the fibre $\sigma = 1$.■

One may ask at this point if notions as *hereditary operators* and the like can be transferred from equation (2.4) to (2.6). This, of course, is possible for any kind of invariant tensor: By the usual procedure [8], [9] we extend the Lie derivative from vector fields and scalar fields to arbitrary tensor fields by requiring the validity of the product rule. Then a tensor $\Phi(u,t)$ is invariant with respect to the flow

$$u_t = K(u,t)$$

if and only if its Lie derivative with respect to the vector field $K(u,t) - u_t$ vanishes. Now, if the meaning of $\Lambda_H$ is extended, such that it stands for the

Lie derivative with respect to $H$ applied to arbitrary tensor fields, then one easily sees that

$$\exp(\Lambda_H)$$

provides an isomorphism for the tensor algebra built up over the vector fields $\mathcal{L}_{ex}$. Hence, by application of this formal isomorphism we may transfer invariant tensors for (2.4) to invariant tensors for (2.6).

# 3 Applications

The examples given in the following sections are based on transforming equations as (2.4) with the help of time-dependent vector fields $H(u,t)$ into new equations like (2.6) and then using the direct transfer for solutions, as given by (2.7), or the transfer of vector fields by (2.5), to obtain information about the new equation. For the transfer of solutions, of course, it is necessary that equation (2.7) is integrable. This is the case whenever the vector field $H(u,t)$ is a scaling symmetry or even a simpler field; hence working with scaling symmetries will make up for a large part of these applications. Even for these simple cases we obtain a large variety of equations usually treated as separate cases in the literature; we shall demonstrate that for the KdV. However, in order to show that also less obvious results can be obtained from theorem 2.1 we start with some examples being more involved from the viewpoint of nonlinear equations.

## 3.1 Finite sums of integrable fields with arbitrary time-dependent coefficients

Consider any hierarchy of commuting vector fields $K_n(u),\ n \in \mathbb{N}$. Then usually the equations

$$u_t = K(u) \tag{3.1}$$

can be solved by standard methods (inverse scattering theory, Hirotas bilinear method, etc.) To facilitate notation we denote by

$$u(x,t) = R_K(U(\vec{x},\tau),t) \tag{3.2}$$

the solution of

$$u_t = K(u,t)$$

for the initial condition $u(x,\tau,t=0) = U(x,\tau)$. Here $\tau$ plays the role of an additional parameter. The crucial operator $R_K$ we call the *resolvent* *operator* of the vector field $K$. An equation is said to be *solvable* if this resolvent operator can be computed somehow.

We are interested wether equations of the form

$$u_t = \sum \phi_n(t) K_n(u) \tag{3.3}$$

are again solvable if the $K_n(u)$ are. Here, of course the sum is assumed to be a finite sum. Indeed, we show as an application of theorem 2.1 that this is the case for arbitrary functions $\phi_n(t)$ and that the solution can be written in terms of the resolvent operators of the $K_n$. It suffices to show a suitable result for the linear superposition of two fields.

**Theorem 3.1:**
*Consider commuting vector fields* $K_1(u,t), K_2(u)$

$$[K_1(u,t), K_2(u)] = 0$$

*where* $K_2(u)$ *is assumed to be time-independent. Let the equations*

$$u_t = K_1(u,t) \tag{3.4}$$

$$u_t = K_2(u) \tag{3.5}$$

*be solvable. Then for an arbitrary function* $\phi(t)$ *in time the equation*

$$u_t = K_1(u,t) + \phi(t) K_2(u) \tag{3.6}$$

*is again solvable. Indeed, let*

$$\frac{d}{dt}\psi(t) = \phi(t)$$

*then*

$$R_{K_2}\left(R_{K_1}(U(x), t_1), t_2\right)_{|\{t_2=\psi(t_1), t_1=t\}} \tag{3.7}$$

*is the solution of* (3.6) *for the initial condition* $u(x,t=0) = U(x)$.
Proof: Put $H(u,t) = -\psi(t)K_2(u)$ then (2.6) carries over in (3.6). Since the flow ( evolution parameter $\sigma$ )

$$U_\sigma = \psi(t_1) K_2(U)$$

has for initial $U(x,t_1,\sigma=0) = u(x,t_1)$ the solution $R_{K_2}(u(x,t_1), \sigma\psi(t_1))$ we find by theorem 2.1 that

$$R_{K_2}(u(x,t), \sigma)_{|\{\sigma=\psi(t)\}}$$

must be a solution of (3.6) whenever $u(x,t)$ is a solution of (3.4). Hence (3.7) gives the solution of (3.6). ■

Obviously, iteration of that result leads to the claim that we may write the solutions of (3.3) in terms of the resolvents of the $K_n$. In case these $K_n$ are from a hierarchy generated by a recursion operator, then it is just a matter of routine to construct the recursion operator of (3.3) by similar methods. This is possible because we have generated the solutions by application of a Lie homomorphism for the underlying tensor bundles.

### 3.2 Solutions for flows given by time-dependent symmetries

As we know, for all known integrable systems, in addition to the usual symmetries, there are those depending explicitly on time. These additional symmetries are either only of first order (conformal symmetries) or, for some equations, also of higher order. Such higher order equations for example can be found for the Benjamin-Ono equation (BO) or the Kadomtsev-Petviashvili equation (KP) [1].

To give a non-trivial example, we consider the Benjamin-Ono equation

$$u_t = Hu_{xx} + 2uu_x \tag{3.8}$$

where $H$ stands for the Hilbert transform

$$(Hf)(x) := \frac{1}{\Pi}\int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x} d\xi \quad \text{(principal value integration)} \tag{3.9}$$

The following equation is derived from the first non-trivial time-dependent symmetry of the BO

$$\begin{aligned} u_t \;=\; & 2(Hu_{xx} + 2uu_x) + 2(u + xu_x + x) \\ & 4t(Hu_{xx} + 2uu_x + u + xu_x) \\ & -4t^2(Hu_{xx} + 2uu_x) \end{aligned} \tag{3.10}$$

We want to solve such an equation for arbitrary initial value. Indeed, this is possible in terms of the iteration of the resolvent operator for the BO. To do this consider

$$K_1(u) = Hu_{xx} + 2uu_x \ . \tag{3.11}$$

We claim that

$$u(x,t) := R_{K_1}(R_{K_1}(U(x) + x, t_1) - x, t_2)_{|\{t_2 = t_1, t_1 = t\}} \tag{3.12}$$

solves (3.10) for the initial value $u(x,t=0)=U(x)$, where $R_{K_1}$ is the resolvent operator of the first member of the hierarchy. To prove this result we observe first that the Virasoro algebra of the BO is easily generated by the first non trivial mastersymmetry and that this mastersymmetry is obtained by commuting $K_2$ with the trivial vector field $G(u)=x$ (see [1]), hence $x$ is a mastersymmetry of second order for the BO. As a consequence of the commutation relation resulting from this observation we represent the right hand side of (3.10) by use of

$$\begin{aligned}\exp(-t\Lambda_{K_1})(\exp(\alpha\Lambda_G)K_1) &= Hu_{xx}+2uu_x+2(u+xu_x+x) \\ &\quad 4t(Hu_{xx}+2uu_x+u+xu_x) \\ &\quad -4t^2(Hu_{xx}+2uu_x)\end{aligned} \tag{3.13}$$

and

$$\exp(\alpha\Lambda_G)K_1=K_1(u(x)+x)\ .$$

Using this we see that the right-hand side of (3.10) is equal to

$$\exp(-t\Lambda_{K_1})(\exp(\Lambda_G)K_2)-\frac{\partial}{\partial t}(-tK_1)$$

and theorem 2.1 may be applied because this field is of the form (2.6). Actually the theorem has to be applied twice. In the first step we find that $R_{K_1}(U(x)+x,t)-x$ solves the initial value problem $u(x,t=0)=U(x)$ for

$$u_t=\exp(\Lambda_G)K_1\ .$$

In the second step a further application of 2.1 gives (3.12).

In this example the decisive tool was that the Virasoro algebra of the BO was generated by a non-trivial symmetry and a constant vector field. Hence, the above arguments can be applied to all equations where this is the case, for example to the KP (see [1] for finding the necessary commutation relations).

### 3.3 Solving lower order equations

Consider for example the KdV or the BO. The problem seems interesting wether we can find explicit formulas between the resolvent operators for different members of the hierarchy.

Indeed sometimes this is possible, so for the KdV, the BO or the KP. Since there, because we have a constant vector field as a descending master symmetry of first order, we can determine explicitly from the resolvent for the n-th member of the hierarchy $u_t = K_n(u)$ the resolvent operators of all lower order flows. We briefly demonstrate that for the case of the KdV, where the trivial field $G(u) = 1$ is a mastersymmetry going, via commutation, from $K_{n+1}$ to $K_n$. To see the details let

$$u_t = K_2(u) := (u_{xxxx} + 5u_x u_x + 10uu_{xx} + 10u^3)_x \tag{3.14}$$

be the second member in the hierarchy. Observe that

$$\Lambda_G K_2 = 10(u_{xxx} + 6uu_x) = 10K_1$$

and

$$(\Lambda_G K_2)^2 K_2 = 60u_x \ .$$

Hence

$$K := \exp(\Lambda_G)K_2 = K_2 + 10K_1 + 60K_0$$

where $K_1$ and $K_0 = u_x$ are the lower order symmetries of (3.14). Now, using 2.1 we find that the resolvent operator for $K_2$ trivially yields the resolvent operator for $K$. Hence by the results in section 3.1 we find the resolvent operator for $K - K_2 = 10K_1 + 60K_0$. Now elementary routine allows us to get rid of the $K_0$, thus we are able to find the resolvent operator for $K_1$ in terms of that for $K_2$. As an exercise the reader may derive the explicit formulas (and check them out, say, for the two-soliton solutions).

### 3.4 Some Tools for working with scaling symmetries

In the following examples we use scaling fields and fields of a similarly simple nature. As a tool, we first need solutions for some special Cauchy problems of first order partial differential equations in the unknown function $F(t, \lambda)$.
**Lemma 3.2:** *Let $\psi(t)$ be some differentiable function, its inverse function we denote by $\psi_{inv}$ and we abbreviate*

$$\tau(t, \lambda) = \psi_{inv}(\psi(t) - \lambda) \tag{3.15}$$

*(a) The initial value problem $F(t, \lambda = 0) = \varphi(t)$ for*

$$-\psi_t^{-1} F_t = F_\lambda \tag{3.16}$$

*has the solution*[2]

$$F(t, \lambda) = \varphi(\tau(t, \lambda)) \tag{3.17}$$

[2] By $\psi_t(\tau(t, \lambda))$ we denote the derivative of $\psi$ at $\tau(t, \lambda)$ whereas $\psi(\tau(t, \lambda))_t$ denotes the $t$-derivative of $\psi(\tau(t, \lambda))$. i.e $\psi(\tau(t, \lambda))_t = \tau_t(t)\psi_t(\tau(t, \lambda))$.

*(b) The initial value problem $F(t, \lambda = 0) = \varphi(t)$ for*

$$-\psi_t^{-2}\psi_{tt}F - \psi_t^{-1}F_t = F_\lambda \tag{3.18}$$

*has the solution*

$$F(t, \lambda) = \psi(t)_t^{-1}\psi_t(\tau(t, \lambda))\varphi(\tau(t, \lambda)) \tag{3.19}$$

*(c) Let $a(t)$ be differentiable in $t$, then the general solution of*

$$F_\lambda + \psi_t^{-1}F_t = \frac{a_t}{\tau_t a} \tag{3.20}$$

*is*

$$F(t, \lambda) = ln(a(t))\psi_t(\tau(t, \lambda)) + Q(\psi(t) - \lambda) \tag{3.21}$$

*for arbitrary $Q$. Hence the solution of the initial value problem $F(t, \lambda = 0) = 0$ is obtained for*

$$Q(\sigma) = -ln(a(\psi_{inv}(\sigma)))\psi_t(\psi_{inv}(\sigma)) \ .$$

*as*

$$F(t, \lambda) = \ln\left(\frac{a(t)}{a(\tau(t, \lambda))}\right)\psi_t(\tau) \ . \tag{3.22}$$

The proof of these results, which were obtained by the well known theory for first order equations [10], can be checked by direct computation. However, one should keep in mind that

$$\tau_t = \frac{\psi_t(t)}{\psi_t(\tau(t, \lambda))} \quad \textit{and} \quad \tau_\lambda = \frac{-1}{\psi_t(\tau(t, \lambda))} \ . \tag{3.23}$$

■

We introduce now the notion of *scaling*. The meaning of $u$-scaling was already introduced in context of example 2.2. By an $x_i$-scaling of a vector field $K$ we understand the successive operations of first replacing $x_i$ by

$$\frac{x_i}{\alpha}$$

and then taking the derivative with respect to $\alpha$ at the point $\alpha = 1$. The operator performing this scaling we denote by $S_{x_i}$. So, for homogeneous terms, the operator $S_{x_i}$ counts the number of $x_i$-derivatives and deducts from that the number of powers in $x_i$. For example,

$$S_x(u_{xxx} + u_x u_{xx}^2 + x^4 u_x) = 3u_{xxx} + 5u_x u_{xx}^2 - 3x^4 u_x \ .$$

In the same manner we introduce the $t$-scaling. A trivial, but nevertheless important observation is that these scalings can be realized by Lie derivatives

$$S_{x_i} = \Lambda_{x_i u_{x_i}}, \quad S_t = \Lambda_{t u_t} \qquad (3.24)\ .$$

This corresponds to the already observed fact for $u$-scaling

$$S = \Lambda_u \qquad (3.25)\ .$$

Hence, it will not come as a surprise that exponentials of these scalings lead to Lie algebra isomorphisms. We compute explicitly some of the Lie algebra isomorphisms given by these scaling quantities.

**Lemma 3.3:**

*(a) Let $K(u,t)$ be a vector field not containing $t$-derivatives and let $a(t)$, $b_i(t)$ be suitable functions in $t$. Consider*

$$H = ln(a(t))u + \sum_i ln(b_i(t))x_i u_{x_i} \qquad (3.26)$$

*then*

$$\exp(\lambda\Lambda_H)K(u,t) = a(t)^{\lambda S}\prod_i b_i(t)^{\lambda S_{x_i}} K(u,t)\ . \qquad (3.27)$$

*(b) Now, consider the functions introduced in context of lemma 3.2, then for*

$$H(u,t) = \psi_t(t)^{-1}u_t \qquad (3.28)$$

*then for some $t$-independent $K(u)$, which does not contain $t$-derivatives, we find*

$$\exp(\lambda\Lambda_H)\varphi(t)K(u) = \varphi(\tau(t,\lambda))K(u)\ . \qquad (3.29)$$

*(c) And for the same $H(u,t) = \psi_t(t)^{-1}u_t$ we have*

$$\exp(\lambda\Lambda_H)\varphi(t)u_t = \psi(t)_t^{-1}\psi_t(\tau(t,\lambda))\varphi(\tau(t,\lambda))u_t \qquad (3.30)$$

*(d) Finally, for*

$$H = ln(a(t))u + \sum_i ln(b_i(t))x_i u_{x_i} + \psi_t(t)^{-1}u_t \qquad (3.31)$$

*we obtain*

$$\exp(\lambda\Lambda_H)u_t = R_a(t,\lambda)u + \sum_i R_{b_i}(t,\lambda)x_i u_{x_i} + \tau_t(t,\lambda)^{-1}u_t \qquad (3.32)$$

*where*

$$R_a(t,\lambda) = \ln\left(\frac{a(t)}{a(\tau(t,\lambda))}\right)\psi_t(\tau)\ . \tag{3.33}$$

*The $R_{b_i}$ are defined accordingly.*

Proof:
The result of (a) follows directly from the interpretation of scaling as the effect of Lie derivatives.
(b): Looking at the power series for $\exp(\lambda\Lambda_H)$ we justify the ansatz

$$\exp(\lambda\Lambda_H)\varphi(t)K(u) = F(t,\lambda)K(u)\ . \tag{3.34}$$

Taking the $\lambda$-derivative yields

$$[H, F(t,\lambda)K(u)] = F_\lambda K(u)\ .$$

Explicit computation leads to

$$-\psi_t(t)^{-1}F_t = F_\lambda\ . \tag{3.35}$$

Furthermore by putting $\lambda = 0$ in (3.34) we see that $F$ must fulfill the initial condition

$$F(t,\lambda=0) = \varphi(t).$$

So, the result follows from lemma 3.2.a.
(c): Here we proceed in the same way. We take the ansatz

$$\exp(\lambda\Lambda_H)\varphi(t)u_t = F(t,\lambda)u_t. \tag{3.36}$$

and obtain by $\lambda$-derivative and $\lambda = 0$ the initial value problem

$$-\psi_t(t)^{-2}\psi_{tt}F - \psi_t(t)^{-1}F_t = F_\lambda,\ \ F(t,\lambda=0) = \varphi(t).$$

The unique solution of which is given by lemma 3.2.b.
(d): Here we make the ansatz

$$\exp(\lambda\Lambda_H)u_t = A(t,\lambda)u + \sum_i B_i(t,\lambda)x_i u_{x_i} + C(t,\lambda)u_t\ . \tag{3.37}$$

Again, $\lambda$-derivative and $\lambda = 0$ lead to the initial value problem

$$C_\lambda + \psi_t^{-1}C_t = -\frac{\psi_{tt}(t)}{\psi_t(t)^2}C,\quad C(t,\lambda=0) = 1\ .$$

From lemma 3.2.b we find its solution as

$$C(t,\lambda) = \tau_t(t,\lambda)^{-1} \ .$$

Using this result we see that the initial value problems for $A$ and the $B_i$ are

$$A_\lambda + \psi_t^{-1} A_t = \frac{a_t}{\tau_t a} \ \textit{with} \ A(t,\lambda=0) = 0$$

and

$$(B_i)_\lambda + \psi_t^{-1} (B_i)_t = \frac{b_{it}}{\tau_t b_i} \ \textit{with} \ B_i(t,\lambda=0) = 0 \ .$$

The solution of these have been given in lemma 3.2.c. Inserting them in (3.37) we obtain the result. ■

## 3.5 Variable coefficient KdV's

We start with the well known KdV

$$u_t = u_{xxx} + 6uu_x \tag{3.38}$$

for which the Virasoro algebra is easily generated by the hereditary recursion operator

$$\Phi(u) = D^2 + 2u + 2DuD^{-1}. \tag{3.39}$$

Application to suitable base elements yields the $K_m$ and $\tau_n$:

$$K_m = \Phi^m u_x, \tau_n = \Phi^n(xu_x + 2u). \tag{3.40}$$

Indeed, the hereditary property of $\Phi$ is equivalent to the Virasoro property of the algebra generated by the $K_m$, $\tau_n$ (see [4]). Expressing this in the extended Lie algebra we know that

$$\Gamma(u) = u_{xxx} + 6uu_x - u_t \tag{3.41}$$

admits a Virasoro algebra as commutant. Defining successively

$$\Gamma_1 = \exp(\Lambda_{\ln(a(t))u + \ln(b(t))xu_x})\Gamma \tag{3.42}$$
$$\Gamma_2 = \exp(\Lambda_{\psi_t^{-1}u_t})\Gamma_1 \tag{3.43}$$
$$\Gamma_3 = \exp(\Lambda_{\ln(\alpha(t))u + \ln(\beta(t))xu_x})\Gamma_2 \tag{3.44}$$

we obtain a vector field $\Gamma_3(u,t)$ which admits a Virasoro algebra as commutant, hence

$$\Gamma_3(u,t) = 0 \tag{3.45}$$

is an integrable equation.
Abbreviating

$$T(t) := \tau(t,1) = \psi_{inv}(\psi(t)-1) \tag{3.46}$$

and carrying out the details of the computation we obtain with the help of the lemma 3.3

$$\Gamma_1 = b(t)^3 u_{xxx} + 6a(t)b(t)uu_x - \frac{a'(t)}{a(t)}u - \frac{b'(t)}{b(t)}xu_x - u_t \tag{3.47}$$

$$\begin{aligned}\Gamma_2 &= b(T(t))^3 u_{xxx} + 6a(T(t))b(T(t))uu_x - \frac{a'(T(t))}{a(T(t))}u - \frac{b'(T(t))}{b(T(t))}xu_x \\ &\quad -(\psi_t(t))^{-1}\psi_t(T(t))u_t\end{aligned} \tag{3.48}$$

$$\begin{aligned}\Gamma_3 &= b(T(t))^3\beta(t)^3 u_{xxx} + 6a(T(t))b(T(t))\alpha(t)\beta(t)uu_x \\ &\quad -\frac{a'(T(t))}{a(T(t))}u - \frac{b'(T(t))}{b(T(t))}xu_x \\ &\quad -(\psi_t(t))^{-1}\psi_t(T(t))\frac{\alpha'(t)}{\alpha(t)}u - (\psi_t(t))^{-1}\psi_t(T(t))\frac{\beta'(t)}{\beta(t)}xu_x - (\psi_t(t))^{-1}\psi_t(T(t))u_t.\end{aligned} \tag{3.49}$$

Observing

$$T_t(t) := \frac{\psi_t(t)}{\psi_t(T(t))} \ , \tag{3.50}$$

multiplying (3.49) with $T_t(t)$ and renaming $b(T(t)) = B(t)$, $a(T(t)) = A(t)$ we obtain that the equation

$$\begin{aligned}u_t &= B(t)^3\beta(t)^3 T_t(t)u_{xxx} + 6A(t)B(t)\alpha(t)\beta(t)T_t(t)uu_x \\ &\quad -\left(\frac{A(t)_t}{A(t)} + \frac{\alpha(t)_t}{\alpha(t)}\right)u - \left(\frac{B(t)_t}{\beta(t)} + \frac{B(t)_t}{\beta(t)}\right)xu_x\end{aligned} \tag{3.51}$$

must be integrable. Introducing now

$$v(t) := B(t)\beta(t),\ w(t) := A(t)\alpha(t) \tag{3.52}$$

we find the integrable equation

$$u_t = v(t)^3 T_t(t)u_{xxx} + 6v(t)w(t)T_t(t)uu_x - \frac{w_t}{w}u - \frac{v_t}{v}xu_x \ . \tag{3.53}$$

I should be observed that the compatibility conditions for this equation are hidden, on one hand, in the interdependence of the coefficients, and on the other hand in the way how the crucial function $T(t)$ was constructed.

From the second part of theorem 2.1 we know that there must be a direct link between the KdV and equation (3.53). In order to find this, we have to solve successively the evolution equations

$$U_{i+1}(x,t,\sigma)_\sigma = -H_i(U_{i+1}(x,t,\sigma),t), \quad i = 1..3 \tag{3.54}$$

for

$$\begin{aligned} H_1 &= \ln(a(t))U + \ln(b(t))xU_x & (3.55) \\ H_2 &= (\psi_t)^{-1}U_t & (3.56) \\ H_3 &= \ln(\alpha(t))U + \ln(\beta(t))xU_x & (3.57) \end{aligned}$$

in case of the initial conditions

$$U_{i+1}(x,t,\sigma=0) = u_i(x,t) := U_i(x,t,\sigma=1) \tag{3.58}$$

where $u_1$ is taken to be a solution of the KdV. Similar as in lemma 3.2 we find

$$\begin{aligned} U_2(x,t,\sigma) &= a(t)^{-\sigma}u_1(xb(t)^{-\sigma},t) & (3.59) \\ U_3(x,t,\sigma) &= u_2(x,\tau(t,\sigma)) & (3.60) \\ U_4(x,t,\sigma) &= \alpha(t)^{-\sigma}u_3(x\beta(t)^{-\sigma},t)\ . & (3.61) \end{aligned}$$

Hence

$$u_4(x,t) = (\alpha(t)a(T(t))^{-1}\, u_1(x(\beta(t)b(T(t))^{-1},T(t))\ . \tag{3.62}$$

As consequence we have that whenever $u(x,t)$ is a solution of the KdV, then

$$u_{new}(x,t) = w(t)^{-1}u(xv(t)^{-1},T(t)) \tag{3.63}$$

must be a solution of (3.53).

**Remark 3.4:** This last equation provides a *direct link* between the KdV and the class of time-dependent coefficient KdV's given by (3.53). Of course, most of the results of this section could have been obtained much simpler by use of this direct link (3.63). However, the derivation we have given yields additional information. First, that the class (3.53) is closed under all Lie homomorphisms being derived from first order problems and that this is the smallest such class of equations containing the KdV. Hence all first order modifications which can be found in the literature must be among these equations[3]. Another decisive advantage of our derivation for these equations is that we constructed them vie a Lie homomorphism for the tensor bundles in the extended Lie algbra. This allows a direct transfer of all notions and quantities formulated in a differential geometric invariant way, for example the Virasoro algebra. We do not give explicitly the Virasoro algebra of equation (3.53), since that is noe a simple exercise.

### 3.5.1 Special cases

Consider the special case

$$\psi(t) = -(\ln(c))^{-1} \ln(\ln(t)) \tag{3.64}$$

then

$$\psi(t) = 1 + \psi(t^c)$$

or

$$T(t) = t^c, \quad T_t(t) = ct^{c-1} \ . \tag{3.65}$$

If now, for example, we choose

$$v(t) = \left(ct^{c-1}\right)^{-1/3}, \quad w(t) = \left(ct^{c-1}\right)^{-2/3} \tag{3.66}$$

this produces the KdV, where the most simple mastersymmetry has been added

$$u_t = u_{xxx} + 6uu_x - (c-1)(2u + xu_x) \ . \tag{3.67}$$

This is Blaszaks [11] extended KdV, for which he studied solitons and the like. Since the symmetry group structure of this equation is isomorphic to that of the KdV, soliton solutions, being solutions obtained by group theoretical reductions, are carried over with the help from formula (3.63). This equation also has been studied, as GKdV, in [12], where pseudopotentials, Lax pairs, and Bäcklund transformations were investigated. For the solution of this equation by inverse scattering see [13], also [14]. Of course all these results can now be obtained by the direct link which preserves all the group theoretic structure.

It should be observed that equation (3.67) is the most simple nonisospectral flow for the usual Lax pair formulation of the KdV. Other nonisospectral flows for other equations (see [15] or [16] for the whole AKNS-class of these equations) can be obtained in the same way.

[3] If one searches the literature with respect to these equations, one really is surprised how much work has been invested in the study of special cases of these equations.

By a different choice of $v(t)$ and $w(t)$ other equations with variable coefficients are obtained. For example, one easily obtains

$$u_t + \beta t^{2n+1} \left(u_{xxx} + 6uu_x\right) - \frac{n+1}{t}(2u + xu_x) = 0 \ . \qquad (3.68)$$

an equation introduced by Nirmala, Vedan and Baby ([17], [18]). Also the other equations studied by Baby, as well as by Li Yi-Shen and Baby, are of the same type (see [19], [20], [21] and [22]). All these equations were introduced in order to explain soliton breaking in variable depth shallow water.

Other special cases of equation (3.53) are some (but not all, see below) of the KdV equations with non-uniformities studied by Brugarino [23]; among them the KdV in nonuniform media with relaxation coefficients.

## 3.6 Other KdV and mKdV-modifications

The same procedure as with the KdV can be done with mKdV, or more generally, with the Gardner equation

$$u_t = u_{xxx} + c_1 uu_x + c_2 u^2 u_x + c_3 u_x \ . \qquad (3.69)$$

We obtain then the integrable modification

$$\begin{aligned} u_t &= v(t)^3 T_t(t) u_{xxx} + c_1 v(t) w(t) T_t(t) uu_x + c_2 w(t)^2 v(t) T_t(t) u^2 u_x \\ &\quad + c_3 v(t) T_t(t) u_x - \frac{w_t}{w} u - \frac{v_t}{v} x u_x \ . \end{aligned} \qquad (3.70)$$

The solutions of (3.69) and (3.70) are again related by formula (3.63).

But also from the KdV itself we can directly obtain more general integrable equations. Let us give a more elaborate example where more complicated actions of exponentials of scalings have to be computed. Starting again with (3.41)

$$\Gamma(u) = u_{xxx} + 6uu_x - u_t$$

and defining

$$\Gamma_1 = \exp(\Lambda_{\ln(a(t))u + \ln(b(t))xu_x + (\psi_t(t))^{-1}u_t})\Gamma$$

we obtain the integrable equation

$$\Gamma_1(u,t) = 0 \ .$$

Abbreviating again $T(t) := \tau(t,1) = \psi_{inv}(\psi(t)-1)$ and carrying out the details of the computation we obtain (with the help of the lemma 3.3.d)

$$\begin{aligned} u_t &= T_t(t)\nu(t)^3 u_{xxx} + 6\omega(t)\nu(t)T_t(t)uu_x \\ &\quad -\psi_t(t)\ln\left(\frac{a(t)}{a(T(t))}\right)u - \psi_t(t)\ln\left(\frac{b(t)}{b(T(t))}\right)xu_x \end{aligned} \tag{3.71}$$

where

$$\begin{aligned} \omega(t) &= \exp\left(\int_t^{T(t)} \psi_s(s)\ln(a(s))ds\right) \\ \nu(t) &= \exp\left(\int_t^{T(t)} \psi_s(s)\ln(b(s))ds\right) . \end{aligned}$$

Using theorem 2.1 we find as before that for solutions $u(x,t)$ of the KdV, the function

$$u_{new}(x,t) = \omega(t)u(x\nu(t), T(t)) \tag{3.72}$$

must be a solution of (3.71). It should be remarked, that equation (3.71) is the most general equation which can be obtained out of the KdV by application of exponentials of scalings. This follows from the fact that the exponentials of the kind

$$\exp(\Lambda_{\ln(a(t))u+\ln(b(t))xu_x+(\psi_t(t))^{-1}u_t})$$

are a subgroup of the exponentials of vector fields.

## 3.7 Cylindrical Equations

In the KdV (for the variable $\tilde{u}$) we substitute

$$\tilde{u} = u - \frac{x}{6t} .$$

Then $u$ evolves with

$$u_t = u_{xxx} + 6u_x u - \frac{1}{t}(u + xu_x) . \tag{3.73}$$

Now, taking the same transformations as in (3.42), (3.43) and (3.44) we obtain the integrable equation

$$u_t = v(t)^3 T_t(t)u_{xxx} + 6v(t)w(t)T_t(t)uu_x - \left(\frac{1}{T(t)} + \frac{w_t}{w}\right)u - \left(\frac{1}{T(t)} + \frac{v_t}{v}\right)xu_x . \tag{3.74}$$

Now, by choice of

$$\psi(t) = \frac{\ln(t)}{\ln(c)} \tag{3.75}$$

we pick out a special case. We get

$$T(t) = \frac{t}{c} \; and \; T_t = \frac{1}{c} \; . \tag{3.76}$$

Putting

$$v(t) = t^{-1/3} \; and \; w(t) = t^{-2/3} \tag{3.77}$$

we obtain

$$u_t = \frac{1}{ct}\left(u_{xxx} + 6u_x u\right) + \left(\frac{2}{3\,t} - \frac{c}{t}\right) u + \left(\frac{1}{3\,t} - \frac{c}{t}\right) x u_x \; . \tag{3.78}$$

Now, performing for the parameter $c$ the limit $c \to 1$ we find the equation

$$u_t = \frac{1}{t}\left(u_{xxx} + 6u_x u - \frac{1}{3}u - \frac{2}{3}x u_x\right) \; . \tag{3.79}$$

Following all the steps, and transforming the solutions accordingly, we find that whenever $u(x,t)$ is a solution of the KdV then

$$u_{new}(x,t) := t^{2/3} u(x t^{1/3}, t) + \frac{x}{6} \tag{3.80}$$

is a solution of (3.79). At this point we transform dependent and independent variables according to

$$u = 6^{-2/3} U, \; x = 6^{1/3} X, \; t = \sigma^{-1/2} \tag{3.81}$$

to find

$$U_\sigma = -\frac{1}{12\sigma}\left(U_{XXX} + 6U_X U - 2U - 4XU_X\right) \; . \tag{3.82}$$

This is a well known equation from the literature [24, p268]. From here the transformation

$$v(\xi,\sigma) = (12\sigma)^{-2/3} U(X,\sigma), \; \xi = (12\sigma)^{1/3} X \tag{3.83}$$

yields a link to the cylindrical KdV

$$v_\sigma + v_{\xi\xi\xi} + v_\xi v + \frac{v}{2\,\sigma} = 0 \; . \tag{3.84}$$

Thus we have found a direct link from KdV to the cylindrical KdV. So, if $u(x,t)$ is a solution of the KdV then

$$v(\xi,t) = \frac{1}{4^{1/3}\sigma} u\left(\frac{\xi}{2^{1/3}\sqrt{\sigma}}, \frac{1}{\sqrt{\sigma}}\right) + \frac{\xi}{12\sigma} \tag{3.85}$$

must be a solution of the cylindrical KdV.

In a similar fashion all the other KdV-modifications with variable coefficients, for example those in [23], can be found. Also the group structure of the cylindrical KdV (see [6]) and similar equations can be derived by the methods in this paper. In particular, the link given in (3.85) is compatible with group theoretical reductions since it was obtained by transfer via a generalized scaling group.

### 3.8 Link from KdV to KP

As for the KdV we may, by the same methods, obtain a link between the Kadomtsev-Petviashvili equation

$$(U_t + 6UU_x + U_{xxx})_x = -3\alpha^2 U_{yy} \tag{3.86}$$

and the Johnson equation ([25])

$$\left(V_\sigma + 6VV_\sigma + V_{\xi\xi\xi} + \frac{V}{2\sigma}\right)_\xi = -3\alpha^2 \frac{V_{\eta\eta}}{\sigma^2}\ , \tag{3.87}$$

which was investigated for its applications in water with variable depth (see [26] and , or [27]. However, this link is well known from the literature, so we may skip it here. In a paper of Lipovskii, Matveev and Smirnov [28] we find that whenever $V(\xi,\eta,\sigma)$ is a solution of Johnsons equation then

$$U(x,y,t) = V\left(x + \frac{y^2}{12\alpha^2 t}, \frac{y}{t}, t\right) \tag{3.88}$$

must be a solution of KP. Now, obviously, any solution of the cylindrical KdV of the form (3.85) is a solution of Johnsons equation (not depending on $\eta$). Hence we have found a direct link from the KdV to the KP. Interestingly, this link, contrary to the trivial link where the $y$-dependence is neglected, yields solutions of KP which genuinely depend on the second spatial variable. To make this link precise, we conclude that whenever $u(x,t)$ is a solution of KdV then

$$U(x,y,t) = \frac{1}{4^{1/3}t} u\left(\frac{x}{2^{1/3}\sqrt{t}} + \frac{y^2}{12(2)^{1/3}\alpha^2 t^{3/2}}, \frac{1}{\sqrt{t}}\right) + \frac{x}{12\,t} + \frac{y^2}{(12\alpha\, t)^2}$$

must be a solution of KP. This fact is easily verified by direct computation. Since the KP is invariant with respect to translation of time we have found the following general class of solutions:

$$\begin{aligned} U(x,y,t) &= \frac{1}{4^{1/3}(t+c)} u\left(\frac{x}{2^{1/3}\sqrt{(t+c)}} + \frac{y^2}{12(2)^{1/3}\alpha^2(t+c)^{3/2}}, \frac{1}{\sqrt{(t+c)}}\right) \\ &\quad + \frac{x}{12\,(t+c)} + \frac{y^2}{(12\alpha\,(t+c))^2}\ . \end{aligned} \tag{3.89}$$

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