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Nonlinear PDE's and recursive flows: Theory

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A class of nonlinear vector fields on infinite dimensional manifolds is introduced such that the corresponding nonlinear partial differential equations are solvable by a generalization of the method of varation of constants. This method also characterizes these equations and it can be used to construct sufficiently many conserved densities to solve them explicitly.

1 Introduction

In the following, three classes of special nonlinear flows are introduced which were discovered in the context of action-angle-variable transformation for soliton equations [2]. A more detailed version and applications [4] of the proposed method will be published in an enlarged paper [3].

For simplicity, the manifold we consider is some function space E and we denote by v its typical element. A vector field $G(v)$ on E is said to be be **nilpotent** if the flow $s_t = G(s)$ admits some N such that identically $(d/dt)^{N+1}s = 0$. N is said to be its order of nilpotency if this relation does not hold for any lower N. The vector field is called **recursive** if $s_t = G(s)$ defines a flow admitting constant coefficients α_n such that

$$(d/dt)^{N+1}s = \sum_{n=0}^{N} \alpha_n (d/dt)^n s \ .$$

Again, the recursiveness is said to be of *order* N if such a relation does not hold for any lower N.

The relevance of this definition stems from the observation that, although $G(v)$ may be nonlinear, in both cases the initial value problem of $s_t = G(s)$ is easily solved. To see this define $\vec{u} = (s, s_t, ..., (d/dt^N s))^T$ then $\vec{u}(t)$ follows a linear ordinary differential equation with constant coefficients. This is easily solved and leads to the solution of $s_t = G(s)$.

Example 1.1: Let s such that $1/s$ vanishes rapidly at $-\infty$. Then the following vector fields

$$\begin{aligned} G(s) &= \sqrt{2sF(x)} && (1.1)\\ G(s) &= sD^{-1}(s^{-1}) && (1.2)\\ G(s) &= sD^{-1}(F(x)s^{-1}) && (1.3)\\ G(s) &= sD^{-1}(s^{-2}\sqrt{ms^2 - s_x^2}) && (1.4) \end{aligned}$$

are nilpotent of second order. The vector field $G(s) = sD^{-1}(F(x)s^{-2})$ is nilpotent of first order. For β different from -1 the equation $s_t = s(D^{-1}s^{-1})^{\beta}$is nilpotent of order N. Observe that these equations, although integrations appear, are easily rewritten in terms of differential equations. For example, (1.2) goes over to the Liouville equation $s\, s_{tx} - s_t s_x = s$, and the above method yields a new method of solution for it [1]. $\square$

However, if D_{x_0} = *integration from* x_0 *to* x then the system $s_t = sD_{x_0}^{-1}s^{-1}F(x) + sa$ is recursive only if $a = a(x_0) = s_t(t, x_0)/s(t, x_0)$. So, in order to include different boundary conditions, we introduce the generalized notion of a vector field $G(v, t)$ being **weakly recursive** if for $s_t = G(s, t)$ there are smooth coefficients $\alpha_n(t)$ such that $(d/dt)^{N+1}s = \sum_{n=0}^{N} \alpha_n(t)(d/dt)^n s$. The smallest integer N fulfilling that, again is the order. Since now the coefficients depend on time t these equations may not be explicitly solvable.

2 Characteristic Operators

For $v(t, x)$ we denote by $v_{t\$n,x\$m}$ its n-th t-derivative and m-th x-derivative. If $s_t = G(s, t)$, then we can replace $s_{t\$(n+1),x\$m}$ by $G_{t\$n,x\$m}$.

Assume now that $G(s, t)$ is weakly recursive of order N, and consider coefficients $a_n = a_n(x, t)$ such that the operator $\Phi = (D^{N+1} + \sum_{n=0}^{N} a_n(x, t)D^n)$ satisfies $\Phi s_{t\$n} = 0$ for $n = 0, 1, \ldots, N$ and for all t. Such an operator is called the **characteristic operator** for the corresponding vector field. An explicit construction for this characteristic operator is easily given. Take the Wronskian determinant

$$\Phi_{op\_(N+1)} = \det \begin{pmatrix} s & s_t & \cdots & s_{t\$N} & D^0 \\ s_x & s_{t,x} & \cdots & s_{t\$N,x} & D^1 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ s_{x\$(N+1)} & s_{t,x\$(N+1)} & \cdots & s_{t\$N,x\$(N+1)} & D^{N+1} \end{pmatrix} \tag{2.1}$$

and define by Φ_{sub_n} the determinant obtained from $\Phi_{op\_(N+1)}$ by eliminating the last column and the $(n+1)$-th row. Then, since determinants having two columns in common are zero, we obtain $\Phi = (\Phi_{sub\_N})^{-1}\Phi_{op\_(N+1)}$ as characteristic operator, and the coefficients of Φ are obviously given by $a_n(x, t) = (-1)^n(\Phi_{sub\_N})^{-1}\Phi_{sub\_n}$.

With respect to the flow $s_t = G(s, t)$, a function $a = a(s, t)$ is said to be an **invariant density** if $0 = d/dt\{a(s, t)\}$. Because of the assumption that $G(s, t)$ is weakly recursive, $span\{s, s_t, ..., s_{t\$N}\}$ has dimension $N + 1$ and equals the kernel of Φ; therefore $s_t = G(s, t)$ defines a transformation in the kernel of the linear operator Φ and we find that the $a_i(s,t)$ are invariant under the flow $s_t = G(s,t)$, hence they are invariant densities for this flow. **Example** 2.1: Consider (1.2), then the characteristic operator is $\Phi = D^3 + a_1 D + a_0$ with coefficients $a_1 = {s_x}^2/s^2 - 2s_{xx}/s$ and $a_0 = a_{1x}/2$. One easily shows that these quantities are indeed invariant densities for (1.2).$\Box$

Now we show that characteristic operators can be factorized. Define, for $w \in E$ the **lowering** $\Psi^{(w)}$ of a differential operator Ψ by w to be the operator

$$\alpha(x) \longrightarrow \Psi^{(w)}\alpha := [\Psi, \textstyle\int^x \alpha(\xi)d\xi]w = \Psi w \int^x \alpha(\xi)d\xi - \int^x \alpha(\xi)d\xi \Psi w \ .$$

This operator is easily computed and we observe that, although it contains integrations, it is a pure differential operator of order less than Ψ. For repeated lowerings we use the notation $\Psi^{()} := \Psi$ and $\Psi^{(w_0,\dots,w_n,w_{n+1})} := (\Psi^{(w_0\dots,w_n)})^{(w_{n+1})}$. If Ψ is of N-th order then the *complete lowering* $\Psi^{(w_0,w_1,\dots,w_{N-1})} = a_N w_0 w_1 \dots w_{N-1}$ is a multiplication operator mapped to zero by one further lowering. The well known method of *variation of constants* can be restated in terms of lowerings since for $\Psi w_0 = 0$ the relation $\Psi^{(w_0)}w_1 = 0$ is equivalent to $\Psi w_0 D^{-1} w_1 = 0$. Hence, the second solution for $\Psi w = 0$ can be computed from one known solution and the solution of a lower order differential operator. This method is easily extended by recursion and leads to the following

Theorem 2.2: *Take $w_0, w_1, \dots, w_n$, then the following are equivalent*

i) $\Psi w_0 = 0$ and $\Psi^{(w_0,\dots,w_k)} w_{k+1} = 0$ for all $k < n$

ii) $\Psi w_0 = 0$ and $\Psi w_0 D^{-1} w_1 \cdots D^{-1} w_m = 0$ for all $m \le n$

A sequence $\{w_0, \cdots, w_n\}$ with property (i) or (ii) is said to be a **factorizing sequence** for Ψ. Indeed, such sequences may be used to factorize and invert operators formally: Let $w_0, \dots, w_n$ be a factorizing sequence for Ψ and denote by $\Psi^{-(w_0,\dots,w_n)}$ the inverse of $\Psi^{(w_0,\dots,w_n)}$ then

$$\Psi^{-1} = w_0 D^{-1} w_1 \cdots D^{-1} w_n D^{-1} \Psi^{-(w_0,\dots,w_n)} \tag{2.2}$$

Restating the equivalence of theorem 2.2 we find that the following are equivalent:

i) $\{w_0, w_1, \dots, w_n\}$ is a factorizing sequence

ii) $\varphi_0 = w_0, \varphi_1 = w_0 D^{-1} w_1, \ \cdots \ , \varphi_n = w_0 D^{-1} w_1 D^{-1} w_2 D^{-1} w_3 \cdots D^{-1} w_n$ are solutions of $\Psi\varphi = 0$

Conversely:

Let $\varphi_0, \cdots, \varphi_n$ be linearly independent solutions of $\Psi\varphi = 0$ and define (on *the intervals where* $w_0,\dots,w_n\neq 0$*, see [5, par.IV.3])*

$$\begin{aligned} w_0 &= \varphi_0, w_1 = Dw_0^{-1}\varphi_1,\cdots,w_k = Dw_{k-1}^{-1}D\cdots Dw_1^{-1}Dw_0^{-1}\varphi_k,\cdots\cdots, \\ w_n &= Dw_{n-1}^{-1}D\cdots Dw_1^{-1}Dw_0^{-1}\varphi_n. \end{aligned}$$

Then the $w_i,\ i=0,\cdots,n$ *define a factorizing sequence.*

Now take the formula for the complete lowering of Ψ and consider the representation for inverses given above. Taking a factorizing sequence $\{w_0,\dots,w_N\}$ for $\Psi=(D^{N+1}+\sum_{n=0}^{N}a_nD^n)$ one obtains

$$\begin{aligned} \Psi^{-1} &= w_0D^{-1}w_1\cdots D^{-1}w_ND^{-1}\{w_0\cdots w_N\}^{-1} && (2.3)\\ \Psi &= w_0\cdots w_{N-1}w_ND\{w_N\}^{-1}D\{w_{N-1}\}^{-1}\cdots D\{w_0\}^{-1}\ . && (2.4) \end{aligned}$$

Since factorizing sequences for differential operators are related one-to-one to the kernel of the operator, and since for characteristic operators of weakly recursive flows the kernel is known, we may apply these results immediately to characterize these flows.

Recall that along the flow defined by $s_t=G(s,t)$ one may replace $s_{t\$(n+1)}$ by $G_{t\$n}(s,t)$. Denote $G_0(s,t):=s$ ... $G_n(s,t):=G_{t\$(n-1)}(s,t)$

Theorem 2.3: *The following are equivalent:*

i) $G(s,t)$ *is weakly recursive of order* $N-1$

ii) There is a differential operator $\Phi=D^N+a_{N-1}(s,t)D^{N-1}+\cdots+a_0(s,t)$ *such that* $\{G_n(s,t):n\in\mathbb{N}_0\}$ *spans the solution space* $\{\varphi|\Phi\varphi=0\}$ *of* Φ .

If either of these equivalent conditions is fulfilled, then in addition: The coefficients $a_0(s,t),\dots,a_{N-1}(s,t)$ *in ii) are invariant densities with respect to the flow* $s_t=G(s,t)$ *and* Φ *is represented by:*

$$\Phi = w_0\cdots w_{N-1}D\{w_{N-1}\}^{-1}D\{w_{N-2}\}^{-1}\cdots D\{w_0\}^{-1} \qquad (2.5)$$

where

$$\begin{aligned} w_0 &= s\\ w_1 &= Dw_0^{-1}G_1(s,t),\cdots\cdots\\ w_k &= Dw_{k-1}^{-1}D\cdots Dw_1^{-1}Dw_0^{-1}G_k(s,t)\cdots\cdots\\ w_{N-1} &= Dw_{N-2}^{-1}D\cdots Dw_1^{-1}Dw_0^{-1}G_{N-1}(s,t)\ . && (2.6) \end{aligned}$$

Conversely, if an N-th order differential operator

$$\Phi_N = D^N+a_{N-1}(s,t)D^{N-1}+a_{N-2}(s,t)D^{N-2}\cdots+a_0(s,t)$$

is factorized by

$$\Phi_N = w_0 \cdots w_{N-1} D\{w_{N-1}\}^{-1} D \cdots D\{w_0\}^{-1}$$

where $w_0 = s$ *and* $w_i = w_i(s,t)$ $, i = 1, 2, \ldots, N-1$ *fulfill in the direction of* $s_t = sD^{-1}w_1$*, as vector fields depending on* $s(t)$*, the conditions*

$$\begin{array}{rcl} D((w_0)_t/w_0) & = & w_1 \\ D((w_1)_t/w_1) & = & w_2 - 2w_1 \\ D((w_k)_t/w_k) & = & w_{k+1} - 2w_k + w_{k-1} \ldots\ldots \\ D((w_{N-1})_t/w_{N-1}) & = & -2w_{N-1} + w_{N-2} \end{array}$$

then $G(s,t) := sD^{-1}w_1$ *is weakly recursive of order* $N-1$ *and* $s_t = G(s,t)$ *leaves the coefficients* $a_i = a_i(s,t)$ $, n = 0, 1, \ldots, N-1$ *invariant.*

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