

Unified Approach to Action-Angle Representation of Real and Complex Multisolitons

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Using a method of complexification a unified approach to action/angle representation of complex and real multisolitons is given. Within a purely algebraic framework the action/angle variables, the interacting solitons as well as the eigenstates of the recursion operator are explicitly expressed in terms of the physical field variable. Several plots of these quantities are given.

Introduction

For real-analytic integrable equations in $(1 + 1)$ -dimensions, which were considered as flows on infinite dimensional manifolds, we constructed in [5],[6],[10] the action/angle variables for multisoliton solutions. Using only the algebraic properties of symmetries and mastersymmetries for these equations, it was possible to give explicit formulas for the action/angle variables, and their corresponding hamiltonian vector fields in terms of the physical field variable. This result supplemented the well known representation of action/angle variables in terms of scattering data obtained by the Inverse Scattering Method.

In this letter we will examine the structure of those complex equations, which, like the Nonlinear Schrödinger Equation (NLS), involve complex conjugation in their tangent fields. Complex conjugation is not an analytic operation, hence these equations are not in the class of equations considered in [5],[6],[10]. Since these equations necessarily involve complex quantities we term them **complex equations**. In most cases, these complex equations can be considered as reductions of complex-analytic systems. Therefore in chapter 1 we will briefly present results for integrable evolution equations on complex manifolds analogous to the ones given in [10]. In chapter 2 the notion of **extended systems** as a special case of complex-analytic equations

on complex manifolds is introduced. For equations like the NLS we propose in this paper a method of complexification, which embeds these and similar equations in a unified theory of *analytic hereditary vector fields* considered in [5]. Let us point out where the value of this method is to be found.

If one compares the qualitative behaviour of soliton solutions of equations like the Korteweg-deVries equation (KdV) and the NLS one discovers considerable differences. In the KdV-case multisolitons asymptotically are waves of translations, only determined by their asymptotic speeds, whereas for NLS the carrier waves, i.e. the waves of translation, occur in superposition with some oscillatory wave. So in this case two parameters characterize an asymptotically appearing soliton.

However, looking at the detailed example in chapter 4, we observe that our method of complexification yields a unified theory which allows to see these two, seemingly, different behaviours under a common viewpoint. This is true insofar, as it turns out that the oscillatory behaviour is related to the imaginary part of some complex asymptotic speed of a multisoliton solution of an equation which is obtained by complexification. This means in addition, that the seemingly more complicated nature of formulas for multisolitons in the NLS-case is only a result of the fact that one has expressed everything in real quantities. If expressed in complex-analytic quantities no difference in the structure of formulas for KdV and NLS can be observed.

In order to allow a comparison we therefore present in chapter 3 in all detail, how one passes from complex-analytic formulas to representation by real quantities. Since in this process derivatives occur, one has to use the symmetry relations represented by the Cauchy-Riemann equations for complex derivatives. The observation, that the structure of the system becomes less symmetric if represented only by real-valued parameters is also seen from the fact that quantities, like the recursion operator, which are diagonal if suitable complex-analytic parameters are used, suddenly become off-diagonal representations if only real quantities are used for parametrization.

Another advantage of complexification is that we obtain in an extremely simple way the densities of action/angle variables, or the eigenstates of the recursion operator, by taking derivatives with respect to asymptotic speeds and corresponding oscillating parameters. The essential quantity, whose derivatives give the action/angle representation, turns out to be the fundamental hamiltonian field given by the scaling symmetry.

1 Notation and Basic Facts

Let $S_{\mathbb{C}}(\mathbb{R})$ be the space of complex-valued functions on the real line such that all derivatives vanish rapidly at $\pm\infty$. For elements $u(\cdot, t), v(\cdot, t)$ of $S_{\mathbb{C}}(\mathbb{R})$ we consider the following system of evolution equations

$$\vec{u}_t = \vec{K}_1(\vec{u}) =: \begin{pmatrix} K_1(\vec{u}) \\ G_1(\vec{u}) \end{pmatrix} . \quad (1.1)$$

Here \vec{u} is an abbreviation of $\begin{pmatrix} u \\ v \end{pmatrix}$. We restrict ourselves to vector fields $\vec{K}_1(\vec{u})$, which depend analytically on u and v and their derivatives w.r.t. the independent variable x . These systems we term **complex-analytic systems**. Furthermore we assume that $\vec{K}_1(\vec{u})$ does not explicitly depend on the variable x . So, in particular, these systems are translation invariant systems. Analogous to [5],[6],[10] we assume that the following conditions hold:

- (C1) The system (1.1) admits a localized hereditary recursion operator $\Phi(\vec{u})$ with an implectic/symplectic factorization ([3]) or equivalent, a compatible pair of hamiltonian (or Poisson) operators ([7],[9])

$$\Phi(\vec{u}) = \Theta_1(\vec{u}) \Theta_0^{-1}(\vec{u}) .$$

- (C2) In addition to the hierarchy of commuting symmetries

$$\vec{K}_n(\vec{u}) := \Phi^n(\vec{u}) \vec{K}_0(\vec{u}) = \Phi^n(\vec{u}) \vec{u}_x$$

the operator $\Phi(\vec{u})$ generates a hierarchy of mastersymmetries ([4])

$$\vec{\tau}_n(\vec{u}) = \Phi^n(\vec{u}) \vec{\tau}_0(\vec{u})$$

which fulfill the commutator relations

$$\begin{aligned} [\vec{K}_n, \vec{K}_m] &= 0, \quad [\vec{\tau}_n, \vec{K}_m] = (m + \alpha) \vec{K}_{n+m}, \\ [\vec{\tau}_n, \vec{\tau}_m] &= (m - n) \vec{\tau}_{n+m}. \end{aligned} \quad (1.2)$$

Here $[\cdot, \cdot]$ denotes the usual commutator between vector fields. The fields \vec{K}_n are assumed to be hamiltonian vector fields w.r.t. Θ_0 and Θ_1 . The constant α is called **scaling degree**, in those examples we consider explicitly, we will have $\alpha = 1$.

(C3) Those \vec{u} fulfilling

$$\sum_{n=0}^N \beta_n(\vec{u}) \vec{K}_n(\vec{u}) = 0 \quad \text{with} \quad \beta_N = 1, \quad (1.3)$$

where the polynomial $P(z) = \sum_{n=0}^N \beta_n z^n$ has only simple zeroes c_1, \dots, c_N , form the manifold of non-degenerated N -soliton solutions of (1.1). This manifold is denoted by M_N ; its dimension as a complex manifold is $2N$ ([5]).

(C4) On M_N the solutions \vec{u}_N decompose for large time t into single soliton solutions of the form

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} \cong \begin{pmatrix} \sum_{k=1}^N s_k(c_k, c_k^\alpha(x + c_k t + q_k)) \\ \sum_{k=1}^N r_k(c_k, c_k^\alpha(x + c_k t + q_k)) \end{pmatrix} \quad (1.4)$$

with velocities $c_k \neq c_l$ and phases $q_k \neq q_l$ for $k \neq l$. In general velocities and phases can be complex quantities.

Of course, the conditions (C1) - (C4) are not independent from each other; for connections between these properties of integrable equations see for example [9],[11].

Generalizing now the purely algebraic construction in [5] we can restrict the system (1.1) to the finite dimensional manifold M_N of all non-degenerated N -soliton solutions \vec{u}_N . In order to get a better understanding of the algebraic structure of the multisoliton manifolds we now consider a reparametrization of these manifolds by asymptotic data c_1, \dots, c_N and q_1, \dots, q_N . These new parameters are found in the following way: We take an arbitrary point \vec{u}_N of the manifold as initial condition for (1.1). Then we take the evolution determined by this initial condition and **measure** the asymptotic speeds and phases of that evolution. Obviously, these parameters characterize the initial condition completely. Hence, they can be taken as coordinate values for the initial condition. Now we can use the well known formulas for transformation of vector fields in order to rewrite the original flow in terms of these new parameters. As a basis of the tangent space we choose

$$\frac{\partial \vec{u}_N}{\partial q_k} \quad \text{and} \quad \frac{\partial \vec{u}_N}{\partial c_k}, \quad k = 1, \dots, N$$

and identify these $2N$ vectors with the standard basis in \mathbb{C}^{2N} .

Observe that due to the complex structure of our initial problem $\frac{\partial \vec{u}_N}{\partial q_k}$ means the complex derivative of \vec{u}_N w.r.t. the complex variable q_k , and

due to the analytic structure of (1.1) these derivatives fulfill the Cauchy-Riemann differential equations.

With the same arguments as in [5],[6],[10] we obtain the following results.

Lemma 1: In terms of the parametrization given by asymptotic data

(a) the restricted evolution on M_N reads

$$\frac{\partial}{\partial t} \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \quad (1.5)$$

(b) the restricted operators Φ , Θ_0 and Θ_1 are of the form

$$\Theta_0 := \begin{pmatrix} \mathcal{O} & \Lambda_{(-\alpha)} \\ -\Lambda_{(-\alpha)} & \mathcal{O} \end{pmatrix}, \quad \Theta_1 := \begin{pmatrix} \mathcal{O} & \Lambda_{(1-\alpha)} \\ -\Lambda_{(1-\alpha)} & \mathcal{O} \end{pmatrix}, \quad (1.6)$$

and

$$\Phi = \Theta_1 \Theta_0^{-1} = \begin{pmatrix} \Lambda_1 & \mathcal{O} \\ \mathcal{O} & \Lambda_1 \end{pmatrix}, \quad (1.7)$$

where α is the scaling degree and Λ_p denotes the diagonal $N \times N$ -matrix

$$\Lambda_p = \begin{pmatrix} c_1^p & 0 & \cdots & 0 \\ 0 & c_2^p & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & c_N^p \end{pmatrix}. \quad (1.8)$$

\mathcal{O} denotes the corresponding zero-matrix.

(c) The symmetries \vec{K}_n and the scaling mastersymmetry $\vec{\tau}_0$ are found to be

$$\begin{aligned} \vec{K}_n &= (c_1^n, c_2^n, \dots, c_N^n, 0, \dots, 0)^T = \Theta_1 \text{grad} \left(\frac{1}{n+\alpha} \sum_{k=1}^N c_k^{n+\alpha} \right) \quad (1.9) \\ &= \Theta_0 \text{grad} \left(\frac{1}{n+1+\alpha} \sum_{k=1}^N c_k^{n+1+\alpha} \right), \end{aligned}$$

$$\vec{\tau}_0 = (-\alpha q_1, \dots, -\alpha q_N, c_1, \dots, c_N)^T = \Theta_1 \text{grad} \left(- \sum_{k=1}^N c_k^\alpha q_k \right) . \quad (1.10)$$

Here T denotes the transposed vector.

(d) For $k = 1, \dots, N$ the vectors

$$\Theta_1 \text{grad} \left(\frac{1}{\alpha} c_k^\alpha \right) \quad \text{and} \quad \Theta_1 \text{grad} (-q_k) \quad (1.11)$$

are eigenstates of the recursion operator for the discrete eigenvalue c_k .

(e) W.r.t. the Poisson bracket defined by Θ_1 the coordinates c_k^α, q_l fulfill for all $k, l = 1, \dots, N$ the following relations

$$\{c_k^\alpha, q_l\}_{\Theta_1} := \langle \text{grad } q_l, \Theta_1 \text{grad } c_k^\alpha \rangle = \alpha \delta_{kl} , \quad (1.12)$$

$$\{c_k^\alpha, c_l^\alpha\}_{\Theta_1} = \{q_k, q_l\}_{\Theta_1} = 0 . \quad (1.13)$$

Hence, $\frac{1}{\alpha} c_k^\alpha, q_l$ are the canonical coordinates corresponding to Θ_1 . They are called canonical action/angle variables.

It has already been stated ([12]) that for all known integrable equations the scaling mastersymmetry $\vec{\tau}_0(\vec{u})$ has a unique hamiltonian formulation

$$\vec{\tau}_0(\vec{u}) = \Theta_1(\vec{u}) \text{grad } F(\vec{u}) ,$$

where $F(\vec{u})$ is some scalar field. If this fact is fulfilled as additional assumption, then, with the same arguments as in [10], and with the help of lemma 1 one can prove the following representation of action/angle variables in terms of the physical variable \vec{u} :

Theorem 1:

(a) For a N -soliton solution \vec{u}_N of (1.1) the canonical action/angle variables w.r.t. $\Theta_1(\vec{u}_N)$ are given by the partial derivatives

$$- \frac{\partial F(\vec{u}_N)}{\partial q_k} \quad \text{and} \quad - \frac{\partial F(\vec{u}_N)}{\partial (c_k^\alpha)} . \quad (1.14)$$

(b) The corresponding hamiltonian vector fields are determined by

$$\alpha \frac{\partial \vec{u}_N}{\partial q_k} = \Theta_1(\vec{u}_N) \text{grad} \left(- \frac{\partial F(\vec{u}_N)}{\partial q_k} \right) , \quad (1.15)$$

$$\alpha \frac{\partial \vec{u}_N}{\partial (c_k^\alpha)} = \Theta_1(\vec{u}_N) \text{grad} \left(\frac{\partial F(\vec{u}_N)}{\partial (c_k^\alpha)} \right) . \quad (1.16)$$

These quantities are eigenstates of the recursion operator $\Phi(\vec{u}_N)$ w.r.t. the discrete eigenvalue c_k .

As we will see in examples there is another suitable and important parametrization of N -soliton solutions which is related to the above via the change of coordinates

$$\begin{aligned} q_k &\rightarrow \hat{q}_k := c_k^\alpha q_k \\ c_k &\rightarrow \hat{c}_k := c_k . \end{aligned} \quad (1.17)$$

With respect to these coordinates the N -soliton solutions (1.4) are given in the following form

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} \cong \begin{pmatrix} \sum_{k=1}^N s_k(\hat{c}_k, \hat{c}_k^\alpha x + \hat{c}_k^{\alpha+1} t + \hat{q}_k) \\ \sum_{k=1}^N r_k(\hat{c}_k, \hat{c}_k^\alpha x + \hat{c}_k^{\alpha+1} t + \hat{q}_k) \end{pmatrix} . \quad (1.18)$$

Since for all known complex examples the scaling degree is equal 1, we explicitly give the transformed quantities only for $\alpha = 1$.

Lemma 2: In terms of the parametrization given by $\hat{q}_k := c_k q_k$ and $\hat{c}_k := c_k$

(a) the restricted evolution on M_N reads

$$\frac{\partial}{\partial t} (\hat{q}_1, \dots, \hat{q}_N, \hat{c}_1, \dots, \hat{c}_N)^\top = (\hat{c}_1^2, \dots, \hat{c}_N^2, 0, \dots, 0)^\top , \quad (1.19)$$

(b) the restricted operators $\hat{\Phi}$, $\hat{\Theta}_0$ and $\hat{\Theta}_1$ are of the form

$$\hat{\Theta}_0 := \begin{pmatrix} \mathcal{O} & \hat{\Lambda}_0 \\ -\hat{\Lambda}_0 & \mathcal{O} \end{pmatrix} , \quad \hat{\Theta}_1 := \begin{pmatrix} \mathcal{O} & \hat{\Lambda}_1 \\ -\hat{\Lambda}_1 & \mathcal{O} \end{pmatrix} , \quad (1.20)$$

and

$$\hat{\Phi} = \hat{\Theta}_1 \hat{\Theta}_0^{-1} = \begin{pmatrix} \hat{\Lambda}_1 & \mathcal{O} \\ \mathcal{O} & \hat{\Lambda}_1 \end{pmatrix}, \quad (1.21)$$

where Λ_p denotes the diagonal $N \times N$ -matrix

$$\hat{\Lambda}_p = \begin{pmatrix} \hat{c}_1^p & 0 & \cdots & 0 \\ 0 & \hat{c}_2^p & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \hat{c}_N^p \end{pmatrix}. \quad (1.22)$$

- (c) The symmetries \hat{K}_n and the scaling mastersymmetries $\hat{\tau}_0$ are found to be

$$\begin{aligned} \hat{K}_n &= (\hat{c}_1^{n+1}, \dots, \hat{c}_N^{n+1}, 0, \dots, 0)^T = \hat{\Theta}_1 \text{grad} \left(\frac{1}{n+1} \sum_{k=1}^N \hat{c}_k^{n+1} \right) \\ &= \hat{\Theta}_0 \text{grad} \left(\frac{1}{n+2} \sum_{k=1}^N \hat{c}_k^{n+2} \right) \end{aligned} \quad (1.23)$$

$$\hat{\tau}_0 = (0, \dots, 0, \hat{c}_1, \dots, \hat{c}_N)^T = \hat{\Theta}_1 \text{grad} \left(- \sum_{k=1}^N \hat{q}_k \right). \quad (1.24)$$

Remark: By transformation laws it is obvious that \hat{q}_l and \hat{c}_k are again canonical coordinates but w.r.t. $\hat{\Theta}_0$. However, in this new parametrization it will be more difficult to express the corresponding quantities in terms of the physical variable \vec{u} . Hence, the parametrization (1.4) of N -soliton solutions by asymptotic data seems to be the most natural one.

2 Extendable Systems

Although we have introduced general complex systems (1.1) we are mostly interested in systems for which the restriction $v = \bar{u}$, where the bar means complex conjugation, is compatible with the time evolution. In other words our interest is focused on complex equations which can be extended to a complex-analytic system (1.1) by setting $v = \bar{u}$ and then treating v as independent of u . The NLS, for example can be considered in that way. We term

these equations **extendable systems** and their complex-analytic counterparts are called **extended systems**. Obviously in the case of an extended system it has to hold

$$G_1(\vec{u}) = \overline{K_1(\vec{u})} . \quad (2.1)$$

Since equations which possess soliton solutions with real asymptotic data were already considered in [5],[10], we have to single out those equations which were not yet covered by the theory presented in these papers. We observe therefore that, whenever $K(u)$ is a real-analytic vector field (i.e. $K(\bar{u}) = \overline{K(u)}$), then the analytic equation $u_t = K(u)$ itself can be understood as a reduction given by $v = \bar{u}$. The corresponding extended system then is the trivial system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} K(u) \\ K(v) \end{pmatrix} . \quad (2.2)$$

So, in order to consider only cases which are not yet covered in [5],[10] we assume in addition that for the extended system we have for all \vec{u}

$$\begin{pmatrix} G_1(\vec{u}) \\ K_1(\vec{u}) \end{pmatrix} \neq \begin{pmatrix} K_1(\vec{u}^*) \\ G_1(\vec{u}^*) \end{pmatrix} , \quad (2.3)$$

where

$$\vec{u}^* = \begin{pmatrix} v \\ u \end{pmatrix} \text{ when } \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} .$$

This condition clearly is not fulfilled for the extension (2.2) of a real-analytic system. In particular, assumption (2.3) implies that for the family of soliton solutions of (1.1) the complex parameters c_k and q_k do not take real values.

We are now going to deduce the structure of extendable systems from the results in chapter 1.

Lemma 3: Consider a complex system

$$u_t = \Gamma(u, \bar{u}) , \quad (2.4)$$

which is extendable, i.e. it can be considered as a reduction $v = \bar{u}$ of a complex-analytic system

$$\vec{u}_t = \vec{K}_1(\vec{u}) , \quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} . \quad (2.5)$$

If the extended system (2.5) fulfills the assumptions (C1)-(C4) then for (2.4) the discrete eigenvalues c_1, \dots, c_N of the recursion operator appear only in complex conjugated pairs and none of these eigenvalues lies on the real line.

Proof: From the first line of

$$0 = \sum_{n=0}^N \beta_n \vec{K}_n = \begin{pmatrix} \sum_{n=0}^N \beta_n K_n \\ \sum_{n=0}^N \beta_n G_n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^N \beta_n \Phi^n K_0 \\ \sum_{n=0}^N \beta_n \Phi^n G_0 \end{pmatrix}$$

it follows that the eigenvalues c_1, \dots, c_N of Φ are the roots of the polynomial (see also (C3))

$$P(z) = \sum_{n=0}^N \beta_n z^n.$$

Since for an extendable system equation (2.1) holds, we also have

$$0 = \sum_{n=0}^N \beta_n G_n = \sum_{n=0}^N \beta_n \bar{K}_n.$$

Hence, c_1, \dots, c_N are roots of $P(\bar{z})$ as well, i.e.

$$\{c_1, \dots, c_N\} = \{\bar{c}_1, \dots, \bar{c}_N\}.$$

Now the proof is already complete, because real velocities are not compatible with assumption (2.3).

Corollary: To obtain soliton solutions for an extendable system one has to choose $N = 2M$. Then the non-degenerated M -soliton solutions are described as solutions sets of the following linear combinations

$$0 = \sum_{n=0}^{2M} \beta_n \vec{K}_n = \prod_{k=1}^M (\Phi - c_k)(\Phi - \bar{c}_k) \vec{K}_0 .$$

3 Real Representation of Extendable Systems

In this section we will give the representation of any extendable system with scaling degree $\alpha = 1$ in terms of the real a_k, γ_k and the imaginary parts b_k, δ_k of the **complex velocities**

$$c_k = a_k + i b_k \quad (3.1)$$

and the **complex phases**

$$q_k = \gamma_k + i \delta_k . \quad (3.2)$$

We confine ourselves to the case $\alpha = 1$, because extendable systems with other scaling degrees are not known so far. In case $\alpha \neq 1$ one has to choose $c_k^\alpha = a_k + i b_k$ and rewrite all the following quantities.

Let \vec{u}_M be a non-degenerated M -soliton solution of an extendable system. According to the results presented above \vec{u}_M depends on $2N = 4M$ complex parameters $c_1, \dots, c_M, \bar{c}_1, \dots, \bar{c}_M, q_1, \dots, q_M, \bar{q}_1, \dots, \bar{q}_M$. Then the tangent space of all M -soliton solutions is spanned by the complex derivatives

$$A_k := \frac{\partial \vec{u}_M}{\partial q_k} , \quad \tilde{A}_k := \frac{\partial \vec{u}_M}{\partial \bar{q}_k} , \quad B_k := \frac{\partial \vec{u}_M}{\partial c_k} , \quad \tilde{B}_k := \frac{\partial \vec{u}_M}{\partial \bar{c}_k} , \quad (3.3)$$

for $k = 1, \dots, M$. Using the relations (which are a consequence of the Cauchy-Riemann equations for the complex derivatives)

$$\begin{aligned} A_k &= \frac{1}{2} \left(\frac{\partial \vec{u}_M}{\partial \gamma_k} - i \frac{\partial \vec{u}_M}{\partial \delta_k} \right) , & \tilde{A}_k &= \frac{1}{2} \left(\frac{\partial \vec{u}_M}{\partial \gamma_k} + i \frac{\partial \vec{u}_M}{\partial \delta_k} \right) \\ B_k &= \frac{1}{2} \left(\frac{\partial \vec{u}_M}{\partial a_k} - i \frac{\partial \vec{u}_M}{\partial b_k} \right) , & \tilde{B}_k &= \frac{1}{2} \left(\frac{\partial \vec{u}_M}{\partial a_k} + i \frac{\partial \vec{u}_M}{\partial b_k} \right) \end{aligned} \quad (3.4)$$

we can perform a change of basis in the tangent space from A_k, \tilde{A}_k, B_k and $\tilde{B}_k, k = 1, \dots, M$, to

$$D_k := \frac{\partial \vec{u}_M}{\partial \gamma_k} = A_k + \tilde{A}_k , \quad E_k := \frac{\partial \vec{u}_M}{\partial \delta_k} = i(A_k - \tilde{A}_k) \quad (3.5)$$

$$G_k := \frac{\partial \vec{u}_M}{\partial a_k} = B_k + \tilde{B}_k , \quad H_k := \frac{\partial \vec{u}_M}{\partial b_k} = i(B_k - \tilde{B}_k) . \quad (3.6)$$

These new basis fields are now determined by derivatives w.r.t. real-valued quantities. Rewriting all quantities of chapter 2 in this new basis one obtains the following representations.

Lemma 4: W.r.t. the basis D_k, E_k, G_k and H_k

(a) the evolution equation (1.5) for an extendable system reads

$$\begin{aligned} \frac{\partial}{\partial t} (\gamma_1, \dots, \gamma_M, \delta_1, \dots, \delta_M, a_1, \dots, a_M, b_1, \dots, b_M)^T &= \\ &= (a_1, \dots, a_M, b_1, \dots, b_M, 0, \dots, 0, 0, \dots, 0)^T \end{aligned} \quad (3.7)$$

(b) the recursion operator is then of the form

$$\Phi = \begin{pmatrix} \mathcal{A} & -\mathcal{B} & \mathcal{O} & \mathcal{O} \\ +\mathcal{B} & \mathcal{A} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{A} & -\mathcal{B} \\ \mathcal{O} & \mathcal{O} & +\mathcal{B} & \mathcal{A} \end{pmatrix}, \quad (3.8)$$

where \mathcal{A} and \mathcal{B} denote the diagonal $M \times M$ -matrices

$$\mathcal{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & a_M \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & b_M \end{pmatrix}, \quad (3.9)$$

and \mathcal{O} is the corresponding zero-matrix.

(c) The symmetries \vec{K}_n and the mastersymmetries $\vec{\tau}_n$ are found to be

$$\vec{K}_n = \Phi^n \vec{K}_0 \quad (3.10)$$

$$= \Phi^n (1, \dots, 1, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)^T$$

$$\vec{\tau}_n = \Phi^n \vec{\tau}_0 \quad (3.11)$$

$$= \Phi^n (-\gamma_1, \dots, -\gamma_M, -\delta_1, \dots, -\delta_M, a_1, \dots, a_M, b_1, \dots, b_M)^T.$$

To represent system (3.7) as a hamiltonian system we theoretically have two possibilities: Either we choose the scalar fields

$$\frac{1}{2(n+1)} \sum_{k=1}^M (c_k^{n+1} + \bar{c}_k^{n+1})$$

as Hamiltonians or we take the following quantities

$$H_n := -\frac{i}{2(n+1)} \sum_{k=1}^M (c_k^{n+1} - \bar{c}_k^{n+1}), \quad n \geq 0. \quad (3.12)$$

Both cases are compatible with the reduction leading from the extended equation to the extendable. For the example which we consider in chapter 4, it turns out that only the scalar fields H_n have a representation in terms of the field variable \vec{u}_M . Therefore we concentrate on the hamiltonian representation given by the H_n . With (3.12) we easily verify the following relations.

Lemma 5:

(a) For $n \in \mathbb{N}_0$ it holds

$$\vec{K}_n = \Theta_1 \text{grad } H_n \quad (3.13)$$

with

$$\Theta_1 := \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} & \mathcal{O} \\ \mathcal{O} & -\mathcal{I} & \mathcal{O} & \mathcal{O} \\ -\mathcal{I} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \quad (3.14)$$

and \mathcal{I} is the $M \times M$ unit matrix. Here H_0, H_1 and H_2 are of the form

$$H_0 = \sum_{k=1}^M b_k, \quad H_1 = \sum_{k=1}^M a_k b_k, \quad H_2 = \frac{1}{3} \sum_{k=1}^M 3a_k^2 b_k - b_k^3. \quad (3.15)$$

(b) The scaling mastersymmetry $\vec{\tau}_0$ can be written as

$$\begin{aligned} \vec{\tau}_0 &= \Theta_1 \text{grad} \left(\frac{i}{2} \sum_{k=1}^M (c_k q_k - \bar{c}_k \bar{q}_k) \right) \\ &= \Theta_1 \text{grad} \left(- \sum_{k=1}^M (a_k \delta_k + b_k \gamma_k) \right) = \Theta_1 \text{grad } S. \end{aligned} \quad (3.16)$$

(c) W.r.t. the Poisson bracket defined by Θ_1 all these brackets between γ_k, δ_l, a_m and b_n vanish except

$$\{a_k, \delta_k\}_{\Theta_1} = 1 = \{b_k, \gamma_k\}_{\Theta_1} \quad (3.17)$$

for $k = 1, \dots, M$. Hence, these coordinates form pairs of canonical real-valued action/angle variables.

(d) For the basis vectors D_k, E_k, G_k and H_k one obtains

$$D_k = \Theta_1 \text{grad} (b_k) \quad , \quad E_k = \Theta_1 \text{grad} (a_k) \quad (3.18)$$

$$G_k = \Theta_1 \text{grad} (-\delta_k) \quad , \quad H_k = \Theta_1 \text{grad} (-\gamma_k) \quad . \quad (3.19)$$

In analogy to theorem 1 we obtain the following representation of the canonical coordinates a_k, δ_k, b_k and γ_k and their corresponding hamiltonian vector fields in the physical variable \vec{u} .

Theorem 2:

(a) For a M -soliton solution \vec{u}_M of an extendable system a set of real canonical coordinates w.r.t. $\Theta_1(\vec{u}_M)$ is given by the partial derivatives

$$-\frac{\partial S(\vec{u}_M)}{\partial \delta_k} \quad \text{and} \quad -\frac{\partial S(\vec{u}_M)}{\partial a_k} \quad , \quad (3.20)$$

$$-\frac{\partial S(\vec{u}_M)}{\partial \gamma_k} \quad \text{and} \quad -\frac{\partial S(\vec{u}_M)}{\partial b_k} \quad , \quad (3.21)$$

where the scalar field $S(\vec{u}_M)$ is determined by

$$\vec{\tau}_0 = \Theta_1(\vec{u}_M) \text{grad} S(\vec{u}_M) \quad .$$

(b) The corresponding hamiltonian vector fields are represented by

$$\frac{\partial \vec{u}_M}{\partial \delta_k} = \Theta_1(\vec{u}_M) \text{grad} \left(-\frac{\partial S(\vec{u}_M)}{\partial \delta_k} \right) \quad , \quad (3.22)$$

$$\frac{\partial \vec{u}_M}{\partial a_k} = \Theta_1(\vec{u}_M) \text{grad} \left(\frac{\partial S(\vec{u}_M)}{\partial a_k} \right) \quad , \quad (3.23)$$

$$\frac{\partial \vec{u}_M}{\partial \gamma_k} = \Theta_1(\vec{u}_M) \text{grad} \left(-\frac{\partial S(\vec{u}_M)}{\partial \gamma_k} \right) \quad , \quad (3.24)$$

$$\frac{\partial \vec{u}_M}{\partial b_k} = \Theta_1(\vec{u}_M) \text{grad} \left(\frac{\partial S(\vec{u}_M)}{\partial b_k} \right) \quad . \quad (3.25)$$

In order to compare the results of the present paper with the literature we perform at the end of this chapter the change of coordinates (1.17) for the real coordinates γ_k, δ_k, a_k and b_k , which is given by

$$\begin{aligned}\gamma_k &\rightarrow \hat{\gamma}_k := a_k \gamma_k - b_k \delta_k \ , \\ \delta_k &\rightarrow \hat{\delta}_k := a_k \delta_k + b_k \gamma_k \ ,\end{aligned}\tag{3.26}$$

$$\begin{aligned}a_k &\rightarrow \hat{a}_k := a_k \ , \\ b_k &\rightarrow \hat{b}_k := b_k \ .\end{aligned}\tag{3.27}$$

In terms of this parametrization the restricted evolution (3.7) reads

$$\begin{aligned}\frac{\partial}{\partial t} (\hat{\gamma}_1, \dots, \hat{\gamma}_M, \hat{\delta}_1, \dots, \hat{\delta}_M, \hat{a}_1, \dots, \hat{a}_M, \hat{b}_1, \dots, \hat{b}_M)^T &= \\ = (\hat{a}_1^2 - \hat{b}_1^2, \dots, \hat{a}_M^2 - \hat{b}_M^2, 2\hat{a}_1 \hat{b}_1, \dots, 2\hat{a}_M \hat{b}_M, 0, \dots, 0, 0, \dots, 0)^T \ .\end{aligned}\tag{3.28}$$

The operators $\hat{\Phi}, \hat{\Theta}_0$ and $\hat{\Theta}_1$ are of the form

$$\begin{aligned}\hat{\Phi} &= \begin{pmatrix} \hat{\mathcal{A}} & -\hat{\mathcal{B}} & \mathcal{O} & \mathcal{O} \\ +\hat{\mathcal{B}} & \hat{\mathcal{A}} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \hat{\mathcal{A}} & -\hat{\mathcal{B}} \\ \mathcal{O} & \mathcal{O} & +\hat{\mathcal{B}} & \hat{\mathcal{A}} \end{pmatrix}, \quad \hat{\Theta}_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} & \mathcal{O} \\ \mathcal{O} & -\mathcal{I} & \mathcal{O} & \mathcal{O} \\ -\mathcal{I} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}, \\ \hat{\Theta}_1 &= \hat{\Phi} \hat{\Theta}_0 \ ,\end{aligned}$$

where \mathcal{I} denotes the $M \times M$ unit matrix and $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are given by

$$\hat{\mathcal{A}} = \begin{pmatrix} \hat{a}_1 & 0 & \dots & 0 \\ 0 & \hat{a}_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & \hat{a}_M \end{pmatrix}, \quad \hat{\mathcal{B}} = \begin{pmatrix} \hat{b}_1 & 0 & \dots & 0 \\ 0 & \hat{b}_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & \hat{b}_M \end{pmatrix}.$$

The scaling mastersymmetry $\hat{\tau}_0$ is found to be

$$\begin{aligned}\hat{\tau}_0 &= (0, \dots, 0, 0, \dots, 0, \hat{a}_1, \dots, \hat{a}_M, \hat{b}_1, \dots, \hat{b}_M)^T \\ &= \hat{\Theta}_1 \text{grad} \left(-\sum_{k=1}^M \hat{\delta}_k \right) \ .\end{aligned}\tag{3.29}$$

Remark: In this parametrization complex soliton systems were examined in [1]. However, there it was not possible to find a representation of the canonical variables $\hat{a}_k, \hat{b}_k, \hat{\gamma}_k$ and $\hat{\delta}_k$ in terms of the physical variable \bar{u} . From the approach presented here not only this difficulty is overcome, but the appropriate parametrization of complex solitons is deduced from structural arguments.

4 Example

Since the Nonlinear Schrödinger Equation (NLS) is the best known example of a complex soliton equation, we illustrate our results by an exhaustive presentation of that equation. Other examples like the derivative Schrödinger equation and spin systems can be treated in analogy.

The NLS is given by

$$u_t = i(u_{xx} + |u|^2u) , \quad (4.1)$$

which determines the extendable system

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix}_t = \begin{pmatrix} i(u_{xx} + u^2\bar{u}) \\ -i(\bar{u}_{xx} + \bar{u}^2u) \end{pmatrix} . \quad (4.2)$$

By setting $v = \bar{u}$ we obtain the corresponding extended system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} i(u_{xx} + u^2v) \\ -i(v_{xx} + v^2u) \end{pmatrix} =: \vec{K}_1(\vec{u}) , \quad (4.3)$$

which has the following compatible bi-hamiltonian structure

$$\begin{aligned} \vec{K}_1(\vec{u}) &= \begin{pmatrix} 0 & -4i \\ 4i & 0 \end{pmatrix} \text{grad} \frac{1}{4} \int_{-\infty}^{+\infty} (u_x v_x - \frac{u^2 v^2}{2}) dx \\ &= \begin{pmatrix} -4uD^{-1}u & 4D + 4uD^{-1}v \\ 4D + 4vD^{-1}u & -4vD^{-1}v \end{pmatrix} \text{grad} \frac{i}{8} \int_{-\infty}^{+\infty} (u_x v - v_x u) dx . \end{aligned}$$

Here D denotes the differential operator w.r.t. x and D^{-1} its inverse. The hereditary recursion operator

$$\begin{aligned} \Phi(\vec{u}) &= \begin{pmatrix} -4uD^{-1}u & 4D + 4uD^{-1}v \\ 4D + 4vD^{-1}u & -4vD^{-1}v \end{pmatrix} \begin{pmatrix} 0 & -\frac{i}{4} \\ \frac{i}{4} & 0 \end{pmatrix} \\ &=: \Theta_1(\vec{u}) \Theta_0^{-1}(\vec{u}) \end{aligned}$$

generates the hierarchy of commuting symmetries

$$\vec{K}_n(\vec{u}) := \Phi^n(\vec{u}) \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

and the hierarchy of the mastersymmetries

$$\vec{\tau}_n(\vec{u}) := \Phi^n(\vec{u}) \begin{pmatrix} xu_x + u \\ xv_x + v \end{pmatrix} .$$

The symmetries and the mastersymmetries fulfill the following commutator relations

$$\begin{aligned} [\vec{K}_n, \vec{K}_m] &= 0, \quad [\vec{\tau}_n, \vec{K}_m] = (m+1) \vec{K}_{n+m}, \\ [\vec{\tau}_n, \vec{\tau}_m] &= (m-n) \vec{\tau}_{n+m}, \end{aligned}$$

hence, the scaling degree α is equal to 1. Furthermore one easily checks that assumption (2.3) is true. The M -soliton solutions of (4.2) are given for large time by ([2])

$$\begin{pmatrix} u_M \\ v_M \end{pmatrix} \cong \begin{pmatrix} \sum_{k=1}^M \sqrt{2} \hat{b}_k \operatorname{sech}(\hat{b}_k x + 2\hat{a}_k \hat{b}_k t + \hat{\delta}_k) \exp(-i(\hat{a}_k x + (\hat{a}_k^2 - \hat{b}_k^2)t + \hat{\gamma}_k)) \\ \sum_{k=1}^M \sqrt{2} \hat{b}_k \operatorname{sech}(\hat{b}_k x + 2\hat{a}_k \hat{b}_k t + \hat{\delta}_k) \exp(i(\hat{a}_k x + (\hat{a}_k^2 - \hat{b}_k^2)t + \hat{\gamma}_k)) \end{pmatrix} .$$

The time evolution of these solutions expressed in asymptotic data reads

$$\begin{aligned} \frac{\partial}{\partial t} (\hat{\gamma}_1, \dots, \hat{\gamma}_M, \hat{\delta}_1, \dots, \hat{\delta}_M, \hat{a}_1, \dots, \hat{a}_M, \hat{b}_1, \dots, \hat{b}_M)^T &= \\ = (\hat{a}_1^2 - \hat{b}_1^2, \dots, \hat{a}_M^2 - \hat{b}_M^2, 2\hat{a}_1 \hat{b}_1, \dots, 2\hat{a}_M \hat{b}_M, 0, \dots, 0, 0, \dots, 0)^T . \end{aligned}$$

One proves that in this parametrization the conservation laws $H_n(\vec{u}_M)$, which are determined by

$$H_n(\vec{u}_M) = \frac{1}{n+2} \langle \operatorname{grad} \frac{1}{4} \int_{-\infty}^{+\infty} u_M v_M dx, \vec{\tau}_n(\vec{u}_M) \rangle$$

are of the form (3.12). Starting with the fundamental field

$$S(\vec{u}_M) = \frac{1}{4} \int_{-\infty}^{+\infty} x u_M v_M dx , \quad (4.4)$$

the scaling mastersymmetry is

$$\vec{\tau}_0(\vec{u}_M) = \Theta_1(\vec{u}_M) \text{grad } S(\vec{u}_M) .$$

In the above parametrization this scaling field is recovered to be (see (3.29))

$$\begin{aligned} \vec{\tau}_0 &= (0, \dots, 0, 0, \dots, 0, \hat{a}_1, \dots, \hat{a}_M, \hat{b}_1, \dots, \hat{b}_M)^T \\ &= \hat{\Theta}_1 \text{grad} \left(- \sum_{k=1}^M \hat{\delta}_k \right) . \end{aligned}$$

Following the theory developed in the preceding chapters we have to invert the change of coordinates (3.26), i.e. we take as new coordinates

$$\begin{aligned} \gamma_k &:= (\hat{a}_k^2 + \hat{b}_k^2)^{-1} (\hat{a}_k \hat{\gamma}_k + \hat{b}_k \hat{\delta}_k) , \\ \delta_k &:= (\hat{a}_k^2 + \hat{b}_k^2)^{-1} (\hat{a}_k \hat{\delta}_k - \hat{b}_k \hat{\gamma}_k) , \\ a_k &:= \hat{a}_k , \\ b_k &:= \hat{b}_k . \end{aligned} \tag{4.5}$$

(4.6)

Then we obtain the action/angle coordinates in the physical field variable simply by taking the partial derivatives

$$- \frac{\partial S(\vec{u}_M)}{\partial \delta_k} \quad \text{and} \quad - \frac{\partial S(\vec{u}_M)}{\partial a_k} , \tag{4.7}$$

$$- \frac{\partial S(\vec{u}_M)}{\partial \gamma_k} \quad \text{and} \quad - \frac{\partial S(\vec{u}_M)}{\partial b_k} . \tag{4.8}$$

of the fundamental field $S(\vec{u}_M)$ as defined in (4.4). The corresponding hamiltonian vector fields determine the eigenvectors

$$A_k , \tilde{A}_k , B_k , \tilde{B}_k$$

(see (3.3)) of the recursion operator $\Phi(\vec{u}_M)$. They can be obtained by application of the implectic operator $\Theta_1(\vec{u}_M)$ or by taking the partial derivatives of \vec{u}_M w.r.t. the action/angle variables as in formulas (3.22)-(3.25). Then equations (3.4) give the desired eigenvectors. Of course, all these quantities are now easily computed.

As an example we take the 2-soliton solution of (4.1) in Hirota Form ([8]). It is given by

$$u_2(x, t) = \frac{G(x, t)}{F(x, t)} \tag{4.9}$$

with

$$G(x, t) = \sqrt{2} b_1 \operatorname{sech} \chi_1 \exp(-i\zeta_1) (\cos \phi_1 - i \sin \phi_1 \tanh \chi_2) \\ + \sqrt{2} b_2 \operatorname{sech} \chi_2 \exp(-i\zeta_2) (\cos \phi_2 - i \sin \phi_2 \tanh \chi_1) \quad (4.10)$$

$$F(x, t) = \cosh(\kappa) + \\ \sinh(\kappa) (\tanh \chi_1 \tanh \chi_2 - \operatorname{sech} \chi_1 \operatorname{sech} \chi_2 \cos(\zeta_1 - \zeta_2)) \quad (4.11)$$

where the following abbreviations are used

$$\exp(\kappa) = \frac{|c_1 - c_2|}{|c_1 - \bar{c}_2|} = \left(\frac{(b_1 - b_2)^2 + (a_1 - a_2)^2}{(b_1 + b_2)^2 + (a_1 - a_2)^2} \right)^{1/2} \\ \phi_1 = \arg \left(\frac{\bar{c}_1 - \bar{c}_2}{\bar{c}_1 - c_2} \right) = \arg \left(\frac{(a_1 - a_2)^2 + b_1^2 - b_2^2 + 2ib_2(a_1 - a_2)}{(b_1 + b_2)^2 + (a_1 - a_2)^2} \right) , \\ \phi_2 = \arg \left(\frac{\bar{c}_2 - \bar{c}_1}{\bar{c}_2 - c_1} \right) = \arg \left(\frac{(a_1 - a_2)^2 - b_1^2 + b_2^2 - 2ib_1(a_1 - a_2)}{(b_1 + b_2)^2 + (a_1 - a_2)^2} \right) ,$$

$$\chi_k = \operatorname{Im} (c_k(x + c_k t + q_k^0)) = b_k x + 2a_k b_k t + (a_k \delta_k^0 + b_k \gamma_k^0) , \\ \zeta_k = \operatorname{Re} (c_k(x + c_k t + q_k^0)) = a_k x + (a_k^2 - b_k^2)t + (a_k \gamma_k^0 - b_k \delta_k^0)$$

for $k = 1, 2$. Here $c_k = a_k + ib_k$ denotes a complex eigenvalue of the recursion operator $\Phi(\vec{u}_2)$, which turns up only together with \bar{c}_k . Of course, the $q_k^0 = \gamma_k^0 + i\delta_k^0$ are not the asymptotic phases, but they are equal to these quantities up to an additive term depending linearly on t and on c_k and \bar{c}_k . So it holds $q_k(t=0) = q_k^0$. For the eigenvalues

$$c_1 = 0.2 + i0.4 \quad \text{and} \quad c_2 = 0.3 + i0.5$$

the corresponding 2-soliton $u_2(x, t)$ is easily derived. Our plot (figure 1) shows the square of the modulus of $u_2(x, t)$, i.e. $|u_2|^2$ with $\gamma_k^0 = 0 = \delta_k^0$. Here and in the next pictures the plots give the modulus of the first component of \vec{u} as well as of the second component, since these are conjugated quantities. Pictures 2-5 show the moduli of the hamiltonian vector fields corresponding to the real action/angle coordinates (4.7) and (4.8). These plots are obtained by taking derivatives with respect to δ_1, a_1, γ_1 and b_1 , respectively. Figures 6 and 7 are plots of the moduli of the eigenstates which are obtained by taking derivatives with respect to q_1 and c_1

$$A_1 = \frac{\partial \vec{u}_2}{\partial q_1} = \frac{1}{2} \left(\frac{\partial \vec{u}_2}{\partial \gamma_1} - i \frac{\partial \vec{u}_2}{\partial \delta_1} \right) \\ B_1 = \frac{\partial \vec{u}_2}{\partial c_1} = \frac{1}{2} \left(\frac{\partial \vec{u}_2}{\partial a_1} - i \frac{\partial \vec{u}_2}{\partial b_1} \right)$$

of the recursion operator $\Phi(\vec{u}_2)$ for the discrete eigenvalue $c_1 = 0.2 + i0.4$. Figure 6 is also known as interacting soliton of (4.1). Of course, these last two plots show the same qualitative behaviour like the plots in [5] reflecting again our unified approach to KdV and NLS solitons. However, the results on NLS given in [5],[10],[1] have to be replaced by the theory presented in this paper.

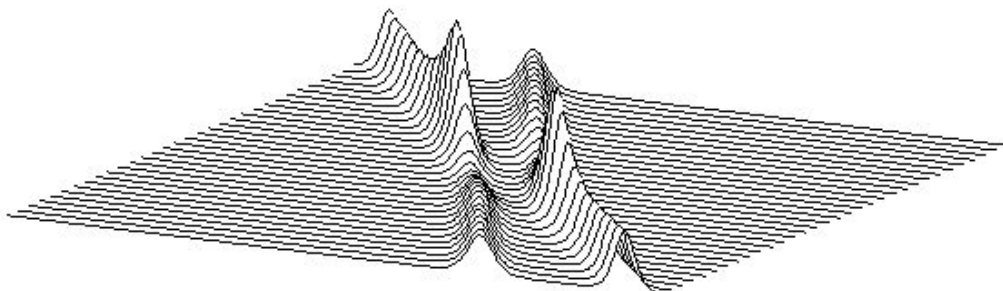


Figure 1 : Two-Soliton of the NLS
(represented by the square of the modulus)

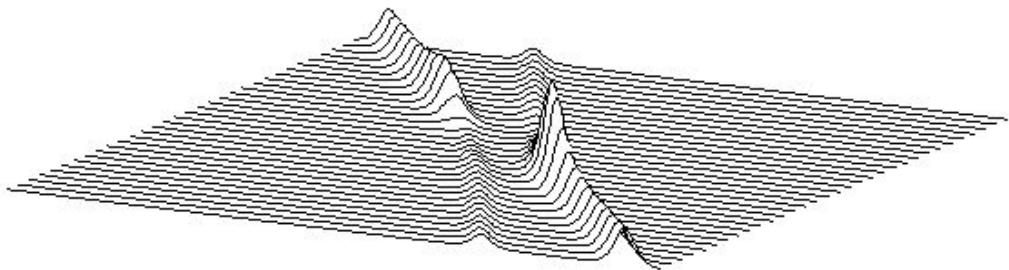


Figure 2 : Vectorfield corresponding to an action variable
(derivative with respect to δ_1)

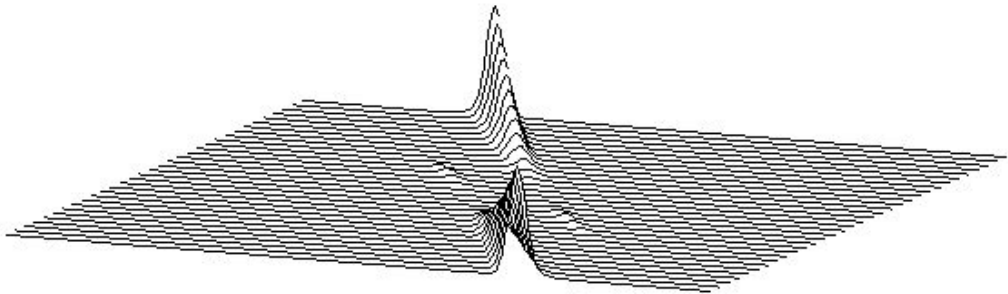


Figure 3 : Vectorfield corresponding to an angle variable
(derivative with respect to a_1)

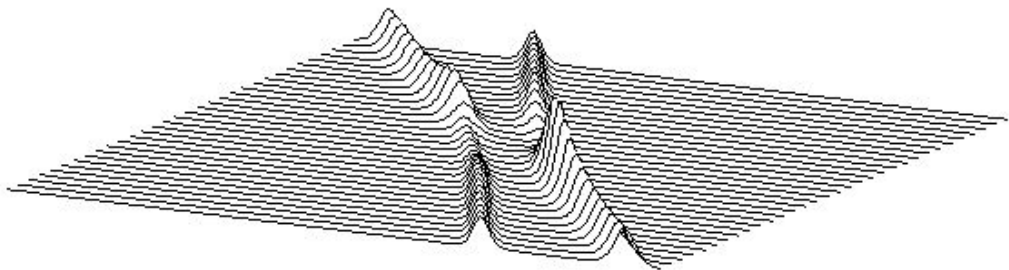


Figure 4 : Vectorfield corresponding to an action variable
(derivative with respect to γ_1)

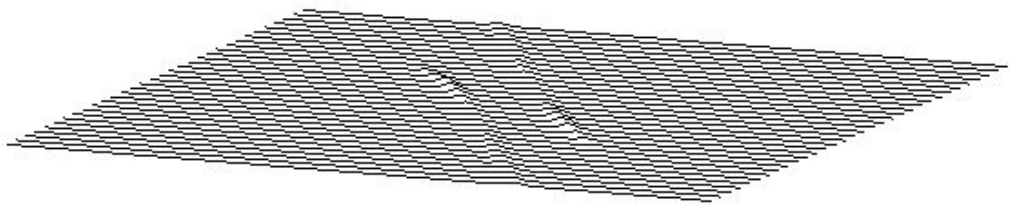


Figure 5 : Vectorfield corresponding to an angle variable
(derivative with respect to b_1)

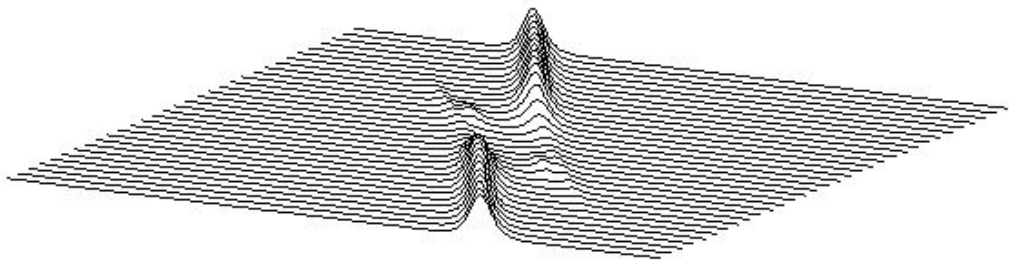


Figure 6 : Interacting Soliton
(derivative with respect to q_1)

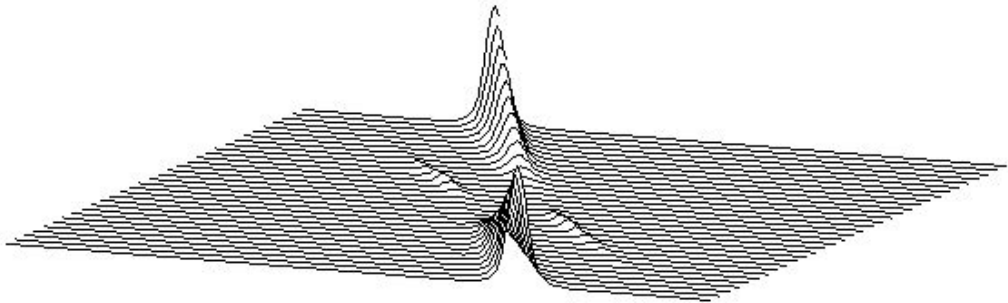


Figure 7 : Second eigenstate of recursion operator
(derivative with respect to c_1)

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