in: Geometry and Analysis: Trends in Teaching and Research
B. Fuchssteiner and W. A. J. Luxemburg, eds.
Bibliographisches Institut Mannheim, p.73-85, 1992

A new class of nonlinear partial differential equations solvable by quadratures [1]

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Abstract

We introduce a class of nonlinear partial differential equations in two independent scalar variables, say x and t, characterized by the property that the initial value problem for given boundary values can be solved by quadratures. The Liouville equation enjoys such a property and seems to be the most simple equation among the elements of this class. Hence we term these equations *generalized Liouville equations*.

We further introduce the *Riccati property*, which refers to nonlinear ordinary differential equations and generalizes a well known property of the Riccati equation. This property requires that, whenever one particular solution of an equation is given, then it is possible to construct from that the general solution by quadratures. Nonlinear ordinary differential equations which enjoy the Riccati property are shown to be related to generalized Liouville equations.

[1] work partially supported by the G.N.F.M. of C.N.R. and by the M.U.R.S.T. project *Geometria e Fisica*. One of the authors (B. F.) wishes to acknowledge his gratitude to the Università "La Sapienza" and the Italian C.N.R. for the kind hospitality he received in Roma.

1 Introduction

A class of nonlinear differential equations, whose independent variables we denote as x and t, is here investigated.

Remarkably, the members of this class admit an explicit solution when the initial values with respect to t and the boundary values with respect to $x = const$ are given. Since the Liouville equation

$$h_{xt} = e^{2h}$$

enjoys such a property, we term the whole class of equations characterized by this property *generalized Liouville equations.* To motivate the notions which are needed to find a solution method valid for all members of this class, we shall at first introduce a new approach to solve the Liouville equation.

The Liouville equation has been widely studied, especially in the last century. It has applications in the theory of of equilibria, the problem of isothermal gas spheres, in magnetohydrodynamics, field theory and many other areas (see [3], [13]). Indeed, it enjoys many interesting properties and, in particular, its "general" solution ([11], [2]) is well known since there is a Bäcklund transformation between this equation and the wave equation ([10], [8]). There are some cases where this general solution is of little use since it cannot be adapted to boundary conditions of physical relevance ([13]). This difficulty stems from the fact that the general solution is characterized by its boundary values on the *light cone* which, for some applications, does not define the relevant initial value problem. However, there are also many cases where the Liouville equation in exactly these light cone variables must be considered, among them interaction of quasi-monochromatic waves with second harmonics in nonlinear media [1], resonant two-photon propagation [12] and the interaction of non relativistic matter with Chern-Simons fields [7].

In this paper, we consider light cone coordiantes (x, t) and a general initial value problem on a curve Γ in the (x, t)-space. This initial value problem is shown to be naturally connected to a Riccati equation whose general solution is, for special cases, obtained in explicit form from the given initial data.

The Riccati equation represents a time invariant density directly related to the Liouville equation, an observation which seems to be new. The use of this density, enables us to show that the Liouville equation subject to general Cauchy data, which are given by prescribing values on a non-characteristic curve cannot be solved explicitly. However, if the non-characteristic Cauchy problem degenerates and the initial data are prescribed on a pair of orthogonal characteristic lines, then, by use of the same conserved density, the well known general solution of the Liouville equation can be retrieved. This degenerate Cauchy problem, here termed *characteristic Cauchy problem*, coincides with an initial value problem for t when boundary values for $x = const$ are given. This analysis is the subject of section 2.

The subsequent section 3 is devoted to analyze the crucial features which allowed us to achieve the result mentioned above. In particular, this section is devoted to the investigation of other nonlinear partial differential equations where the same method can be applied. Thus, what we shall term *Riccati property*, is introduced and discussed.

In section 4 examples of nonlinear ordinary differential equations enjoying the Riccati Property are given. They are shown to be related to nonlinear evolution equations which, under our viewpoint, represent generalizations of the Liouville equation. The solution method for such generalized Liouville equations is, then, immediately obtained from the previous study.

2 A new approach to the "General solution" of the Liouville equation

In this section the Liouville equation will be briefly reconsidered. In particular, the related well known "general" solution will be retrieved. Specifically, we analyze its Cauchy problem.

Let us consider the Liouville equation in the following form

$$h_{xt} = e^{2h} \ . \tag{2.1}$$

Previous to the study of the general Cauchy problem we shall consider very special initial data, namely those $h(x, t = 0)$ such that e^h is a square integrable function at $-\infty$, but this restriction is only assumed for the moment. It is worthwhile to remark how, given such initial values, the Liouville equation can be trivially solved. Introduce $s = -1/2e^{-2h}$, then, rewriting (2.1) in terms of s, we obtain

$$s_t = sD^{-1}(s^{-1}) \tag{2.2}$$

where D^{-1} represents integration from $-\infty$ to x. This integration is well defined because of the special requirement imposed on the initial value data. It should be observed that (2.2) is a nilpotent flow of the second order

[4], namely: $s_{ttt} = 0$. In fact, such a result is readily obtained on use of differentiation and integration by parts

$$\begin{aligned} s_{tt} &= s_t D^{-1}(s^{-1}) - sD^{-1}(s_t s^{-2}) \\ &= s\left(D^{-1}(s^{-1})\right)^2 - sD^{-1}\left(s^{-1}D^{-1}s^{-1}\right) \\ &= \frac{1}{2}s\left(D^{-1}s^{-1}\right)^2 = \frac{1}{2}{s_t}^2 s^{-1} \ . \end{aligned}$$

Hence

$$s_{ttt} = 2s_t s_{tt} s^{-1} - {s_t}^3 s^{-2}$$

which, on use of the last formula, equates to zero.

This shows in particular that the Taylor series in t for $s(x,t)$ truncates after the third term and we obtain:

$$\begin{aligned} s(x,t) &= s(x,0) + s_t(x,0)t + \frac{1}{2}s_{tt}(x,0)t^2 \\ &= s(x,0) + ts(x,0)D^{-1}\left(s(x,0)^{-1}\right) + \frac{t^2}{4}s(x,0)\left(D^{-1}(s(x,0)^{-1})\right)^2 \ . \end{aligned}$$

Thus, when the initial conditions are assigned on $t = const$, by transformation of the function $s(x,t)$ into $h(x,t)$, the solution of the Liouville equation follows imediatly.

Our aim is to generalize this method to arbitrary boundary values on a given line $x = const$, or more generally, to arbitrary Cauchy data.

Non-characteristic Cauchy problem:

We wish to find $h(x,t)$ such that h satisfies (2.1), subject to the following initial conditions:

- given a curve $\Gamma = \Gamma(\sigma)$ which is parametrized by σ

$$\Gamma : \begin{cases} x &= \xi(\sigma) \\ t &= \tau(\sigma) \end{cases}$$

  where Γ is assumed to have only noncharacteristic tangents;

- assign values along Γ by

$$h|_\Gamma := f(\xi(\sigma), \tau(\sigma))$$

  for given f;

- assign, furthermore, the directional derivative of $h(x,t)$ along this curve in a direction $\eta(\sigma)$ which is supposed not to be tangential to $\Gamma(\sigma)$.

We observe that, since the characteristic lines of the Liouville equation are parallel to the x- and t-axis, respectively, the curve Γ does neither have horizontal nor vertical tangents. So, for simplicity we can choose the special parametrization $\tau(\sigma) = \sigma$

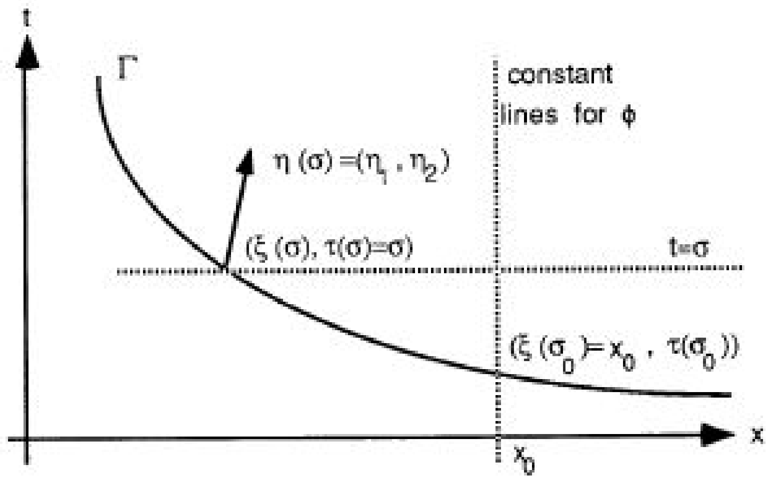

Fig. 1: Noncharacteristic curve

Lemma 2.1: *Whenever a function $h(x,t)$ fulfills the Liouville equation, then the function:*

$$\phi(x,t) := h_{xx} - h_x^2 \tag{2.3}$$

follows to be t-independent.
Proof:
Partial differentiation of (2.3) with respect to t gives:

$$\phi_t = h_{xxt} - 2h_{xt}h_x \tag{2.4}$$

thus, $h_{xt} = e^{2h}$ gives

$$h_{xxt} = 2h_x e^{2h}. \tag{2.5}$$

On substitution of h_{xt} and h_{xxt} into (2.4), it follows: $\phi_t = 0$. ■

Since the function $\phi(x) := h_{xx} - h_x^2$ is independent of t, this quantity will be addressed to as a *t-invariant density.* The existence of such a time-independent quantity implies that the non-characteristic Cauchy problem reduces to an ordinary differential problem.

To see this consider again the non-characteristic curve Γ parametrized by the variable σ. We claim that the Cauchy problem with non-characteristic data allows us to determine explicitly the time-invariant density ϕ on Γ.

As suggested by Figure 1 the quantities $\xi(\sigma)$ and $\tau(\sigma)$ denote the x and t coordinates of the curve points corresponding to the the parameter value σ. Let $\eta(\sigma)$ be the direction in which the directional derivative of h is given on Γ. The coordinates of η are denoted by η_1 and η_2. If the function h is assigned on Γ, it is also possible to evaluate the derivative of this function with respect to the parameter σ. Thus, the following two linear equations in the unknowns h_x and h_t, again, both restricted to Γ, are readily obtained:

$$h_\sigma = h_x \xi_\sigma + h_t \tau_\sigma \tag{2.6}$$

$$h_\eta = h_x \xi_\sigma + h_t \tau_\sigma \ . \tag{2.7}$$

The latter can be solved with respect to h_x and h_t if and only if the determinant of the coefficients does not vanish. Indeed, such a requirement is equivalent to say that η is never tangential to Γ. Therefore, we can compute both the x- and t-derivatives of h on Γ. Derivation, with respect to σ, of the h_x readily delivers:

$$h_{x\sigma} = h_{xx} \xi_\sigma + h_{xt} \tau_\sigma \tag{2.8}$$

which gives h_{xx} since h_{xt} is known in terms of h from the Liouville equation itself. Thus, the time-invariant density ϕ follows since, on Γ, we have computed both h_{xx} and h_x

$$\phi(\xi(\sigma), \tau(\sigma)) = h_{xx}(\xi(\sigma), \tau(\sigma)) - h_x(\xi(\sigma), \tau(\sigma))^2 \ .$$

To solve the Liouville equation, we now consider the line

$$t = \sigma \ ,$$

which is parallel to the x-axis since we have chosen σ such that $\tau(\sigma) = \sigma$ (see Fig. 1). We observe that, since ϕ is constant on the line $x = x_0$, parallel to the t-axis, the quantity $\phi(x_0, \sigma) = \left(h_{xx} - {h_x}^2\right)_{|(x_0,\sigma)}$ is obtained in (x_0, σ) by its value in the point of intersection $(\xi(\sigma_0) = x_0, \tau(\sigma_0))$ of $x = x_0$ with the curve Γ. Hence, by the initial values the function $\phi(x,t)$ is known for all t and those x being in the projection of Γ onto the x-axis. Since this function only depends on the value x we denote it by

$$J(x) := \phi(x,t)$$

and obseve that it can be computed by the given Cauchy data.

Now, in order to find h along the line $t = \sigma$ we only need to solve the ordinary differential equation

$$y_{xx} - {y_x}^2 = J(x) \tag{2.9}$$

subject to the family of initial conditions

$$y(x(\sigma)) = h(\xi(\sigma), \tau(\sigma)) \text{ and } y_x(x(\sigma)) = h_x(\xi(\sigma), \tau(\sigma)) \tag{2.10}$$

given by the values of h and h_x on the intersections of Γ with the lines $t = \sigma$. We put $h(x,\sigma) := y(x,\sigma)$, then the family of solutions of equation (2.9) allows to obtain the solution of the Liouville equation which corresponds to the given Cauchy data.

Observe that (2.9) is the Riccati equation in normal form, observe furthermore that by suitable initial data we can generate every form of inhomogenuity $J(x)$ in (2.9). Now, because the general Riccati equation cannot be solved by quadratures [9], the following result has been obtained:

Observation 2.2: *In the case of the Liouville equation the solution of the general non-characteristic Cauchy problem cannot be obtained by quadratures.*

This not only means that our method fails for the solution of the general Cauchy problem of the Liouville equation but also that every other method will fail as well. However, if we allow the introduction of new transcendents defined by the Riccati equation itself, then a recipe to solve the general Liouville non-characteristic Cauchy problem follows from our considerations

RECIPE 1:

1) Compute, on use of the Cauchy data, the function h_x as well as

$$\phi = h_{xx} - {h_x}^2$$

on the curve Γ. Then, determine the function ϕ in the whole plane by using the fact that it is constant along the lines $x = const$. Let $J(x) = \phi(x,t)$ for arbitrary values of t.

2) Solve, for arbitrary σ, the initial value problem

$$y_{xx} - {y_x}^2 = J(x) \tag{2.11}$$

for

$$y(x(\sigma)) = h(\xi(\sigma), \tau(\sigma)) \text{ and } y_x(x(\sigma)) = h_x(\xi(\sigma), \tau(\sigma)) \ .$$

3) Then,

$$h(x, \sigma) := y(x, \sigma)$$

is the solution of the Liouville equation corresponding to the prescribed Cauchy data.

The nonlinear Riccati equation has the remarkable property that it can be transformed into a second order *linear* problem. Therefore, via such a transformation, it follows that, whenever one solution of the Riccati equation is known, then the general solution can be obtained by quadratures.

Thus, we find that the method we presented above delivers the explicit solution of the following:
Characteristic Cauchy problem: The values of h are given on two orthogonal characteristic lines $x = x_0$ and $t = t_0$.

To see that in this case we can find the explicit solution, we fix the notation and define the following functions:

$$f(x) := h(x, t_0) \quad , \quad g(t) := h(x_0, t) \ .$$

We define furthermore a function $G(t)$ as solution of the following first order problem

$$G_t = e^{2g}, \ G(t = t_0) = f_x(x_0) \ .$$

Then obviously

$$G(t) = h_x(x_0, t) \ .$$

The time invariant density ϕ is readily obtained via differentiation along the line $t = t_0$

$$\phi = f_{xx} - {f_x}^2 \ . \tag{2.12}$$

The same arguments as before imply that the solutions of

$$y_{xx} - {y_x}^2 = \phi \ \ where \ \ \phi = f_{xx} - {f_x}^2 \tag{2.13}$$

for

$$y(x_0, t) = g(t) \text{ and } y_x(x_0, t) = G(t) \tag{2.14}$$

give the solution $h(x,t) := y(x,t)$ of the Liouville equation which satisfies the prescribed data. Again, this is a Riccati equation, however, it can be solved since we already know one particular solution $\tilde{y}$, namely

$$\tilde{y} = f \ .$$

To find the general solution of (2.13) we use the following transformation of the dependent variable:

$$\frac{u_x}{u} = -y_x$$

which, by use of $\phi = y_{xx} - y_x^2$, delivers the equation

$$u_{xx} + \phi(x)u = 0 \ . \qquad (2.15)$$

A special solution of this equation is

$$u_0 = ce^{-f}$$

and, therefore, the subsequent application of the method of variation of constants delivers the general solution:

$$u = c_1 e^{-f} + c_2 e^{-f} D_{x_0}^{-1} e^{2f} \qquad (2.16)$$

where $D_{x_0}^{-1}$ denotes integration from x_0 to x. Hence, by definition of u, the general solution of (2.13) is

$$y = -ln\left(c_1 e^{-f} + c_2 e^{-f} D^{-1} e^{2f}\right) \ . \qquad (2.17)$$

Observe that c_1 and c_2 are only constants with respect to x, not with respect to t. Insertion of the initial data leads to the determination of these quantities and we obtain

$$y(x,t) = -ln\left(e^{\alpha - g(t) - f(x)} - e^{-2\alpha}(\partial_{t_0}^{-1} e^{2g})(D_{x_0}^{-1} e^{2f})\right)$$

where $\alpha = g(t_0) = f(x_0)$ and where $\partial_{t_0}^{-1}$ denotes t-integration from t_0 to t. Thus, the characteristic Cauchy problem is completely solved.

3 The Riccati Property

In this section the key facts which allowed to establish the result for the Liouville equation are reconsidered. Indeed, they indicate a novel and interesting connection between nonlinear partial differential equations and ordinary differential equations. It will be shown how the essential features of the procedure outlined in the preceding section can be generalized in order to obtain analogous results for other nonlinear evolution equations.

The major ingredients of our method were that we had:

1) a t-invariant density for the equation under consideration

2) such that this density equated to some given function, reduces the nonlinear partial differential equation to a Riccati ordinary differential equation.

The crucial property of the Riccati equation is that its general solution can be determined explicitly when only one particular solution is known ([6], [9]). Thus, we introduce the following generalization

Definition 3.1: *An ordinary differential equation is said to possess the* **Riccati property** *if, given one particular solution, it is possible to find its general solution explicitly by quadratures.*

Trivial examples of differential equations which enjoy the Riccati property are all linear second order ordinary differential equations. Indeed, the *variation of constants method* implies that, known one particular solution, the general one can be obtained in explicit form.

Here some further examples of ordinary differential equations which enjoy the same property:

Example 3.2:
We claim that the following linear equation in the dependent variable s

$$s_{xxx} + 4us_x + 2u_x s = 0 \tag{3.1}$$

enjoys the Riccati property. Here u is some arbitrarily given function in x. To prove our claim, let us consider (3.1) as an ordinary differential equation for the function u. Solving this with respect to u, gives:

$$u \quad = \quad \frac{1}{4s^2}({s_x}^2 - 2ss_{xx}) + \frac{c_1}{s^2} \tag{3.2}$$

where c_1 is a constant of integration. We choose the value of c_1 such that u is expressed in terms of s as a *homogeneous* function, namely, it must be invariant under the substitution $s \to \alpha s$, $\alpha \in \mathbb{R}$. In this case $c_1 = 0$ and (3.2) reads:

$$u = \frac{1}{4s^2}({s_x}^2 - 2ss_{xx}) \tag{3.3}$$

Observe that this can be considered as a second order equation for s, given u. Now, we look for a one-parameter group which transforms the s-variable in such a way that the corresponding u-variable is unchanged. Let us denote by τ such a group parameter and let $v = s_\tau$ be the infinitesimal generator of that group. Then, obviously, v is subject to satisfy a linear differential equation obtained by derivation of (3.3) with respect to τ

$$(ss_{xx} - {s_x}^2)v - s^2 v_{xx} + ss_x v_x = 0 \ . \tag{3.4}$$

The homogenuity of (3.3) implies that the special one-parameter group $s \to \tau s$ trivially leaves u unchanged. The infinitesimal symmetry generator of this special group is s itself, thus, $v = s$ is a particular solution of (3.4), which, interpreted as a differential equation in the unknown v, enjoys the Riccati property and therefore allows to find explicitly its general solution. The second solution of (3.4) obtained by variation of constants is

$$v = \frac{1}{2} s \left(D_{x_0}^{-1}(s^{-1}) \right) \tag{3.5}$$

where again

$$D_{x_0}^{-1} = \int_{x_0}^{x} \cdots dx \ .$$

Now, since any solution s of (3.3) is a solution of (3.2), the group $s \to s(\tau)$, for which the generator is determined, leaves also (3.2) invariant. Furthermore, the linearity of (3.1) implies that the related solution space is a linear space; consequently, any infinitesimal generator of a group acting in its solution space is itself a vector which belongs to the same space of solutions. Therefore, whenever s in (3.4) is a solution of (3.1) then any solution v of (3.4), satisfies (3.1) as well. Hence, (3.5) must be another solution of (3.1). Finally, given two independent solutions of a third order equation, a third solution, independent from the previous ones, can be obtained by the method of variation of constants. On use of this method, it follows that if s is a solution of (3.1), then both:

$$sD_{x_0}^{-1}(s^{-1}) \ \ and \ \ \frac{1}{2} s \left(D_{x_0}^{-1}(s^{-1}) \right)^2 \tag{3.6}$$

are again solutions of (3.1). Equation (3.1), thus, is proved to enjoy the Riccati property. □

The analysis of the last example yields a method to construct third order linear equations which enjoy the Riccati property.

Method:

1) Consider a third order linear equation in the unknown function s such that:

   - the coefficients depend on a given function u;
   - if the equation is considered as an equation for the given function u, instead of s, then it is possible to determine explicitly its solution u as a function of s and its derivatives *up to second order*, say
   $$u = F(c_1, \ldots, c_n, s, s_x, s_{xx})$$
   where the $c_1, \ldots, c_n$ are suitable constants of integration.

2) In the representation $u = F(c_1, \ldots, c_n, s, s_x, s_{xx})$, choose the integration constants such that F is invariant under the substitution $s \to \alpha s, \alpha \in \mathbb{R}$. Such a choice is possible since the original equation was linear in s and, hence, invariant under this substitution.

3) Then evaluate the variational derivative of F
$$F'(s)[v] = \frac{\partial}{\partial \epsilon} F(s + \epsilon v, s_x + \epsilon v_x, s_{xx} + \epsilon v_{xx})_{|\epsilon=0} \tag{3.7}$$
and consider $F'(s)[v] = 0$ as a second ordinary differential equation in the unknown function v. The homogenuity immediatly implies that one solution is given by $v = s$ and a second independent solution
$$v = G(s) \tag{3.8}$$
can be computed by application of the variation of constants method. This $G(s)$ represents also a solution of the original linear problem.

4) Compute, now, the third solution of the original linear problem via the known two solutions by further application of the variation of constants method.

Example 3.3:
The method, we just presented, can be applied to

$$u_x(s_{xx} - \lambda s) - u(s_{xxx} - \lambda s_x) - 4u^3 s_x = 0 \tag{3.9}$$

which implies

$$2u \;=\; \frac{\lambda\ s - \ s_{xx}}{\sqrt{\lambda\ s^2 - \ {s_x}^2 + c_1}}$$

where a suitable choice of c_1 makes $u(s)$ homogeneous in s. This results in

$$2u \;=\; \frac{\lambda\ s - \ s_{xx}}{\sqrt{\lambda\ s^2 - \ {s_x}^2}}\ . \tag{3.10}$$

Carrying out the described steps, it follows that if s is a solution then

$$sD_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\ s^2 - \ {s_x}^2}\right) \tag{3.11}$$

is again such a solution. The third independent solution can be obtained when we consider (3.11) as an infinitesimal generator of a one-parameter group acting on the space of solutions of (3.11) itself. Namely, on use of the group

$$s_\mathcal{T} = sD_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\ s^2 - \ {s_x}^2}\right) \tag{3.12}$$

we find by linearity of the solution space, that also $s_{\mathcal{T}\mathcal{T}}$ must be a solution of (3.9). Explicit computation delivers:

$$s_{\mathcal{T}\mathcal{T}} = \frac{1}{2}s\left\{D_{x_0}^{-1}\left(s^{-2}\sqrt{\lambda\ s^2 - \ {s_x}^2}\right)\right\}^2 - s\left(D_{x_0}^{-1}(s_x s^{-3})\right)\ . \tag{3.13}$$

Having now two solutions of (3.9), namely the right hand sides of (3.12) and (3.13), we find the third solution by variation of constants. □

4 Generalized Liouville equations

We will show that equations with the Riccati property naturally lead to a class of equations sharing with the Liouville equation the property that the related initial value problem can be solved explicitly. In particular, those third order problems enjoying the Riccati property which were considered in the last section, will be shown to be connected to these *generalized Liouville* equations.

A key role in reducing the Cauchy problem for a partial differential equation to the solution of an ordinary differential equation was played by the existence of an invariant density which depended explicitly only on one of the two independent variables. Then, to obtain the solution of the initial boundary value problem it was crucial that the reduced differential equation, i.e. the equation obtained by equating the conserved density to a given function, was of a special type, namely of Riccati type as defined in 3.1. We define:

Definition 4.1: *A differential equation in two independent variables is said to be a* **generalized Liouville** *equation if it admits a conserved density wherein only derivatives with respect to one independent variable occur, and equating this conserved density to an arbitrary given function must produce an ordinary differential equation which enjoys the Riccati property.*

In analogy to the procedure in Section 2 the initial-boundary value problem related to such an equation is explicitly solved in the following way

RECIPE 2:
Initial-boundary value problems for generalized Liouville equations:
Consider a partial differential equation for $s(x,t)$ which is of Liouville type. Assume that $H(s)$ is a conserved density, where only derivatives in x appear such that the highest order of these derivatives is N. For fixed t_0, let $s(x,t_0)$ be given and for x_0, and let the quantities $s(x_0,t), s_x(x_0,t), s_{xx}(x_0,t), ..$ be given up to order $(N-1)$ such that these data are compatible at (x_0,t_0) with the partial differential equation. Then, the solution of that initial-boundary value problem is obtained by following the subsequent steps:

- Compute, on use of the given data $s(x,t_0)$, the quantity $H(s)$ in the whole plane and denote the result by ϕ. Recall that the computation relies on the fact that $H(s)$ does not change along the lines $t = const$.
- For fixed but arbitrary t, consider the Riccati equation $H(y) = \phi$. One solution of this equation is known, namely $\tilde{y} = s(x,t_0)$. Thus, by the Riccati property the general solution can be obtained. Choose in this general solution the integration constants (i.e. constants with respect to x, not with respect to t) with such a dependence of t that the resulting function $y(x,t)$ attains the data given on the line $x = x_0$
- Then $s(x,t) := y(x,t)$ solves the given initial-boundary value of the original equation.

Although this seems to be a straight-forward method for solving initial-boundary value problems, we have to show that there are really other equations of that type apart from the ordinary Liouville equation. Therefore we exhibit a general method for generating generalized Liouville equations.

We assume that a linear differential equation, in the function s, enjoys the Riccati property. Furthermore, we assume that this linear differential equation depends on a given function u such that, whenever considered as a differential equation in the unknown function u instead of s, it can be completely solved by quadratures. Consider this general solution for u to be

$$u = F(c_1, \ldots, c_n, s, s_x, \ldots) \tag{4.1}$$

where $c_1, \ldots, c_n$ are integration constants (i.e. constants with respect to x).

Now, consider $G_0(s) = s, G_1(s), \ \ldots, \ G_n(s)$ to be a basis for the solution space of the linear problem. Such a basis, which can be expressed in terms of s alone, exists since the linear problem in the unknown function s has the Riccati property and because we already know one solution, namely $G_0(s) = s$. Let $G(s)$ be a linear combination in the $G_k(s)$ then we have the surprising:

Main Theorem:
For the equation

$$s_t = G(s) \tag{4.2}$$

there is a choice $c_1(t), \ldots, c_n(t)$ *in (4.1) such that*

$$H(s) = F(c_1(t), \ldots, c_n(t), s, s_x, \ldots) \tag{4.3}$$

is a conserved density. Furthermore,

$$H(s) = R(x), \tag{4.4}$$

enjoys for arbitrarily given $R(x)$ *the Riccati property.*

Proof: Observe that, for arbitrary but fixed $u(x)$, and variable set of constants $c_1, \ldots, c_n$, then equation (4.1) describes the same solution space as the one spanned by the solutions of the linear problem in s. Since (4.2) is the infinitesimal form of a group under which that solution space in invariant, it follows that there is a suitable choice of $c_1, \ldots, c_n$ (which depends on the value of the group parameter t) such that u is unchanged. Choosing these $c_1(t), \cdots, c_n(t)$ and inserting them into (4.1) obviously leads to a conserved density, since u, by definition, does not change. Observe that the original equation had the Riccati property, and had the same solution space as (4.1) (admitting variable coefficients). Therefore equation (4.4) has the Riccati property as well; it is only needed to pick among all solutions those which lead to the $c_1(t), \cdots, c_2(t)$. ■

Remark 4.2: *Thus, equation (4.2) is a generalized Liouville equation and, correspondingly, the related initial-boundary value problem can be solved by quadratures for given $s(x, t_0)$ and suitable data on (x_0, t).*

Example 4.3:
Let s be a solution of (3.1). Cconsider the second solution

$$G(s) = \frac{1}{2} s {D_{x_0}}^{-1} s^{-1}$$

of (3.1) given by (3.6). Rewriting

$$s_t = G(s)$$

as a differential equation, it follows

$$\left(\frac{2 s_t}{s} \right)_x = \frac{1}{s}$$

or

$$2 s_{tx} s - 2 s_t s_x = s \ . \tag{4.5}$$

This equation represents a generalized Liouville equation. The related conserved density is obtained by assuming t-dependent $c_1(t)$ and demanding that u in (3.2) is independent of t. In this case, it results $c_1 = 0$, and, hence

$$\phi = \frac{{s_x}^2}{s^2} - 2 \frac{s_{xx}}{s} \tag{4.6}$$

is a conserved density under the flow (4.5). The latter, considered as an equation in the unknown function s, certainly enjoys the Riccati property since, indeed, it is a Riccati equation.

Hence, the complete solution of (4.5), corresponding to assigned boundary values $s(x, t_0)$ and initial values $s(x_0, t)$ are obtained by Recipe 2. □

Example 4.4:

Consider the further solution

$$G(s) = s\left(D_{x_0}^{-1}(s^{-1})\right)^2$$

of (3.1) given by (3.6). Then

$$s_t = G(s)$$

can be rewritten in the form

$$(s_{tx}s - s_t s_x)^2 = 4 s_t s \ . \tag{4.7}$$

This equation is a generalized Liouville equation. The related conserved density is obtained by time-dependent $c_1(t)$ and demanding that u in (3.2) is independent on t. In this case, this results in the following invariant density

$$\phi = \frac{s_{xx}}{s} - \frac{{s_x}^2}{2s^2} - \frac{3t}{s^2} \ . \tag{4.8}$$

The latter considered as an equation in the unknown function s, again, is a Riccati equation.

Hence, the complete solution of (4.7), corresponding to assigned boundary values $s(x, t_0)$ and initial values $s(x_0, t)$ are, once more, obtained by Recipe 2. □

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