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Nilpotent and Recursive Flows

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In this paper we introduce a class of nonlinear vector fields on infinite dimensional manifolds such that the corresponding evolution equations can be solved with the same method one uses to solve ordinary differential equations with constant coefficients. Mostly, these equations are nonlinear partial differential equations. It is shown that these flows are characterized by a generalization of the 'method of variation of constants' which is widely used for second order problems to find general solutions out of particular ones. Invariant densities are constructed for these flows in a natural way. These invariant densities are providing an essential tool for solving initial value and boundary value problems for the equations under consideration. Many applications are presented

1 Introduction

In this section we give the essential definitions and illustrate them by a number of examples.

Definition 1.1:
We consider a manifold given by some vector space E, and we denote by v the typical element of E. A vector field $G(v)$ on E is said to be be

- **nilpotent** *if, whenever the equation $s_t = G(s)$ defines a flow $(s,t) \mapsto s(t)$, then this flow satisfies*

$$(\frac{d}{dt})^{N+1} s = 0 \tag{1.1}$$

for some[1] $N \in \mathbb{N}_0$.
More general, the vector field $G(s)$ *will be said* **nilpotent** *when its* N*-th directional derivative, in the direction of the field itself,is zero identically on* E.
The nilpotency is said to be of order N *if this relation does not hold for any lower* N.

- **recursive** *if the* N*-th derivative of* G *in the direction of* G *is a linear combination of the derivatives of lower order,or in particular, whenever* $s_t = G(s)$ *defines a flow such that there are constant coefficients* α_n *such that (identically)*

$$(\frac{d}{dt})^{N+1} s = \sum_{n=0}^{N} \alpha_n (\frac{d}{dt})^n s \ . \tag{1.2}$$

Again, the recursiveness is said to be of order N *if such a relation does not hold for any lower* N.

In the following, we shall say that $s_t = G(s)$, and possibly its flow, are recursive (or nilpotent) when the corresponding vector field has this property.
In all examples which follow, the space E will consist of functions of an independent variable x. We remark that the α_n cannot depend on t when the vector field $G(v)$ is not depending explicitly on t. In both cases of this definition the initial value problem of $s_t = G(s)$ is easily solved, although $G(v)$ may be nonlinear. To see this we define for recursive $G(v)$

$$\vec{u} = \begin{pmatrix} s \\ s_t \\ \vdots \\ (\frac{d}{dt})^N s \end{pmatrix} \tag{1.3}$$

then the flow for $\vec{u}_t$ follows a linear equation

$$\vec{u}_t = \begin{pmatrix} 0 & 1 & 0 & .. \\ 0 & 0 & 1 & ... \\ . & . & . & ... \\ \alpha_0 & \alpha_1 & ... & \alpha_N \end{pmatrix} \vec{u} \tag{1.4}$$

[1]By $\mathbb{N}$ we denote the positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

and for the solution of the initial value problem we only need to compute the exponential function of that matrix.

Example 1.2:
Let D^{-1} denote integration from $-\infty$ to x, and assume that s is such that $1/s$ vanishes rapidly at $-\infty$. Then the following vector fields

$$G(s) = \sqrt{2sF(x)} \tag{1.5}$$

$$G(s) = sD^{-1}(s^{-1}) \tag{1.6}$$

$$G(s) = sD^{-1}(F(x)s^{-1}) \tag{1.7}$$

$$G(s) = sD^{-1}(s^{-2}\sqrt{ms^2 - s_x^2}) \tag{1.8}$$

$$G(s) = sD^{-1}(s^{-2}\sqrt{ms^2 + s_x^2}) \tag{1.9}$$

are nilpotent of second order. The vector field

$$G(s) = sD^{-1}(F(x)s^{-2}) \tag{1.10}$$

is nilpotent of first order. □

Proof: We show for (1.6) that $s_{ttt} = 0$. To see this we observe

$$\begin{aligned} s_{tt} &= s_t D^{-1}(s^{-1}) - sD^{-1}(s_t s^{-2}) \\ &= s(D^{-1}(s^{-1}))^2 - sD^{-1}\left(s^{-1}D^{-1}s^{-1}\right) \\ &= \frac{1}{2}s\left(D^{-1}s^{-1}\right)^2 = \frac{s_t^2}{2s}. \end{aligned}$$

Now, taking a further t-derivative of this representation of s_{tt} we obtain $s_{ttt} = 0$. In case of the other vector fields the proof follows exactly the same lines. ■

Example 1.3:
A simple computation yields a nilpotent vector field of N-th order. For β different from -1 consider the equation

$$s_t = s(D^{-1}s^{-1})^{\beta} \ . \tag{1.11}$$

Then

$$s_{tt} = \frac{s_t^2}{(1+\beta)s} \tag{1.12}$$

and

$$s_{t\$n} = \frac{(1-\beta)(1-2\beta)...(1-(n-2)\beta)s_t^n}{(1+\beta)^{n-1}s^{n-1}}$$

where $s_{t\$n}$ stands for the n-th t-derivative of s. This clearly shows that

$$s_t = s(D^{-1}s^{-1})^{\frac{1}{N-1}} \qquad (1.13)$$

is nilpotent of order N. In the same way one can see that

$$s_t = sD^{-1}s^{-\frac{2}{N}} \qquad (1.14)$$

defines another example of N-th order nilpotent flow . □

Example 1.4:
Observe that, under the assumed boundary condition, the equations $s_t = G(s)$ for (1.6) and (1.7) are equivalent to

$$s\ s_{tx} - s_t s_x = s \qquad (1.15)$$

and

$$s\ s_{tx} - s_t s_x = sF(x)\ , \qquad (1.16)$$

respectively. Now, consider (1.15) under a different boundary condition, say,

$$s_t(t, x_0)/s(t, x_0) = a\ ,$$

where a is independent of t. Then integration from x_0 to x gives

$$s_t = sD_{x_0}^{-1}s^{-1}F(x) + sa \qquad (1.17)$$

where $D_{x_0}^{-1}$ stands for integration from x_0 to x. Then

$$s_{tt} = \frac{1}{2}s^{-1}\left((s_t)^2 + a^2s^2\right) \qquad (1.18)$$

and we obtain a recursive system, because $s_{ttt} = a^2 s_t$. Observe that the generalized Liouville equation

$$h_{xt} = \frac{1}{2}F(x)e^{-2h} \qquad (1.19)$$

allows a Bäcklund transformation to (1.16). For

$$s = e^{2h} \text{ or } h = \log(\sqrt{s})$$

the equation (1.19), in terms of s, yields (1.16). Hence if $h_t(t, x_0) = constant$ then the initial value problem for this equation is easily solved by linearization (see [3] in order to find how, for general boundary conditions, the initial value problem of that equation is solved by use of ideas similar to those in this paper). □

In order to include also different boundary conditions, we introduce a generalized notion

Definition 1.5: *Consider a one-parameter family of vector fields $G(v,t)$ on E. Then this family is said to be* **weakly recursive** *of order N if, whenever $s_t = G(s,t)$ defines a flow $(s,t) \mapsto s(t)$ then this is such that there are smooth coefficients $\alpha_n(t)$ such that (identically)*

$$(\frac{d}{dt})^{N+1} s = \sum_{n=0}^{N} \alpha_n(t) (\frac{d}{dt})^n s \ . \tag{1.20}$$

Here again we assume in addition that there is no linear dependence of lower order between the $(\frac{d}{dt})^n s$.

Example 1.6:
Consider again (1.16), but this time with arbitrary boundary condition $s(t, x_0) = \phi(t)$. Let $g(t) = s_t(t, x_0)/s(t, x_0)$, then integration from x_0 to x gives

$$s_t = s D_{x_0}^{-1} s^{-1} F(x) + s g(t) \ . \tag{1.21}$$

Computing s_{tt} we obtain

$$\frac{(s_t - sg)_t}{(s_t - sg)^2} = \frac{1}{2s} \tag{1.22}$$

and for s_{ttt} we get

$$s_{ttt} = (g(t)^2 + 2g_t) \ s_t + g_{tt} \ s \ . \tag{1.23}$$

Hence (1.16), for general boundary conditions at $x = x_0$, yields a weakly recursive flow. □

2 Characteristic Operators

Assume that the manifold E consists of smooth functions of an independent variable x, and, as before, denote by D the differential operator with respect to x. For any flow on $E, v = v(t)$, we use the following notation for the derivations with respect to t

$$v_{t\$0} := v \ , \ v_{t\$(n+1)} := \frac{d}{dt} v_{t\$n} \ , n = 1, 2, \ldots \ ,$$

and $v_{t\$n,x\$m}$ stands for the n-th t-derivative and m-th x-derivative of v. Moreover, let $A(v)$ be a smooth field defined on E ; we denote by $A'(v)[B(v)]$ the variational derivative of A with respect to v in the direction of $B(v)$, that is

$$A'(v)[B(v)] = \frac{\partial}{\partial \epsilon}_{|\epsilon=0} A(v + \epsilon B(v)) \ ,$$

and in case that A or B depend on further parameters, then the prime is reserved for the variational derivative with respect to v. Hence, for a vector field $A(v,t)$ one may use

$$A_t(v,t) = A'(v,t)[v_t] + \frac{\partial}{\partial t} A(v,t)$$

and, of course,

$$DA \equiv A_x(v,t) = A'(v,t)[v_x]$$

to compute the n-th t-derivative and m-th x-derivative of A, which we again denote by $A_{t\$n,x\$m}$. Consequently, if s follows the flow defined by a given vector field $G(v,t)$, then it is possible to replace $s_{t\$(n+1),x\$m}$ with the corresponding expressions $G_{t\$n,x\$m}$, $n, m = 0, 1 \ldots$ (which depend on s, t and suitable x-derivatives and x-integrals of s).
Assume now that $G(s,t)$ is weakly recursive of oder N , and that it admits a flow $s = s(t)$. We consider coefficients $a_n = a_n(x,t)$ such that the operator

$$\Phi = (D^{N+1} + \sum_{n=0}^{N} a_n D^n) \tag{2.1}$$

satisfies (for each t)

$$\Phi s_{t\$n} = 0 \text{ for } n = 0, 1, \ldots, N \ . \tag{2.2}$$

Then, whenever the Wronskian determinant constructed with the functions $(s_{t\$0}, \ldots, s_{t\$N})$ is different from zero, then the $N+1$ quantities $a_0, \cdots, a_N$ uniquely exist and satisfy $N+1$ independent equations [1 , Thm. III.6.2]; moreover the replacement of $s_{t\$(n+1)}$ with $G_{t\$n}$, $n = 0, 1, \ldots, N-1$, allows to determine these $a_n(x,t)$ as functions of s and t

$$a_i = a_i(s,t) \ , i = 0, 1, \ldots, N \ .$$

Of course, these representations may also contain x-derivatives and x-integrals of s. Furthermore, if $G = G(s)$, i.e. if G does not explicitly depend on t, then the a_n are independent of t.

We call the operator Φ introduced in (2.1), **characteristic operator** for the flow $s = s(t)$. Formally, this operator is easily computed. Take

$$\Phi_{op\_(N+1)} = \det \begin{pmatrix} s & s_t & \cdots & s_{t\$N} & D^0 \\ s_x & s_{t,x} & \cdots & s_{t\$N,x} & D^1 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ s_{x\$(N+1)} & s_{t,x\$(N+1)} & \cdots & s_{t\$N,x\$(N+1)} & D^{N+1} \end{pmatrix} \tag{2.3}$$

and

$$\Phi_{sub\_N} = \det \begin{pmatrix} s & s_t & \cdots & s_{t\$N} \\ s_x & s_{t,x} & \cdots & s_{t\$N,x} \\ \vdots & \vdots & \cdots & \vdots \\ s_{x\$N} & s_{t,x\$N} & \cdots & s_{t\$N,x\$N} \end{pmatrix} \tag{2.4}$$

then application of $\Phi_{op\_(N+1)}$ to any of the $s_{t\$n}$, $n \leq N$ gives zero because of equality between two columns. Hence the characteristic operator is

$$\Phi \equiv \Phi_{(N+1)} = (\Phi_{sub\_N})^{-1}\Phi_{op\_(N+1)} \tag{2.5}$$

and the coefficients a_n are obviously given by

$$a_n(x,t) = (-1)^n(\Phi_{sub\_N})^{-1}\Phi_{sub\_n} \tag{2.6}$$

where Φ_{sub_n} is the determinant obtained from $\Phi_{op\_(N+1)}$ by eliminating the last column and the $(n+1)$-th row.

In Theorem 3.9 we present a different explicit computation of this operator, in terms of s, t and $G(s,t)$, which allows the direct reconstruction of the corresponding $G(s,t)$.

Recall that a function $a = a(s,t)$ is said to be an **invariant density** with respect to the flow assigned by $s_t = G(s,t)$ if it does not change under this evolution, i.e. if

$$0 = \frac{d}{dt}a(s,t) = a'(s,t)[G(s,t)] + \frac{\partial}{\partial t}a(s,t) \ . \tag{2.7}$$

Because of the assumption that $G(s,t)$ is weakly recursive, span$\{s, s_t, ..., s_{t\$N}\}$ has exactly dimension $N+1$[2] and equals the kernel of Φ; therefore we have

[2] In the following, the vector space structure of E will be considered as dependent on the free parameter t, hence when we refer to vector relations among the elements of E we mean that these relations have to be satisfied for all the (allowed) values of the time variable t.

that $s_t = G(s,t)$ defines a transformation in the kernel of the linear operator Φ. Hence, we find that the $a_i(s,t)$ are invariant under the flow of $s_t = G(s,t)$.

Remark 2.1: *If the flow of $s_t = G(s,t)$ is weakly recursive then the coefficients of the characteristic operator are invariant densities for that flow.*

Example 2.2:
Consider the equation (1.6). Then the characteristic operator is

$$\Phi = D^3 + a_1 D + a_0 \tag{2.8}$$

where

$$a_1 = \frac{{s_x}^2}{s^2} - \frac{2s_{xx}}{s} \text{ and } a_0 = \frac{a_{1x}}{2} \ . \tag{2.9}$$

One easily shows that these quantities are indeed invariant densities for (1.6).

In case that the flow under consideration is (1.21) instead of (1.6) then we obtain the same characteristic operator, a fact which is easily understood by the observation that the recursiveness of (1.21) was obtained out of (1.6) by some linear transformation in the span of the $\{s, s_t, s_{tt}\}$. □

Example 2.3:
Slightly more complicated is the computation of the invariant densities given by the characteristic operators in case of equation (1.8). The second t-derivative of s which is needed to evaluate (2.5) is

$$s_{tt} = \frac{1}{2s}(s_t^2 + 1) \ . \tag{2.10}$$

If we abbreviate

$$u = \frac{ms - s_{xx}}{2\sqrt{ms^2 - s_x^2}} \tag{2.11}$$

then the coefficients of the characteristic operator

$$\Phi = D^3 + a_2 D^2 + a_1 D + a_0 \tag{2.12}$$

are

$$a_0 = m\frac{u_x}{u}, \ a_1 = 4u^2 - m, \ a_2 = -\frac{u_x}{u} \tag{2.13}$$

□

3 Variation of Constants for Differential Operators

Let $G(s,t)$ be weakly recursive,e.g. of order $(N-1)$. We want to show that characteristic operators completely characterize these $G(s,t)$ (up to linear transformations in the span of $\{s, s_t, \dots, s_{t\$(N-1)}\}$). In order to see that, we study the well known method of *variation of constants* for differential equations.

Let $\Phi = a_N D^N \cdots + a_0$ be a differential operator, where the a_n are functions of x and, possibly, further parameters. Define, for $w \in E$, the **lowering** $\Phi^{(w)}$ of Φ by w to be the operator

$$\alpha(x) \longrightarrow [\Phi, \int^x \alpha(\xi) d\xi] w = \Phi w \int^x \alpha(\xi) d\xi - \int^x \alpha(\xi) d\xi \Phi w \ .$$

This operator is easily computed. For example if $\tilde{\Phi} = a_N D^N$ then we obtain

$$\begin{aligned} \tilde{\Phi}^{(w)} \alpha &= a_N [D^N, \int^x \alpha(\xi) d\xi] w \\ &= a_N \{ w \alpha_{x\$(N-1)} + \binom{N}{1} w_x \alpha_{x\$(N-2)} \cdots + \binom{N}{N-1} \alpha w_{x\$(N-1)} \ . \end{aligned}$$

Hence we find

$$(a_N D^N)^{(w)} = a_N \sum_{k=0}^{N-1} \binom{N}{k} w_{x\$k} D^{N-k-1} \qquad (3.1)$$

and we observe that this is a purely differential operator. This result may be extended by taking linear combinations.

We now use the following notation for lowerings

$$\Phi^{()} := \Phi \quad (\textit{empty lowering})$$

$$\Phi^{(w_0, \dots, w_n, w_{n+1})} = \left(\Phi^{(w_0 \dots, w_n)}\right)^{(w_{n+1})} \ .$$

All these operators are again purely differential operators. If $\Phi = a_N D^N + \cdots + a_0$ is of N-th order then

$$\Phi^{(w_0, w_1, \dots, w_{N-1})} = a_N w_0 w_1, \dots w_{N-1} \qquad (3.2)$$

and this is a multiplication operator, implying $\Phi^{(w_0, w_1, \dots, w_N)} = 0$ for any $(N+1)$-tuple of functions. The name *lowering* is chosen because this operation lowers the degree of a differential operator by one.

The well known method of variation of constants can now be expressed in terms of lowerings:

Example 3.1:
If $\Phi w_0 = 0$ then the following are equivalent:

i) $\Phi^{(w_0)} w_1 = 0$

ii) $\Phi w_0 D^{-1} w_1 = 0$

Proof:
Assume $\Phi^{(w_0)} w_1 = [\Phi, D^{-1}(w_1)] w_0 = 0$. With $\Phi w_0 = 0$, this implies ii)

$$\Phi D^{-1}(w_1) w_0 = 0 \; .$$

And this argument can be reversed, in the sense that where the ratio of two solutions of $\Phi w = 0$ is N-times differentiable, then its derivative is a solution of $\Phi^{(w_0)} w = 0$. ■

This method is easily extended and leads to the following

Theorem 3.2: *Take $w_0, w_1, \ldots, w_n$ in E, then the following are equivalent*

i) $\Phi w_0 = 0$ *and* $\Phi^{(w_0,\ldots,w_k)} w_{k+1} = 0$ *for all* $k < n$

ii) $\Phi w_0 = 0$ *and* $\Phi w_0 D^{-1} w_1 \cdots D^{-1} w_m = 0$ *for all* $m \leq n$

Proof: Using the equivalence from Example 3.1, we obtain subsequently for $m \leq n$

$$\begin{aligned} 0 = \Phi^{(w_0,\ldots,w_{m-1})} w_m &= \Phi^{(w_0,\ldots,w_{m-2})} w_{m-1} D^{-1} w_m \\ &= \cdots = \\ &= \Phi^{(w_0,w_1)} w_2 D^{-1} w_3 \cdots D^{-1} w_m \\ &= \Phi^{(w_0)} w_1 D^{-1} w_2 \cdots D^{-1} w_m \\ &= \Phi w_0 D^{-1} w_1 \cdots D^{-1} w_m \; . \end{aligned}$$

Hence if we assume (i) then (ii) must hold, and conversely, in the same sense as said above. ■

We call a sequence $\{w_0, \cdots, w_n\}$ which has property (i) or, equivalently (ii), a **factorizing sequence** for Φ. The reason is that these sequences may be used to invert operators formally, where by *formally* we mean that the inverse is determined modulo elements of the kernel of Φ.

Theorem 3.3: *Let $w_0, \dots, w_n$ be a factorizing sequence and denote by $\Psi^{(w_0,\dots,w_n)}$ the inverse, if it exists, of $\Phi^{(w_0,\dots,w_n)}$ then*

$$\Phi^{-1} = w_0 D^{-1} w_1 \cdots D^{-1} w_n D^{-1} \Psi^{(w_0,\dots,w_n)}$$

Proof: Using Theorem 3.2 we obtain, when $\Phi^{(w_0,\dots,w_k)} w_{k+1} = 0$ for $k < n < N$ the following

$$\begin{aligned}
\Phi w_0 D^{-1} w_1 \cdots D^{-1} w_n \Psi^{(w_0,\dots,w_n)} &= \Phi^{(w_0)} w_1 D^{-1} w_2 \cdots D^{-1} w_n D^{-1} \Psi^{(w_0,\dots,w_n)} \\
&= \Phi^{(w_0,w_1)} w_2 D^{-1} w_3 \cdots D^{-1} w_n D^{-1} \Psi^{(w_0,\dots,w_n)} \\
&= \cdots = \\
&= \Phi^{(w_0,\dots,w_n)} \Psi^{(w_0,\dots,w_n)} = I
\end{aligned}$$

■

Restating the equivalence of Theorem 3.2 we have

Observation 3.4: *Take $\{w_0, w_1, \dots, w_n\}$ in E and let Φ be some differential operator, then the following are equivalent*

i) $\{w_0, w_1, \dots, w_n\}$ *is a factorizing sequence,*

ii) $\varphi_0 = w_0, \varphi_1 = w_0 D^{-1} w_1, \cdots, \varphi_n = w_0 D^{-1} w_1 D^{-1} w_2 D^{-1} w_3 \cdots D^{-1} w_n$ *are solutions of* $\Phi\varphi = 0$

This statement may be reversed:

Corollary 3.5: *Let $\varphi_0, \cdots, \varphi_n$ be linearly independent solutions of $\Phi\varphi = 0$ and define (on the intervals where $w_0, \dots, w_n \neq 0$, see [8, par.IV.3])*

$$w_0 = \varphi_0, w_1 = D w_0^{-1} \varphi_1, \cdots, w_k = D w_{k-1}^{-1} D \cdots D w_1^{-1} D w_0^{-1} \varphi_k, \\ \cdots\cdots, w_n = D w_{n-1}^{-1} D \cdots D w_1^{-1} D w_0^{-1} \varphi_n \quad .$$

Then the $w_i,\ i = 0, \cdots, n$ define a factorizing sequence.

Proof: The proof of the corollary is obvious because the very definition

of the w_i together with the fact that the φ_i are solutions of $\Phi\varphi = 0$ immediately implies $\Phi w_0 D^{-1} w_1 \cdots D^{-1} w_m = 0$ for all $m \leq n$. ■
Now take formula (3.2) for the complete lowering of Φ and take the result of Theorem 3.3 for the case that Ψ is the inverse of that complete lowering then one obtains the following inversion formula

Observation 3.6: *When*

$$\Phi = (D^{N+1} + \sum_{n=0}^{N} a_n D^n)$$

is a differential operator of order $N+1$ and $\{w_0, \ldots, w_N\}$ is a factorizing sequence, then

$$\Phi^{-1} = w_0 D^{-1} w_1 \cdots D^{-1} w_N D^{-1} \frac{1}{w_0 \cdots w_N}$$

or

$$\Phi = w_0 \cdots w_{N-1} w_N D \frac{1}{w_N} D \frac{1}{w_{N-1}} \cdots D \frac{1}{w_0}$$

Observation 3.7: *Consider variable $M \in I\!N$ and take $\Phi_{(M)}$ as defined in (2.5). In order to compare $\Phi_{(N+1)}$ and $\Phi_{(N)}$ let $\varphi_n = s_{t\$n}, n = 0, \cdots, N$ and define the $w_i, i = 0, \cdots, N$ as in Corollary 3.5. Obviously, $\{w_0, \cdots, w_{N-1}\}$ and $\{w_0, , \cdots, w_N\}$ are factorizing sequences for $\Phi_{(N)}$ and $\Phi_{(N+1)}$, respectively. Now, we take the representation for $\Phi_{(N)}$ and $\Phi_{(N+1)}$ as given in Observation 3.6 and obtain easily, by comparison of this representation with the recursion formula for the w_i, the following recursive identities:*

$$w_0 w_1 \cdots w_N = \Phi_{(N)} s_{t\$N} \tag{3.3}$$

and

$$\Phi_{(N+1)} = \Phi_{(N)} s_{t\$N} D (\Phi_{(N)} s_{t\$N}\ .)^{-1} \Phi_{(N)} \tag{3.4}$$

Directly from their definition it follows that the linear dependence of $\varphi_0, \cdots, \varphi_n$ is a necessary and sufficient condition in order that at least one of the $w_1, \ldots, w_n$ is *identically* zero. However, the functions φ are assumed to be solutions of $\Phi\varphi = 0$, hence their linear dependence is equivalent to the vanishing of their Wronskian determinant. On the other hand, (3.3) implies that whenever the functions w are defined, then they are such that those Wronskians are given by $\Phi_{sub\_k} = w_0^{k+1} w_1^k \cdots w_{k-1}^2 w_k \quad ; \; k = 0, 1, \cdots, n$. Hence we deduce that $w_1, \cdots, w_n$ are defined and nonzero whenever Φ_{sub_n} is nonzero. (This result is clearly independent of the meaning which the functions $s_{t\$n}$ have with respect to the variable t, it only refers to the fact that they are solutions of the linear ODE $\Phi\varphi = 0$).

Example 3.8:
Let $\Phi = D^2 + u$, where $u \in E$. Consider some w with $\Phi w = 0$. Then

$$\Phi^{(w)} = wD + 2w_x$$

Hence $w_1 = \frac{1}{w^2}$ fulfills the requirement $\Phi^{(w)} w_1 = 0$ and (w, w^{-2}) is a factorizing sequence. Thus we obtain:

$$\Phi^{-1} = wD^{-1}\frac{1}{w^2}D^{-1}\frac{1}{ww^{-2}} = wD^{-1}\frac{1}{w^2}D^{-1}w$$

or

$$\Phi = \frac{1}{w}Dw^2D\frac{1}{w} \; .$$

The kernel of this operator obviously is spanned by w and $wD^{-1}w^{-2}$. □

Using all the results we have so far, we obtain a complete characterization of recursive flows. Let $G(v, t)$ be a one-parameter family of vector fields on E and recall that along the flow defined by $s_t = G(s, t)$ one may replace $s_{t\$(n+1)}$ with $G_{t\$n}(s, t)$, $n = 0, 1, \ldots$. We shall denote, for short,

$$G_0(s,t) := s \;\; , \; G_n(s,t) := G_{t\$(n-1)}(s,t) \;\; , n = 1, 2, \ldots, \; . \tag{3.5}$$

Theorem 3.9: *The following are equivalent*

i) $G(s, t)$ *is weakly recursive of order* $N - 1$

ii) There is a differential operator

$$\Phi = D^N + a_{N-1}(s,t)D^{N-1} + a_{N-2}(s,t)D^{N-2} \cdots + a_0(s,t)$$

such that $\{G_n(s, t) : n \in I\!N_0 \}$ *spans the solution space* $\{\varphi | \Phi\varphi = 0\}$ *of* Φ .

If either of these equivalent conditions is fulfilled, then we have in addition: The coefficients $a_0(s,t),\ldots,a_{N-1}(s,t)$ in ii) are invariant densities with respect to the flow of $s_t = G(s,t)$ and for Φ we obtain the representation:

$$\Phi = w_0 \cdots w_{N-1} D \frac{1}{w_{N-1}} D \frac{1}{w_{N-2}} \cdots D \frac{1}{w_0} \tag{3.6}$$

where

$$\begin{array}{lcl} w_0 & = & s \\ w_1 & = & D w_0^{-1} G_1(s,t) \\ & & \cdots\cdots \\ w_k & = & D w_{k-1}^{-1} D \cdots D w_1^{-1} D w_0^{-1} G_k(s,t) \\ & & \cdots\cdots \\ w_{N-1} & = & D w_{N-2}^{-1} D \cdots D w_1^{-1} D w_0^{-1} G_{N-1}(s,t) \; . \end{array}$$

Proof: Remark at first that (i) also is equivalent to assume that the Wronskians Φ_{sub_n} satisfy identically:

$$\Phi_{sub\_N} = 0 \;\; ; \;\; \Phi_{sub\_k} \neq 0 \;\; ; \;\; k = 1,\ldots,(N-1).$$

We prove (i) $\Rightarrow$ (ii) by taking the characteristic operator for $G(s,t)$. Conversely, (ii) $\Rightarrow$ (i) follows from the fact that the kernel of Φ is N-dimensional. So, if that kernel is spanned by $\{s_{t\$n} | n \in \mathbb{N}_0\}$, then at least N of them must be linearly independent. However, for no $j < N$ it can be that $\sum_{i=0}^{j} \alpha_i(t) s_{t\$i} = 0$ because this would imply

$$(\Phi_{sub\_(j-1)})^{-1} \Phi_{sub\_j} = \Phi_{(j)} s_{t\$j} = 0$$

thus yielding $\Phi_{(k)} s_{t\$k} = 0$ for all $k > j$. Therefore exactly the first N of $(s, s_t, s_{tt}, \ldots)$ must be linearly independent and all others can be expressed by them. This is the same as saying that $s_t = G(s,t)$ is weakly recursive of order $N-1$.

Now, according to Section 2 the $a_k(s,t)$ are invariant densities and the representation for Φ is the same as the one given in Corollary 3.6. ■

Corollary 3.10: *Conversely, if an N-th order differential operator*

$$\Phi_N = D^N + a_{N-1}(s,t) D^{N-1} + a_{N-2}(s,t) D^{N-2} \cdots + a_0(s,t)$$

is factorized by

$$\Phi_N = w_0 \cdots w_{N-1} D \frac{1}{w_{N-1}} D \cdots D \frac{1}{w_0}$$

where the fields $w_0 = s, w_i = w_i(s,t)$ $, i = 1, 2, \ldots, N-1$*, satisfy the condition*

$$\begin{cases} D\frac{(w_0)_t}{w_0} & = & w_1 \\ D\frac{(w_1)_t}{w_1} & = & w_2 - 2w_1 \\ D\frac{(w_k)_t}{w_k} & = & w_{k+1} - 2w_k + w_{k-1} \ , \ k = 2, \ldots, N-2 \\ D\frac{(w_{N-1})_t}{w_{N-1}} & = & -2w_{N-1} + w_{N-2} \end{cases} \tag{3.7}$$

in the direction of

$$s_t = sD^{-1}w_1$$

then $G(s,t) := sD^{-1}w_1$ *is weakly recursive of order* $N-1$ *and, if* $s_t = G(s,t)$ *defines a flow, then this leaves invariant the corresponding coefficients* $a_i = a_i(s,t)$ $, n = 0, 1, \ldots, N-1$ *.*
Proof: The condition is required in order that, for $n = 0, 1, \ldots, N-2$,

$$\frac{d}{dt}\left(w_0 D^{-1} w_1 D^{-1} \cdots D^{-1} w_n\right) = w_0 D^{-1} w_1 D^{-1} \cdots D^{-1} w_{n+1} \ ;$$

moreover the $(N-1)$-th order weak recursiveness is equivalent to say $w_N = 0$. ■

Example 3.11:
Consider the recursive flow

$$G(s) = sD^{-1}(s^{-1})$$

given in (1.6). Then its characteristic operator (2.8)

$$\Phi = D^3 + \left(\frac{{s_x}^2}{s^2} - \frac{2s_{xx}}{s}\right) D + \frac{1}{2}\left(\frac{{s_x}^2}{s^2} - \frac{2s_{xx}}{s}\right)_x$$

has, according to the factorization given in the last theorem, the following form

$$\Phi = s^{-1} DsDsDs^{-1} \ . \tag{3.8}$$

Conversely, this operator allows to reconstruct $G(s,t) = sD^{-1}s^{-1}$ as corresponding vector field since one may check that $w_0 = s$ and $w_1 = s^{-1}$ satisfy the condition (3.7) □

4 Applications

4.1 Cauchy problems for Nonlinear Wronskian PDE's

The applications considered in this subsection are related to [3]. Let us first give an example which shows that the structural properties of recursive and weakly recursive flows yield methods to give explicit solutions for a wide class of initial value problems.

Example 4.1:
For

$$s\ s_{tx} - s_t s_x = sF(x) \tag{4.1}$$

we proved that arbitrary boundary conditions at $x = x_0$, say $s(t, x_0) = \phi(t)$, gave the weakly recursive flow

$$s_t = sD_{x_0}^{-1} s^{-1} F(x) + sg(t) \tag{4.2}$$

where $g(t) = s_t(t, x_0)/s(t, x_0)$. The recursiveness followed from the fact that s_{ttt} could be expressed in linear terms of lower derivatives

$$s_{ttt} = (g(t)^2 + 2g_t)\ s_t + g_{tt}\ s\ . \tag{4.3}$$

¿From the information contained in (4.3) one easily finds the complete solution for (4.1) in case that we are given arbitrary boundary values $s(t, x_0) = \phi(t)$ and initial values at $t = 0$. To see this, take (4.3) as an ordinary differential equation and observe that ϕ itself must be a solution of that equation because (4.3) holds for each x, and in particular for x_0. Furthermore, by differentiating with respect to x one may see that $\sigma = s_x(t, x_0)$ must be another solution. This second solution is easily computed from the boundary data on the line $x = x_0$, because there we find from $s\ s_{tx} - s_t s_x = sF(x)$ that $\phi\sigma_t - \sigma\phi_t = F(x_0)\phi$. Hence, knowing two independent solutions of the linear differential equation (4.3) we are able to compute a third one by the method of *variation of constants*. Thus all solutions of $s\ s_{tx} - s_t s_x = sF(x)$, satisfying the given boundary values, are found and this, obviously, suffices to solve the full initial value problem. □

We want to generalize this method. Consider a (possibly nonlinear) PDE of the following type

$$F(s, \cdots, s_{t\$n,x\$m}, \cdots, s_{t\$M,x\$M}) = 0 \tag{4.4}$$

where M is a finite positive integer, and F locally is a C^∞ function in all its entries. Such an equation is said to be of *Wronskian type* if it implies that along its flow the following identity holds for some positive integer N

$$\det \begin{pmatrix} s & s_t & \cdots & s_{t\$(N+1)} \\ s_x & s_{t,x} & \cdots & s_{t\$(N+1),x} \\ \vdots & \vdots & \cdots & \vdots \\ s_{x\$(N+1)} & s_{t,x\$(N+1)} & \cdots & s_{t\$(N+1),x\$(N+1)} \end{pmatrix} = 0 \qquad (4.5)\ .$$

For those PDE's which do not have any solution, for example the equation given by Lewy [9] (see also [2, p.81]), we remark that this definition may still have a precise meaning in terms of ideals. In order to see that, we take the algebra $\mathcal{A}(x,t,s)$ generated by all C^∞ functions of s and of its arbitrary multiple derivatives. They may also depend on x and t but in such a way that this algebra is closed under total differentiation with respect to x and t. An ideal $\mathcal{F}$ in that algebra, again required to be closed under application of derivatives with respect to x t, is said to be a *differential ideal*. Such a differential ideal $\mathcal{F}$ will be called *Wronskian ideal* if there is some N such that $\mathcal{F}$ contains the differential ideal generated by

$$\det \begin{pmatrix} s & s_t & \cdots & s_{t\$(N+1)} \\ s_x & s_{t,x} & \cdots & s_{t\$(N+1),x} \\ \vdots & \vdots & \cdots & \vdots \\ s_{x\$(N+1)} & s_{t,x\$(N+1)} & \cdots & s_{t\$(N+1),x\$(N+1)} \end{pmatrix} \ , \qquad (4.6)$$

and the smallest N for which that is true is said to be the *order* of $\mathcal{F}$. So, more in general, we shall say that equation (4.4) is of *Wronskian type* if the differential ideal generated by the function F is a Wronskian ideal.

Clearly, if equation (4.4) admits a flow, then this definition reduces to the preceding one. Indeed $F = 0$ yields the trivial ideal, and hence at least one Wronskian (of order N) will necessarily be zero along the flow.

Example 4.2:

The following are Wronskian PDE's

$$\left(\frac{s_t}{s}\right)_x = \frac{1}{s^2} \tag{4.7}$$

$$s\ s_{tx} - s_t s_x = sF(x) \tag{4.8}$$

$$s_{tt} = \frac{1}{2} s^{-1}\left((s_t)^2 + a^2 s^2\right) \tag{4.9}$$

$$\left(\frac{s_t}{s}\right)_x = s^{-2}\sqrt{ms^2 - s_x^2} \tag{4.10}$$

$$(N-1)\left(\frac{s_t}{s}\right)^{(N-2)}\left(\frac{s_t}{s}\right)_x = \frac{1}{s}. \tag{4.11}$$

Equation (4.7) is of order 1 and (4.8) to (4.10) are of order 2, whereas (4.11) is of order N.□

The reason why these equations have the Wronskian property is a consequence of the following remark.

Remark 4.3: *Observe that all the recursive and weakly recursive flows we have seen so far, were given by an equation of the form*

$$s_t = sf(h_1 D^{-1} h_2) \tag{4.12}$$

where h_1, h_2 are elements in $\mathcal{A}(x,t,s)$ and where f is a suitable C^∞-function. Indeed this form is necessary, however not sufficient, for recursiveness, as can be seen from the representation by characteristic operators and from the fact that

$$D\frac{G}{w_0} = w_1 = \frac{ss_{tx} - s_t s_x}{s^2}$$

belongs to $\mathcal{A}$.
Observe that when we check, by means of characteristic operators, whether or not the vector field $G(s,t) = sf(h_1 D^{-1} h_2)$ defines a weakly recursive flow, we only use the properties of the functions f, h_1, h_2 and the fact that D^{-1} is the right inverse of the differential operator D. Hence replacing D^{-1} by any other right inverse (i.e by $D_{x_0}^{-1}$ or by $D^{-1} + function(t)$) we again obtain a weakly recursive flow. So, from this viewpoint it is quite obvious, that the flows (1.17) and (1.21) were weakly recursive since they were related to the fundamental flow (1.7) by use of exactly that substitution.
Observe, in addition, that (4.12) may easily be rewritten as a PDE

$$D\left(h_1^{-1} f^{-1}(s_t/s)\right) = h_2\ , \tag{4.13}$$

which will be called the **associated PDE**.
Now, if the flow of (4.12) is weakly recursive, or so if it is the flow of an equation obtained from (4.12) under the substitution

$$D^{-1} \longrightarrow D^{-1} + function(t) \ , \tag{4.14}$$

then (4.13) obviously implies (4.5), because this last is a consequence of recursiveness, and hence (4.13) must be of Wronskian type.
Finally, one easily observes that all equations (4.7)-(4.11) are the associated PDE of some weakly recursive flow, or are obtained from them by differentiation with respect to t along the corresponding vector field, hence they must be of Wronskian type.

This is not the only way how weakly recursive flows lead to Wronskian PDE's. Let the coefficients of the characteristic operator for some weakly recursive flow be elements of $\mathcal{A}(x,t,s)$. Another element of $\mathcal{A}(x,t,s)$ is said to be a *generic quantity* for that characteristic operator if it generates the same differential ideal as the one generated by the set of these coefficients. Observe that (2.11) is a generic quantity for the characteristic operator of (2.10) and that a_1, as given in (2.9), is a generic quantity for (2.8).

Take a generic quantity u for the characteristic operator $\Phi_{(N+1)}$ of some weakly recursive flow of order N. Then the PDE $u_t = 0$ is of Wronskian type of order N. In particular the equations

$$\left\{ \frac{{s_x}^2}{s^2} - \frac{2s_{xx}}{s} \right\}_t = 0 \tag{4.15}$$

and

$$\left(\frac{ms - s_{xx}}{2\sqrt{ms^2 - s_x^2}} \right)_t = 0 \tag{4.16}$$

are Wronskian PDE's.

Indeed, let u be a generic quantity for the characteristic operator $\Phi_{(N+1)}$ of some weakly recursive flow, and observe that all the coefficients of the derivative ${\Phi_{(N+1)}}_t$ belong to the the differential ideal $\mathcal{F}(u_t)$ generated by u_t. Hence, for any $A \in \mathcal{A}(x,t,s)$ the quantity ${\Phi_{(N+1)}}_t A$ must be in the ideal $\mathcal{F}(u_t)$ as well. So, from $\Phi_{(N+1)} s_{t\$N} = 0$ we find, by taking the t-derivative, that ${\Phi_{(N+1)}}_t s_{t\$N} = -\Phi_{(N+1)} s_{t\$(N+1)}$, hence $\Phi_{(N+1)} s_{t\$(N+1)} \in \mathcal{F}(u_t)$.

Consider now the partial differential equation (4.4) and assume that it is Wronskian of order N. We want to give solutions for two different initial-boundary value problems for this equation. First the

Non-characteristic Cauchy problem:
We wish to find $s(x,t)$ such that s satisfies (4.4), subject to th e following initial conditions:

- A curve $\Gamma = \Gamma(\sigma)$ is assigned in the (x,t)-plane, parametrized by σ

  $$\Gamma : \begin{cases} x &= \xi(\sigma) \\ t &= \tau(\sigma) \end{cases}$$

  such that $\xi'(\sigma) \neq 0$ for all σ.

- Cauchy data are given on the curve Γ which are *"sufficiently rich"* to allow the unique determination of all the x- and t-derivatives of s on the curve Γ itself. By this we mean that enough x- and t-derivatives on the curve Γ are assigned such that all the others can be uniquely determined by use not only of the PDE (4.4) itself, but possibly also by use of the linear relations such as

  $$s_\sigma = \sigma_x s_x + \sigma_t s_t \ , \tag{4.17}$$

  which one may deduce for the x- and t-derivatives by taking the σ-derivatives of the Cauchy data on Γ.

We observe that if the PDE is of Wronskian type then we can solve this Cauchy problem solely in terms of linear ordinary differential equations. In fact we use the following

Recipe 1: By means of the given Cauchy data, compute all x- and t-derivatives for s on Γ up to order $(N+1)$. Since the differential equation is Wronskian of order N we can find a linear representation on Γ

$$\vec{s}_{t\$(N+1)} = \sum_{n=0}^{n=N} \alpha_n(t) \vec{s}_{t\$n} \ , \tag{4.18}$$

where

$$\vec{s} = \begin{pmatrix} s \\ s_x \\ \vdots \\ s_{x\$N} \end{pmatrix} \ . \tag{4.19}$$

This representation can be determined from the computed data on Γ solely by linear algebra. Now, solve along each fixed line $x = \xi$ the ODE

$$s_{t\$(N+1)} = \sum_{n=0}^{n=N} \alpha_n(t) s_{t\$n} \tag{4.20}$$

with known initial values $s, s_t, \cdots, s_N$ on the intersection of that line with Γ. Denote the solution obtained this way as $s(t, \xi)$. Then, taking these solutions for all ξ and replacing ξ by x, we have a solution of the original PDE for the given Cauchy data.

However it should be remarked that the coefficients in the linear dependence (4.18) may depend on t in such a way that an explicit solution for the Cauchy problem on the curve Γ cannot be given. This difficulty can be overcome if we consider, instead, the

Characteristic Cauchy problem:
Fix x_0 and t_0 and give *"sufficiently rich"* Cauchy data on the two lines $x = x_0$ and $t = t_0$. By *"sufficiently rich"* we mean that on the line $x = x_0$ enough x-derivatives of s are given such that, by taking the t-derivatives

of these data along the line, and by use of the differential equation under consideration, we can uniquely compute all x-derivatives up to $s_{x\$(N+1)}$ on $x = x_0$. Similarly, assume that the Cauchy data on $t = t_0$ are *"sufficiently rich"* to compute uniquely the corresponding t-derivatives on that line. Moreover, we have to require that the Cauchy data are compatible in the point (x_0, t_0).

We claim that if the PDE is of Wronskian type then there is an explicit solution to this characteristic Cauchy problem. In order to find the solution we use the following

Recipe 2: Compute all x-derivatives up to $s_{x\$(N+1)}$ along the line $x = x_0$ (using as said before the Cauchy data on this line). Then, because the equation is assumed to be Wronskian of order N, there must be a representation:

$$\vec{s}_{t\$(N+1)} = \sum_{n=0}^{n=N} \alpha_n(t) \vec{s}_{t\$n} \tag{4.21}$$

where again $\vec{s}$ is given by (4.19) This representation is found by linear algebra. Now, consider the corresponding equation

$$s_{t\$(N+1)} = \sum_{n=0}^{n=N} \alpha_n(t) s_{t\$n} \tag{4.22}$$

as an ordinary differential equation with respect to t. Indeed, we know a complete basis for the solutions of this ODE, although its coefficients are t-dependent.This basis is given by the already computed

$$\{s(t, x = x_0), s_{x\$1}(t, x = x_0), s_{x\$2}(t, x = x_0), \cdots, s_{x\$N}(t, x = x_0), \} \ .$$

Since $s(x, t)$, for any fixed x, has to fulfill the same ordinary differential equation, we then find that $s(x, t)$ is a linear combination over the elements of this basis, i.e. $s(x, t)$ admits a representation of the following type

$$s(t, x) = \sum_{n=0}^{n=N} \beta_n(x) s_{x\$n}(t, x = x_0) \ . \tag{4.23}$$

In order to find the coefficients β_n we take the t-derivatives of this equation computed along the line $t = t_0$. Thus we have to solve the following linear system

$$\begin{aligned} s(t_0, x) \quad &= \textstyle\sum_{n=0}^{n=N} \beta_n(x) s_{x\$n}(t = t_0, x = x_0) \\ s_{t\$1}(t_0, x) \quad &= \textstyle\sum_{n=0}^{n=N} \beta_n(x) s_{t\$1,x\$n}(t = t_0, x = x_0) \\ \ldots \quad &\quad \ldots \\ s_{t\$N}(t_0, x) \quad &= \textstyle\sum_{n=0}^{n=N} \beta_n(x) s_{t\$N,x\$n}(t = t_0, x = x_0) \end{aligned} \tag{4.24}$$

with respect to the unknowns $\beta_n(x)$. Inserting finally these $\beta_n(x)$ into (4.23) we obtain the explicit solution of the general problem.

Example 4.4:
Consider the Cauchy data $s(0, x) = f(x)$ and $s(t, 0) = g(t)$ for equation (4.7). These data, of course, have to fulfill the compatibility condition $a := f(0) = g(0)$. By integration with respect to x we obtain

$$s_t = s g_t g^{-1} + s D_0^{-1} s^{-2} \tag{4.25}$$

where D_0^{-1} stands for integration from 0 to x. Following the steps of Recipe 2 one finds by use of the Cauchy data on the line $x = 0$

$$s_{tt} = \left\{ \left(\frac{g_t}{g} \right)^2 + \left(\frac{g_t}{g} \right)_t \right\} s \ . \tag{4.26}$$

We know that $s(t,0) = g(t)$ and $s_x(t,0)$ must be solutions of this equation. We solve for $s_x(t,0)$ the equation (4.7), hence out of the Cauchy data we find that

$$g_1(t) := g(t) \int_0^t g^{-2}(\tau) d\tau \tag{4.27}$$

also solves (4.25). Hence the general solution must be of the form

$$s(t,x) = \beta_0(x) g(t) + \beta_1(x) g(t) \int_0^t g^{-2}(\tau) d\tau. \tag{4.28}$$

Computing from here $s_t(t,x)$, and evaluating at $t = 0$

$$s_t(t,0) = \beta_0(x) g_t(0) + \beta_1(x)/a \ ,$$

we find

$$\beta_0(x) = a^{-1} f(x) \text{ and } \beta_1(x) = a f(x) \int_0^x f^{-2}(\xi) d\xi \ .$$

This finally yields the general solution for (4.7)

$$s(t,x) = \frac{f(x)g(t)}{a} + a f(x) g(t) \int_0^t g^{-2}(\tau) d\tau \int_0^x f^{-2}(\xi) d\xi \ . \tag{4.29}$$

Observe that substitution $s = \exp(h)$ transforms (4.7) into the Liouville equation

$$h_{xt} = e^{-2h} \ . \tag{4.30}$$

Hence substitution in the general solution (4.29) must yield the well known general solution for the Liouville equation (see [11] or [7]). □

Example 4.5:
For the noncharacteristic Cauchy problem for (4.7), a general solution cannot be given in explicit form. However, that problem can be reduced to a linear second order ODE. To see this, consider a curve

$$\Gamma = \{(\xi(t), t) : t \in \mathbb{R}\} \ ,$$

parametrized by, say, t. Give on that curve Cauchy data

$$s(\xi(t), t) = \psi_1(t), \ \ s_t(\xi(t), \psi_2(t)$$

and compute $\phi(t) = \psi_1(t)/\psi_2(t)$. Then by Recipe 1 one obtains on the lines $x = \textit{constant}$ the following ODE

$$s_{tt} = (\phi^2 + \phi_t) s \ . \tag{4.31}$$

In case of arbitrary ϕ, and arbitrary curves Γ, we cannot find explit solutions of this ODE. $\square$

Example 4.6:
Let us give another example, which is considerably more involved than the Liouville equation. Take (4.10)

$$\left(\frac{s_t}{s}\right)_x = s^{-2}\sqrt{ms^2 - s_x^2} \tag{4.32}$$

and prescribe the Cauchy data

$$s(0,x) = f(x), \quad s(t,0) = g(t)$$

together with the compatibility condition $a_0 := f(0) = g(0)$. Integrating from 0 to x and defining, for short, $\varphi := g_t/g$ we obtain

$$s_t = s\varphi + sD_0^{-1}s^{-2}\sqrt{ms^2 - s_x^2}\ . \tag{4.33}$$

Differentiation with respect to t, integration by parts and use of (4.32) gives

$$s_{tt} = s\left(\varphi_t + \frac{1}{2}\varphi^2 - \frac{1}{2g^2}\right) + \frac{1}{2s} + \frac{s_t^2}{2s}\ . \tag{4.34}$$

¿From here, one further t-differentiation leads to

$$s_{ttt} = 2s_t U + sU_t \tag{4.35}$$

where

$$U = \left(\varphi_t + \frac{1}{2}\varphi^2 - \frac{1}{2g^2}\right) = \left(\frac{g_{tt}}{g} - \frac{1}{2}\frac{g_t^2}{g^2} - \frac{1}{2g^2}\right)\ . \tag{4.36}$$

This equation indeed shows that (4.32) is Wronskian of order 2. From the prescribed Cauchy data we can find a basis of the solution space for the linear equation (4.35). The first solution, obviously, must be g itself.And computation, with help of (4.32), of first and second x-derivative on the boundary line $t = 0$ gives the further solutions

$$g_1(t) := g(t)\sin\left(\int_0^t \frac{d\tau}{g(\tau)^2}\right) \quad \text{and} \quad g_2(t) := g(t)\cos\left(\int_0^t \frac{d\tau}{g(\tau)^2}\right)\ .$$

So the general solution of (4.32) must be of the form

$$s(x,t) = \beta_0(x)g(t) + \beta_1(x)g_1(t) + \beta_2(x)g_2(t)\ . \tag{4.37}$$

For the determination of $\beta_0, \beta_1, \beta_2$ we have to determine the t-derivatives of the function s on the line $t = 0$. To do this, we introduce the following abbreviations

$$f_1(x) := s_t(t = 0, x), \quad f_2(x) := s_{tt}(t = 0, x), \text{ and } a_k := g_{t\$k}(0) \ .$$

Observe that $f_k(0) = a_k$. Evaluation of (4.33) at $t = 0$ gives

$$f_1(x) = a_1 + f(x)F(x) \ , \tag{4.38}$$

where

$$F(x) = \int_0^x f(\xi)^{-2} \sqrt{m f(\xi)^2 - f_\xi(\xi)^2} d\xi \tag{4.39}$$

Further evaluation of (4.34), again at $t = 0$, produces

$$f_2(x) = \left(\frac{a_2}{a_0} - \frac{1}{2} \frac{a_2}{a_0^2} - \frac{1}{2a_0^2} \right) f(x) + \frac{1}{2f(x)} + \frac{f_1(x)^2}{2f(x)}. \tag{4.40}$$

Finally, taking the first three t-derivatives of (4.37) at $t = 0$ we obtain the linear system

$$\begin{array}{rcl} f(x) & = & \beta_0(x)a_0 + \beta_2(x)a_0 \\ f_1(x) & = & \beta_0(x)a_1 + \beta_2(x)a_1 + \beta_1(x){a_0}^{-1} \\ f_2(x) & = & \beta_0(x)a_2 + \beta_2(x)a_2 - \beta_2(x){a_0}^{-3} \ . \end{array}$$

This system is easily solved. Inserting then the solution into (4.37) delivers the complete solution of this characteristic Cauchy problem. □

4.2 Further Applications

There is another wide class of applications, which we want to mention briefly at the end of this paper (details will be published separately). Observe first that whenever a weakly recursive flow is Hamiltonian, with a differential operator in x as implectic operator (see [10], [4] for definitions), then this flow is completely integrable.

This is easily seen, take the coefficients of the characteristic operator, then evaluation of each coefficient at any x gives a conserved quantity. And these conserved quantities are indeed in involution (since their Poisson brackets vanishes). Hence, the flow must have infinitely many commuting symmetry groups. Now, take the case when the characteristic operator of such a flow admits a generic quantity, say $u(x,t)$. We then consider the variable transformation from s to u, and we want to make out of the symmetry groups for the s-dynamics corresponding symmetry groups for the u-dynamics. In general we cannot conclude that such a transformation produces commuting symmetries for the variable u; this because s is not defined uniquely by u since the transformation is given by a differential equation. However, since u is assumed to be generic, we know that s is unique up to an application of the dynamics given by the original flow for s. Since this flow commutes with all symmetry groups under consideration, we have that all corresponding u symmetries do commute.

If one applies that recipe for, say, the flow given by the vector field (1.6 then the hierarchy of the well known KdV is produced. In case of (1.8) this leads to the hierarchy of the modified KdV. And the interrelations one obtains this way, are even more surprising since the characteristic operator then turns out to be related to the recursion operator of these new hierarchies for the u-dynamics, and the corresponding variable s is given by the eigenfunctions of this recursion operator. The weakly recursive flow we started with, then is uniquely related to the well known transformation whichs maps, for interacting solutions, the action variables into angle variable (see([6])).

References

[1] E.A. Coddington and N. Levison: *Theory of Ordinary Differential Equations*, E. Krieger, Malabar, Florida, 1984

[2] G. B. Folland: *Introduction to partial differential equations*, Princeton University Press, Princeton, 1976

[3] B. Fuchssteiner and S. Carillo: *Generalized Louville equations and the Riccati property*, preprint, 1991

[4] B. Fuchssteiner and A. S. Fokas: *Symplectic Structures, Their, Bäcklund Transformations and Hereditary Symmetries*, Physica, 4 D, p.47-66, 1981

[5] B. Fuchssteiner: *Solitons in Interaction*, Progress of Theoretical Physics, 78, p.1022-1050, 1987

[6] B. Fuchssteiner and S. Carillo: *The Action-Angle transformation for Soliton Equations*, Physica, 166 A, p.651-675, 1990

[7] E. Goursat: *A course in mathematical Analysis, Vol II, part 2*, Dover Publications, Inc., New York, 1964

[8] P. Hartman: *Ordinary Differential Equations*, Birkhäuser, Boston - Basel - Stuttgart, 1982

[9] H. Lewy: *An example of a smooth linear partial differential equation without solution*, Ann. of Math., 66, p.155-158, 1957

[10] C. Rebbi and G. Soliani: *Solitons and Particles*, World Scientific, Singapore, 1984

[11] C. Rogers and W. F. Ames: *Nonlinear Boundary Value Problems in Science and Engineering*, Academic Press, Boston - San Diego - New York - Berkeley - London - Sydney - Tokyo - Toronto, 1989