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Nonlinear PDE's and recursive flows: Applications

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The theory developed in a preceeding paper is applied for the solution of the Cauchy problem for so called Wronskian partial differential equations. These are equations having the property that a Wronskian determinant, formed by the x- and t-derivatives, is equalt to zero on the solution manifold of the equation.

1 Introduction

In [4] (see also [3]) we introduced weakly recursive vector fields or flows: A vector field $G(v,t)$, where v denotes the variable on a manifold formed by a space of functions in x, is called **weakly recursive** if for $s_t = G(s,t)$ there are smooth coefficients $\alpha_n(t)$ such that $(d/dt)^{N+1}s = \sum_{n=0}^{N}\alpha_n(t)(d/dt)^n s$. The smallest integer N fulfilling that relation is the order of recursiveness. When the coefficients are constant, then the field is said to be **recursive**, and it is said to be **nilpotent** if all α_n are zero. Recursive and nilpotent flows can be explicitly solved.

The applications considered in this paper are related to [2]. Let us first give an example which shows that the structural properties of recursive and weakly recursive flows give access to explicit solutions for a wide class of initial value problems.

Example 1.1: For arbitrary boundary conditions $s(t,x_0) = \phi(t)$, integration of $s\ s_{tx} - s_t s_x = sF(x)$ gives the weakly recursive flow $s_t = sD_{x_0}^{-1}s^{-1}F(x) + sg(t)$ where $g(t) = s_t(t,x_0)/s(t,x_0)$. The recursiveness follow from the fact that s_{ttt} can be expressed in linear terms of lower derivatives: $s_{ttt} = (g(t)^2 + 2g_t)\ s_t + g_{tt}\ s$. Consider this last equation as an ordinary differential equation. Observe that it must have ϕ as one solution. Differentiating with respect to x one sees that $\sigma = s_x(t,x_0)$ must be another solution. This second solution is easily computed from the boundary data on the line $x = x_0$, because there we find from $s\ s_{tx} - s_t s_x = sF(x)$ that $\phi\sigma_t - \sigma\phi_t = F(x_0)\phi$. Now, knowing two independent solutions of our linear differential equation, we are able to compute a third one by the method of *variation of constants*. Thus all solutions of $s\ s_{tx} - s_t s_x = sF(x)$ are found and this suffices to solve the full initial value problem provided sufficient Cauchy data at t= 0 are supplied. □

We generalize this method. We denote by $s_{t\$n,x\$m}$ the result of taking n t-derivatives and m x-derivatives. We consider a partial differential equation

$F(s, \cdots, s_{t\$n,x\$m}, \cdots, s_{t\$M,x\$M}) = 0$ where M is a finite positive integer, and F locally is a C^∞-function in all its entries. This equation may be nonlinear. Such an equation is said to be of **Wronskian type** if for some positive integer N we have

$$\det \begin{pmatrix} s & s_t & \cdots & s_{t\$(N+1)} \\ s_x & s_{t,x} & \cdots & s_{t\$(N+1),x} \\ \vdots & \vdots & \cdots & \vdots \\ s_{x\$(N+1)} & s_{t,x\$(N+1)} & \cdots & s_{t\$(N+1),x\$(N+1)} \end{pmatrix} = 0 \qquad (1.1)\ .$$

on its solutions. The smallest N for which that is true is called the *order*. For those PDE's which do not have any solution, for example the equation given by Lewy [6] (see also [1, p.81]), we remark that this definition may still have a precise meaning in terms of ideals: We take the algebra $\mathcal{A}(x, t, s)$ generated by all C^∞-functions of s, x and t and arbitrary derivatives of s, and closed under differentiation with respect to x and t. An ideal $\mathcal{F}$ in that algebra, again required to be closed under application of derivatives, is said to be a *differential ideal*. To make the definition precise, we require that differential ideal generated by F contains the differential ideal generated by the determinant.

Example 1.2: The following are Wronskian PDE's

$$\begin{aligned} (s_t/s)_x &= s^{-2} && (1.2) \\ sF(x) &= s\ s_{tx} - s_t s_x && (1.3) \\ s_{tt} &= (1/2)s^{-1}((s_t)^2 + a^2 s^2) && (1.4) \\ (s_t/s)_x &= s^{-2}\sqrt{ms^2 - s_x^2} && (1.5) \\ (1/s) &= (N-1)(s_t/s)^{(N-2)}(s_t/s)_x. && (1.6) \end{aligned}$$

Equation (1.2) is of order 1 and (1.3) to (1.5) are of order 2, whereas (1.6) is of order N.□

The connection between Wronskian PDE's and weakly recursive flows is seen from the fact that the latter are of the form $s_t = sf(h_1 D^{-1} h_2)$. Indeed this form is necessary, however not sufficient, for recursiveness, as can be seen from the representation by characteristic operators (see [3] for details). Observe that this equation may be easily rewritten as $D\left(h_1^{-1} f^{-1}(s_t/s)\right) = h_2$, a form which will be called the **associated PDE**. Now, if the above flow is weakly recursive, then the theory on recursive flows implies (1.1). So weakly recursive flows give rise to Wronskian PDE's and those in (1.2)-(1.6) are the associated PDE's of some of the weakly recursive flows given in [4].

However, there is an alternative way how weakly recursive flows lead to Wronskian PDE's. Consider the differential ideal generated by the coefficients of the characteristic operator for some weakly recursive flow. Then an element of $\mathcal{A}(x,t,s)$ is said to be a *generic quantity* for that characteristic operator if it generates the same differential ideal.

Example 1.3: Observe that $u=(ms-s_{xx})/(2\sqrt{ms^2-s_x^2}\,)$ is a generic quantity for the characteristic operator of the weakly recursive flow $s_{tt}=(1/2s)(s_t^2+1)$ and that $(s_x{}^2/s^2)-(2s_{xx})/s$ is a generic quantity for the characteristic operator of $s_t=sD^{-1}(s^{-1})$. If u is a generic quantity for the characteristic operator Φ of some weakly recursive flow of order N, then the PDE $u_t=0$ is of Wronskian type of order N. In particular, the equations $(ms-s_{xx})/(2\sqrt{ms^2-s_x^2}\,)_t=0$ and $\left((s_x{}^2/s^2)-(2s_{xx})/s\right)_t=0$ are Wronskian PDE's. As a result one can see that there are many Wronskian PDE's related to recursive flows.□

1.1 Cauchy problems for Nonlinear Wronskian PDE's

Consider now the partial differential equation $F(s,\cdots,s_{t\$n,x\$m},\cdots,s_{t\$M,x\$M})=0$ and assume that it is of N-th order Wronskian. We want to give solutions for two different initial-boundary value problems for this equation. First the **Non-characteristic Cauchy problem:** We want to find $s(x,t)$ such that s satisfies the given PDE, subject to the following initial conditions:

- A curve $\Gamma=\Gamma(\sigma)=\{(\xi(\sigma),\tau(\sigma))|\sigma\in$ *parameter space*$\}$ is given in the (x,t)-plane such that $\tau'(\sigma)\neq 0$ for all σ.
- Cauchy data are given on the curve Γ which are *"sufficiently rich"* to allow the unique determination of all the x- and t-derivatives of s on the curve Γ itself. By this we mean that enough x- and t-derivatives on the curve Γ are assigned such that all the others can be uniquely determined by use of the PDE itself and by use of those linear relations, such as $s_\sigma=\sigma_x s_x+\sigma_t s_t$, which one can deduce by taking the σ-derivatives of the Cauchy data on Γ.

We observe that if the PDE is of Wronskian type then we can solve this Cauchy problem solely in terms of linear ordinary differential equations:

Method 1: By means of the given Cauchy data, compute all x- and t-derivatives for s on Γ up to order $(N+1)$. Since the differential equation is Wronskian of order N we can find a linear representation $\vec{s}_{t\$(N+1)}=\sum_{n=0}^{n=N}\alpha_n(t)\vec{s}_{t\$n}$ on Γ. Here $\vec{s}=(s,s_x,...,s_{x\$N})^T$. This representation can be determined by linear algebra. Now, solve along each fixed line $x=\xi$ the ordinary differential equation $s_{t\$(N+1)}=\sum_{n=0}^{n=N}\alpha_n(t)s_{t\$n}$ with known initial values $s,s_t,\cdots,s_N$ on the intersection of that line with Γ. Denote the solution obtained this way as $s(t,\xi)$. Then, taking these solutions for all ξ, and replacing ξ by x, we have a solution of the original PDE for the given Cauchy data.

However it should be remarked that the coefficients in the ODE above may depend on t in such a way that an explicit solution for the Cauchy problem on the curve Γ cannot be given. This difficulty can be overcome if we consider, instead, the

Characteristic Cauchy problem: Fix x_0 and t_0 and give *"sufficiently rich"* Cauchy data on the two lines $x = x_0$ and $t = t_0$, i.e the condition above must be fulfilled on both lines. Moreover, we require that the Cauchy data are compatible in the point (x_0, t_0).

If the PDE is of Wronskian type then we find the explicit solution by:

Method 2: Using the Cauchy data, compute all x-derivatives up to $s_{x\$(N+1)}$ along the line $x = x_0$. Then, because the equation is assumed to be Wronskian of order N, there must be a representation: $\vec{s}_{t\$(N+1)} = \sum_{n=0}^{n=N} \alpha_n(t)\vec{s}_{t\$n}$ where $\vec{s}$ is given as before. Consider the same ODE as before in method 1. Although its coefficients are t-dependent we already know a complete basis for its solutions, namely

$$\{s(t,x=x_0), s_{x\$1}(t,x=x_0), s_{x\$2}(t,x=x_0), \cdots, s_{x\$N}(t,x=x_0),\}\ .$$

Since $s(x,t)$, for any fixed x, has to fulfill the ordinary differential equation above, we then find that $s(x,t)$ is a linear combination over the elements of this basis, i.e. $s(x,t)$ admits a representation $s(t,x) = \sum_{n=0}^{n=N} \beta_n(x) s_{x\$n}(t, x = x_0)$. In order to find the coefficients β_n we take the t-derivatives of this equation computed along the line $t = t_0$. Thus we have to solve the linear system $s_{t\$n}(t_0,x) = \sum_{n=0}^{n=N} \beta_n(x) s_{t\$n,x\$n}(t = t_0, x = x_0)$ where $n = 0, ..., N$. Inserting finally the $\beta_n(x)$ into the representation for $s(t,x)$ we obtain the explicit solution of the general problem.

Example 1.4: Consider the Cauchy data $s(0,x) = f(x)$ and $s(t,0) = g(t)$ for (1.2). These data have to fulfill the compatibility condition $a := f(0) = g(0)$. By integration with respect to x we obtain $s_t = sg_tg^{-1} + sD_0^{-1}s^{-2}$ where D_0^{-1} stands for integration from 0 to x. Following the steps of method 2 one finds, by use of the Cauchy data on the line $x = 0$ $s_{tt} = \{(g_t/g)^2 + (g_t/g)_t\}s$. We know that $s(t,0) = g(t)$ and $s_x(t,0)$ must be solutions of this equation. Using the original equation and the Cauchy data we determine the latter solution to be $g_1(t) := g(t)\int_0^t g^{-2}(\tau)d\tau$. Hence the general solution must be of the form $s(t,x) = \beta_0(x)g(t) + \beta_1(x)g(t)\int_0^t g^{-2}(\tau)d\tau$. Computing from here $s_t(t,x)$, and evaluating at $t = 0$ we find $s_t(t,0) = \beta_0(x)g_t(0) + \beta_1(x)/a$, hence $\beta_0(x) = a^{-1}f(x)$ and $\beta_1(x) = af(x)\int_0^x f^{-2}(\xi)d\xi$.

This finally yields the general solution

$$s(t,x)=\frac{f(x)g(t)}{a}+af(x)g(t)\int_0^t g^{-2}(\tau)d\tau\int_0^x f^{-2}(\xi)d\xi\ .$$

Observe that the substitution $s=\exp(h)$ transforms (1.2) into the Liouville equation $h_{xt}=e^{-2h}$. Hence substitution in the general solution (1.2) must yield the well known general solution for the Liouville equation (see [7] or [5]). □

Example 1.5: Let us give another example, which is more involved than the Liouville equation. Take (1.5), an equation playing a role in the soliton theory of the mKdV, and prescribe the Cauchy data $s(0,x)=f(x),\ \ s(t,0)=g(t)$ together with the compatibility condition $a_0:=f(0)=g(0)$. Integrating from 0 to x we obtain for $\varphi:=g_t/g,\ s_t=s\varphi+sD_0^{-1}s^{-2}\sqrt{ms^2-s_x^2}$. Differentiation with respect to t, integration by parts and use of (1.5) leads to the associated linear equation $s_{tt}=s\varphi_t+s/(2\varphi^2)-s/(2g^2)+1/(2s)+s_t^2/(2s)$. From here, one further t-differentiation leads to $s_{ttt}=2s_tU+sU_t$ where

$$U=\left(\varphi_t+\frac{1}{2}\varphi^2-\frac{1}{2g^2}\right)=\left(\frac{g_{tt}}{g}-\frac{1}{2}\frac{g_t^2}{g^2}-\frac{1}{2g^2}\right)\ . \tag{1.7}$$

This equation indeed shows that (1.5) is Wronskian of order 2. From the prescribed Cauchy data we find a basis of the solution space for the associated linear equation. The first solution, obviously, must be g itself. Computation of first and second x-derivatives on the boundary line $t=0$ yields further solutions

$$g_1(t):=g(t)\sin\left(\int_0^t\frac{d\tau}{g(\tau)^2}\right)\quad\text{and}\quad g_2(t):=g(t)\cos\left(\int_0^t\frac{d\tau}{g(\tau)^2}\right)$$

and the general solution is $s(x,t)=\beta_0(x)g(t)+\beta_1(x)g_1(t)+\beta_2(x)g_2(t)$ for suitable β's. To find these we have to determine the t-derivatives of s on the line $t=0$. We abbreviate for known data $a_k:=g_{t\$k}(0)$ and introduce $f_1(x):=s_t(t=0,x)$, and $f_2(x):=s_{tt}(t=0,x)$. These have to be determined by known data. Observe that $f_k(0)=a_k$. Evaluation of the integrated form of (1.5) at $t=0$ gives $f_1(x)=a_1+f(x)F(x)$ where $F(x)=\int_0^x f(\xi)^{-2}\sqrt{mf(\xi)^2-f_\xi(\xi)^2}d\xi$. Further evaluation of (1.5) at $t=0$ produces

$$f_2(x)=\left(a_2/a_0-a_2/(2a_0^2)-1/(2a_0^2)\right)f(x)+1/(2f(x))+(f_1(x)^2)/(2f(x)). \tag{1.8}$$

Finally, taking at $t = 0$ the first three t-derivatives of s expressed by the β's we obtain the linear system

$$\begin{aligned} f(x) &= \beta_0(x)a_0 + \beta_2(x)a_0 \\ f_1(x) &= \beta_0(x)a_1 + \beta_2(x)a_1 + \beta_1(x){a_0}^{-1} \\ f_2(x) &= \beta_0(x)a_2 + \beta_2(x)a_2 - \beta_2(x){a_0}^{-3} \ . \end{aligned}$$

where only the β's are unknown. This system is easily solved and delivers the complete solution of this characteristic Cauchy problem. □

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