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Symmetries, Conserved Quantities, and Hierarchies for some Lattice Systems with Soliton Structure

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Basic invariants, such as conserved quantities, symmetries, mastersymmetries and recursion operators are explicitly constructed for the following nonlinear lattice systems: the modified Korteweg-de Vries lattice, the Ablowitz-Ladik lattice, the Brusci-Ragnisco lattice, the Ragnisco-Tu lattice and some cases of the class of integrable systems introduced by Bogoyavlensky. The algorithmic basis for obtaining these quantities is described and the interrelation between the underlying mastersymmetry approach and the Lax pair analysis is discussed. By explicit presentation of the higher order members of the corresponding hierarchies new completely integrable lattice flows are found. For all systems multi-Hamiltonian formulations are given.

1 Introduction

Since the work on the Fermi-Pasta-Ulam problem [1] discrete nonlinear systems have been in the focus of nonlinear studies. Many interesting physical phenomena are modeled by them. Nowadays the range of applications of these systems seems to be unlimited.

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Apart from their classical area of applications they are applied in new areas like: Cellular automata [2], [3], [4], [5], vertex models and quantum spin chains [6], [7], [8], [9] solitons in molecular systems [10] (for example in polypeptides [11]), Knot theory [12], [13] not to mention network theory, neurons and many other areas. And apart from describing phenomena of physical relevance such systems (with special structural properties) are interesting from the mathematical viewpoint since they usually provide excellent discrete approximations for special nonlinear partial differential equations.

Of particular interest are those systems admitting large symmetry groups since contrary to early expectations usually no equipartition of energy between the different modes of the system takes place. This shows that systems with nontrivial nonlinear interactions can behave like linear systems, i. e. they are completely integrable.

Usually, the complete integrability of such a system is shown via the exhibition of a Lax pair which mostly is found by ingenuity, and no algorithm seems to be known telling whether or not a given system is completely integrable [14]. But even if the Lax pair is known then the explicit treatment of a completely integrable nonlinear lattice system is difficult because the computation of the relevant quantities (symmetry generators, conservation laws, angle variables) out of a given Lax pair is a nontrivial task.

We have developed algorithms [15], [16], [17] which tell whether a system (of a certain class) is completely integrable or not. Furthermore, if the answer is affirmative, then the recursive structure of the system under consideration is computed automatically. Such an algorithm is briefly sketched in this paper. Since we have implemented a prerelease version of this algorithm, by means of computer algebra, the algorithm allows us to exhibit easily the recursive structure and the explicit form of relevant quantities for many nonlinear systems. The results of these studies are given in this paper where a catalogue of invariants can be found. This article may be regarded as part II of [18] where more elementary systems were treated.

The paper is built in the following way: In section 2) we shall briefly review the basic notions such as mastersymmetries, recursion operators etc. Such objects give rise to a recursive scheme, from which symmetries as well as conserved quantities for the dynamical system under consideration can be calculated in a simple way. In section 3) we shall briefly indicate how such structures can be found systematically. If no additional information (such as a Lax formulation) about the given system is known, then algorithms implemented in Symbolic Manipulation Systems may be used for a systematic search for symmetries and mastersymmetries. The main ideas of these algorithms will be outlined in section 3a). Once a single (non-trivial) mastersymmetry has been found this vector field usually can be used in a straightforward way to derive further invariants of the dynamical system. Thus often several Hamiltonian formulations of these equations can be constructed. A severe computational difficulty may arise when searching for master symmetries, as these vector fields often turn out to be non-local expressions of the field variables, e.g. they may contain infinite summations or even more complicated inverses of difference operators. In section 3.2) it will be shown for a fairly general class of scattering problems, how in such a situation a Lax formulation for the equation under consideration can be exploited to derive symmetries and mastersymmetries in an algorithmic way. The results of these studies presented in section 4) include the recursive structures (mastersymmetries and recursion operators) of the modified Korteweg-de Vries lattice (mKdV [19]), the Ablowitz-Ladik lattice [20], the Bruschi-Ragnisco lattice [21], the Ragnisco-Tu lattice [22] and some cases of the class of integrable lattice systems introduced by Bogoyavlensky [23].

2 Basic notation

We will consider evolution equations of the form

$$u_t = K(u) \ , \tag{2.1}$$

where u is a point in a space $S = \{u = (u_1, ..., u_m) | u : \mathbb{Z} \to \mathbb{R}^m\}$ of vector valued sequences. The map $K : S \to S$ is to be regarded as a smooth vector field over S. Assuming suitable boundary conditions for these sequences we take the dual space S^* to be again a space of sequences acting on S via the pairing

$$< u^*, u >= \sum_{n \in \mathbb{Z}} (u^*(n), u(n)) \quad ; \quad u \in S \quad , \quad u^* \in S^* \quad , \tag{2.2}$$

where $(.,.)$ is the usual euclidean scalar product on $\mathbb{R}^m$. Regarding S as a manifold the vector fields over S form a Lie algebra, the commutator (Lie derivative) of two vector fields K_1, K_2 being given by

$$L_{K_1} K_2 = [K_1, K_2] = K_2'[K_1] - K_1'[K_2] \quad . \tag{2.3}$$

Here the prime indicates the usual directional derivative of some function A of u into the direction of a vector $v \in S$:

$$A'(u)[v] = \frac{\partial}{\partial\epsilon|_{\epsilon=0}} A(u + \epsilon v) \quad . \tag{2.4}$$

We will be interested in constructing *symmetries* of equation (2.1), i.e. vector fields commuting with K, as the corresponding flows are one-parameter symmetry transformations for the dynamical system given by K. A useful tool for this construction are *recursion operators* [24],[25], i.e. operators $\Phi(u) : S \to S$ satisfying

$$L_K\Phi = \Phi'[K] - K'[\Phi\,\cdot\,] + \Phi K'[\,\cdot\,] = 0 \quad . \tag{2.5}$$

Obviously such an operator maps symmetries to symmetries.

Another powerful tool for the construction of symmetries had been introduced in [26],[27] with the notion of *mastersymmetries*. A mastersymmetry τ for a given vector field K is a vector field satisfying $[K,[\tau,K]] = 0$ and $[\tau,K] \neq 0$, i.e. it sends K to a (non-vanishing) symmetry $[\tau,K]$ of K via the commutator. From the Jacobi identity of the Lie bracket $[.,.]$ one immediately concludes that $[\tau,[\tau,K]]$ is again a further symmetry of K. Without additional assumptions no further algebraic relations can be derived, hence a mastersymmetry essentially has the property of generating 2 symmetries out of K by applying the Lie derivative L_τ. But assuming the group of (time-independent) symmetries of K to be abelian, i.e. all the symmetry generators of K have to commute, one trivially derives that the Lie derivative into the direction of such a mastersymmetry always maps a symmetry of K to another symmetry. Of course, for a given K it is hard to show a priori that all its symmetries will commute. But for integrable equations it is known that a large set of commuting symmetries (corresponding to action variables in involution) exists, i.e. the Lie derivative into the direction of a mastersymmetry will be a selfmap on this set of vector fields. Hence, for integrable K, a mastersymmetry as defined above will be an important *heuristic* tool: find a mastersymmetry τ for K, construct further vector fields by iteratively applying L_τ to known symmetries of K (e.g. K itself) and then try to verify *a posteriori* that these vector fields form an abelian set of symmetries for K. A typical way of such an a posteriori proof is given for Hamiltonian systems, as very often the mastersymmetries lead to the construction of Hamiltonian pairs [28], [29], [30] and hereditary [25], [31], [32] recursion operators from which the commutativity of the constructed vector fields can be derived easily.

We briefly review the necessary notation: The *gradient* of a scalar valued function $f : S \to \mathrm{I\!R}$ is the element of S^* given by

$$< \nabla f(u), v >= f'(u)[v] \quad . \tag{2.6}$$

A vector field of the form $K = P\nabla f$ is called *Hamiltonian*, where P is a *Poisson* (Hamiltonian, implectic [31]) operator, i.e. $P(u) : S^* \to S$ is a skewsymmetric linear operator satisfying the *Jacobi-identity*

$$< a^*, P'[Pb^*]c^* > \;+\; \text{cyclic permutations}(a^*, b^*, c^*) \;=\; 0 \tag{2.7}$$

for arbitrary elements $a^*, b^*, c^* \in S^*$. As a consequence the *Poisson bracket* $\{f_1, f_2\} =< \nabla f_2, P\nabla f_1 >$ defines a Lie algebra structure on the space of scalar fields over S. A *conservation law* for (2.1) is a scalar valued function f such that $f(u(t))$ is constant for all solutions of (2.1), i.e. $< \nabla f, K >= 0$. For a Hamiltonian vector field $K = P\nabla f$ the function f automatically is a conservation law and P will map the gradient of any conservation law to a symmetry for K. The *Lie derivative* of such an operator P into the direction of a vector field τ is given by

$$L_\tau P \quad = \quad P'[\tau] - \tau' P - P\tau'^* \quad , \tag{2.8}$$

where τ'^* denotes the adjoint (w.r.t. the duality (2.2)) of the linearization τ'. If this resulting operator turns out to be Poisson again, then P and $L_\tau P$ automatically form a *compatible Hamiltonian pair* [28], [29], i.e. their sum is again a Poisson operator.

In the examples of the next section the relevant vector fields and operators will be formulated in terms of the following basic operations $S \rightarrow S$:

An element $a \in S$ (or S^*) gives rise to a multiplication operator

$$a: \; u \rightarrow (au)(n) \quad = \quad a(n)u(n) \quad , \tag{2.9}$$

and by $[n]$ we will denote the multiplication

$$[n]: \quad u \quad \rightarrow \quad ([n]u)(n) \; = \; nu(n) \quad . \tag{2.10}$$

Let T_+ and T_- be the shift operators given by $(T_\pm u)(n) = u(n \pm 1)$. Apart from these local operations we will need the linear operator $(1 - T_-)^{-1}$ which may be understood as

$$((1 - T_-)^{-1} u)(n) \quad := \quad \sum_{k=-\infty}^{n} u(k) \tag{2.11}$$

(assuming suitable boundary conditions for u). Note that $(1 - T_-)(1 - T_-)^{-1} = (1 - T_-)^{-1}(1 - T_-) = 1$, i.e. (2.11) indeed defines the inverse of the difference operator $1 - T_-$

when acting on elements of S. For a constant sequence $c(n) = 1$ we formally define $((1 - T_-)^{-1}c)(n) = n$.

All these operations can also be applied to elements of S^*. For the transposed operators (w.r.t. (2.2)) one finds $T_\pm^* = T_\mp$, all multiplication operators are symmetric.

3 Computational aspects

3.1 Searching for symmetries and mastersymmetries via Computer Algebra

In this section we briefly describe the simple ideas which lead to the algorithm by which the results of the following section have been obtained. The algorithm will be reported in full detail elsewhere and its implementation will then be discussed. Eventually, after serious testing, the computer algebra package containing the implementations of this algorithm will be made available.

We essentially rely on the concept of mastersymmetries, i.e. for a given dynamical system $u_t = K(u)$ we look for a vector field τ, say, having the property that $\tilde{S} := [\tau, K]$ ($\neq 0$) is a symmetry of K.

Such mastersymmetries exist for most of the known integrable equations [26],[33], [34], [35], [27], [15], they are an important tool to construct higher invariants for integrable systems by applying Lie derivatives to simple invariants [36]. E.g. the commutator of such a mastersymmetry with a symmetry yields a new symmetry and the Lie derivative of a

Poisson operator yields a second Hamiltonian formulation and hence a hereditary recursion operator for the considered equation. So, knowing a single (non-trivial) mastersymmetry gives immediate access to almost the entire algebraic structure of the equation. Actually, this was the way how the multi-Hamiltonian formulations of the following examples have been found: exploiting the computer algebra algorithms to be described below we found the first non-trivial mastersymmetry τ_1. Then, using a first Hamiltonian formulation of the equation we constructed further Poisson operators by applying the Lie derivative into the direction of this mastersymmetry to the first Poisson operator. Once a second Hamiltonian formulation and hence a recursion operator is constructed the results can be summarized by the theorem of section 3 in [18].

Hence the computational effort essentially consists of finding one mastersymmetry, i.e. for a given dynamical system $u_t = K(u)$ we have to "solve" the equation

$$[K, [K, \tau]] \; = \; 0 \tag{3.1}$$

for τ.

Before going into this problem we have to mention some technical details: With respect to computational aspects a crucial role is played by a *highest-order term projection* for vector fields. Recall that the vector fields under consideration are polynomial such that if one evaluates the vector field at some place n then the field variable evaluated at other places does enter thus describing the interaction between n and its neighbor points. Now, restricting a vector field K onto those terms with farthest reaching interaction and then picking out of that result those terms with highest polynomial degree then gives what we call the *highest order term projection*. The result of this projection is denoted by $ho(K)$.

Now an important role in the program package is played by an *approximate* solution of the division problem in the Lie algebra of vector fields. By *approximate* we mean that given vector fields K and R we are able to find a vector field X such that

$$ho[ho(K), X] \;=\; ho(R) \tag{3.2}$$

This routine is called $CS(K, R)$ (commutator solution) and this subroutine is the heart of the whole matter. The reason why such *commutator solutions* can be found lies in the fact that restricting the considerations to terms less that a fixed degree more or less simulates the situation of a finite dimensional Lie algebra.

Another important point that is essential to find solutions of (3.2) in an algorithmic way is the fact that the explicit form of the mastersymmetries is known to a certain extent: from experience we know that the typical form of these vector fields is given by

$$\tau(u) \;=\; [n]S(u) \;+\; Z(u) \;\;, \tag{3.3}$$

where S and Z are translation invariant (i.e. do not explicitly depend on the lattice point n) and S is a symmetry of K. The following algorithms use this structure by starting with a *first approximation* $\tau \;=\; [n]symmetry$ for the wanted solution of (3.2).

For a given symmetry S of K we now attack (3.2) by splitting it into 2 parts:

1) First we determine the new symmetry $\tilde{S} \;=\; [\tau, K]$. Based on the observation that the highest order term of τ can be assumed to be given by the highest order term of $[n]S$, we know that the highest order term of $\tilde{S} \;=\; [\tau, K]$ is given by the highest order term of $[[n]S, K]$ and we can use the following algorithm SYM of successive approximation to find $\tilde{S} \;=\; [\tau, K] \;\;=:\; SYM(K, S)$:

PROCEDURE SYM(K,S) :

{ *The procedure SYM determines that symmetry* $\tilde{S}$ *of* K *with ho(*$\tilde{S}$*) = ho(*$[[n]S, K]$*).* }

- **Step 0:** *Put* $\tilde{S} := [[n]S, K]$ *.*

- **Step 1:** *Put* $R := [\tilde{S}, K]$ *. If* $R = 0$ *then RETURN(*$\tilde{S}$*) else GOTO Step 2.*

- **Step 2:** *Determine* $\delta S := CS(K, R)$ *, where* $CS(\ )$ *is applied by restricting the considerations to terms of degree less than the degree of* $\tilde{S}$*. If there is no solution then RETURN (There is no symmetry of this form) else GOTO Step 3.*

- **Step 3:** *Put* $\tilde{S} := \tilde{S} + \delta S$ *and GOTO Step 1.*

Obviously, in *Step 0* the wanted new symmetry $\tilde{S}$ is computed correctly in its highest order and each run computes $\tilde{S}$ correctly up to one order less. Hence the algorithm either has to stop after a number of runs given by the degree of $\tilde{S}$ thus finally giving the correct symmetry $\tilde{S}$, or it stops before by telling us that for the given S there is no symmetry which highest order is generated by $[[n]S, K]$. Although, as a language problem, this algorithm has not finite length it terminates since all descending chains (with respect to degree) are finite.

Of course, this algorithm is based on a symmetry S which has to be known already. But observe that one can always use $S = K$.

2) Now, once the new symmetry $\tilde{S}$ has been found we can use the following algorithm to solve $\tilde{S} = [\tau, K]$ for τ:

PROCEDURE MAS(K,S) :

{ *The procedure MAS determines the mastersymmetry* τ *with* $[\tau, K] = SYM(K,S)$. }

- **Step 0:** *Put* $\tilde{S} := SYM(K,S)$, $\tau := [n]S$.
- **Step 1:** *Put* $R := [\tau, K] - \tilde{S}$. *If* $R = 0$ *then RETURN*(τ) *else GOTO Step 2.*
- **Step 2:** *Determine* $\delta\tau := CS(K,R)$, *where CS( ) is applied by restricting the considerations to terms of degree less than the degree of* τ. *If there is no solution then RETURN (There is no solution) else GOTO Step 3.*
- **Step 3:** *Put* $\tau := \tau + \delta\tau$ *and GOTO Step 1.*

The program package is implemented in MAPLE [37], a formula manipulation system developed by the University of Waterloo. The choice for a formula manipulation system was mainly based on our desire for rapid prototyping and on the fact that for these systems many sophisticated algorithms are available.

3.2 Searching for symmetries and mastersymmetries via spectral problems

In this section we present a procedure for deriving both isospectral and non-isospectral hierarchies of equations starting from the following discrete spectral problem

$$T_+\psi = U\psi, \tag{3.4}$$

where $\psi = \psi(n,t) = (\psi_1, \cdots, \psi_N)^T$ is an N-vector and $U = U(\lambda, u)$ is an $N \times N$ matrix depending on a field vector $u = u(n,t) \equiv (u_1, \cdots, u_p)^T$ and a spectral parameter λ.

All quantities depend on a lattice parameter n, T_+ is the shift operator as introduced before. We call (3.4) an isospectral problem relative to a given dynamics of the fields u, if $d\lambda/dt = 0$. If the time dependence of u is such that $d\lambda/dt \neq 0$, then this dynamics shall be called a non-isospectral flow. It turns out that the non-isospectral flows associated to (3.4) are given by the mastersymmetries for the corresponding isospectral equations for (3.4).

In the isospectral case, a systematic procedure had been proposed in [38] to derive the corresponding hierarchy of equations. The procedure for deriving the corresponding non-isospectral hierarchy is similar and can be described as follows.

First we consider the equation

$$\tau(dU/d\lambda) - (T_+W)U + UW = 0 \quad , \tag{3.5}$$

where $\tau = 0$ or 1, and W is an unknown $N \times N$ matrix to be fixed. As we shall see later on the choice $\tau = 0$ will lead to the isospectral hierarchy whereas the choice $\tau = 1$ will lead to the non-isospectral dynamics. To solve the above equation for W, we substitute the expansion $W = \sum_{i\geq 0} W_i\lambda^{-i}$ into (3.5), and then compare the coefficients of λ^{-i} on both sides. Thus one obtains a recurrence relation for the W_i's which can be used to solve (3.5) by finding all W_i's recursively from a suitable chosen W_0.

Once the solution of (3.5) is found we multiply both sides of (3.5) by λ^k and obtain

$$\tau\lambda^k(dU/d\lambda) - T_+(\lambda^k W)\cdot U + U(\lambda^k W) = 0 \quad . \tag{3.6}$$

For any quantity $F = \sum f_i\lambda^i$ given as a *Laurent* expansion of the spectral parameter we define its *positive* and *negative* parts by

$$F_+ = (\sum f_i \lambda^i)_+ \equiv \sum_{i \geq i_0} f_i \lambda^i \quad , \quad F_- = F - F_+ = \sum_{i < i_0} f_i \lambda^i \quad , \tag{3.7}$$

where i_0 is a fixed integer. Inserting

$$\begin{aligned} \lambda^k W &= (\lambda^k W)_+ + (\lambda^k W)_- \\ &= \textstyle\sum_{i \leq k - i_0} \lambda^{k-i} W_i \ + \ \sum_{i > k - i_0} \lambda^{k-i} W_i \end{aligned} \tag{3.8}$$

into (3.6) one finds

$$\tau \lambda^k (dU/d\lambda) - T_+ (\lambda^k W)_+ \cdot U + U (\lambda^k W)_+ = T_+ (\lambda^k W)_- \cdot U - U (\lambda^k W)_- \quad . \tag{3.9}$$

Suppose that the potential U of the scattering problem (3.4) depends only on a finite number of different powers of the spectral parameter, i.e.

$$U(\lambda, u) = \sum_{i=l}^{h} \lambda^i U_i(u) \quad , \quad l \leq h \quad , \tag{3.10}$$

with coefficients $U_i(u)$ not depending on λ. Now we see that the right hand side of (3.9) contains only the powers λ^i with $i \leq h + i_0 - 1$, while the left hand side contains only the powers λ^i with $i \geq l + min(k-1, i_0)$. Thus both sides of (3.9) will contain only powers λ^i with $l + min(k-1, i_0) \leq i \leq h + i_0 - 1$.

We want to define a time evolution for the fields u in the potential U by the left hand side of (3.9). Hence the λ-dependence of (3.9) has to match the form (3.10) of the potential U, which in some cases may be accomplished by a suitable choice of the integer i_0. Still, inserting the solution W of (3.5) the form of the expression (3.9) in general will not be compatible with the dependence of U on the fields u. Hence we search for a sequence of matrices $\Delta_k = \Delta_k(u, \lambda)$, such that the expression

$$\tau \lambda^k (dU/d\lambda) - T_+ V^{(k)} \cdot U + U V^{(k)} \tag{3.11}$$

with

$$V^{(k)} = (\lambda^k W)_+ + \Delta_k \tag{3.12}$$

is tangent to the submanifold of matrices $U(\lambda, u)$ spanned by the fields u. To be more precise, we suppose that

$$U(\lambda, u) = e_0(\lambda) + \sum_{j=1}^{p} u_j \; e_j(\lambda) \quad , \tag{3.13}$$

where $e_j(\lambda), j = 0, 1, \cdots, p$ are matrices depending on λ. Hence we have to search for Δ_k such

$$\tau\lambda^k (dU/d\lambda) - T_+ V^{(k)} \cdot U + UV^{(k)} = -\sum_{j=1}^{p} f_{kj} \; e_j(\lambda) \tag{3.14}$$

for some expressions $f_{kj}(u)$. Once we succeed in finding such a sequence $\{\Delta_k\}$, the corresponding hierarchy of dynamical systems will be given by

$$du/dt_k = f_k(u) \quad , \quad (f_k = (f_{k1}, \cdots, f_{kp})^T) \quad , \tag{3.15}$$

being equivalent to

$$U_{t_k} + \tau\lambda^k (dU/d\lambda) - T_+ V^{(k)} \cdot U + UV^{(k)} = 0 \quad . \tag{3.16}$$

Here $U_{t_k} = \sum_{j=1}^{p} (\partial U/\partial u_j)(du_j/dt_k)$ is the part of the dynamics of U coming from the evolution of the fields u according to (3.15). Since the total time derivative of U is given by $dU/dt_k = U_{t_k} + (\partial U/\partial\lambda)(d\lambda/dt_k)$, equation (3.16) can be rewritten as the *discrete zero-curvature* equation

$$dU/dt_k - T_+ V^{(k)} \cdot U + UV^{(k)} = 0 \quad . \tag{3.17}$$

Hence for fixed k (3.17) is the compatibility condition of the following couple of linear problems

$$T_+\psi = U\psi \quad , \quad d\psi/dt_k = V^{(k)}\psi \tag{3.18}$$

under the constraint

$$d\lambda/dt_k = \tau\lambda^k \quad , \tag{3.19}$$

corresponding to the isospectral case for the choice $\tau = 0$ and corresponding to the non-isospectral case when choosing $\tau = 1$.

Example 1: As an illustrative example of the above procedure, we first discuss the spectral problem (3.4)

$$U = \begin{bmatrix} \lambda + s & q \\ r & 1 \end{bmatrix} = \lambda U_0 + U_1 \tag{3.20}$$

with

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad U_1 = \begin{bmatrix} s & q \\ r & 1 \end{bmatrix} \quad . \tag{3.21}$$

We will see that we have to impose the constraint $s = qr$, then (3.20) is the scattering problem of the isospectral hierarchy discussed in [22]. We shall also give some additional results in section 4.4.

Since $dU/d\lambda = U_0$, we first have to find a solution W of the equation

$$\tau U_0 - (T_+W)U + UW = 0 \quad . \tag{3.22}$$

Introducing the matrix V by $W = VU$ we have

$$(T_+W)U - UW = ((T_+ - 1)W - [U, V])U \quad , \tag{3.23}$$

with $[U,V] = UV - VU$, i.e.

$$(T_+ - 1)W = [U,V] + (\tau U_0)U^{-1} \quad . \tag{3.24}$$

From the last equation we deduce that

$$(T_+ - 1)(tr\, W) = tr((\tau U_0)U^{-1}) = \tau/\lambda \quad , \tag{3.25}$$

where tr is the usual trace of the matrices. Therefore the dependence of $tr(W)$ on the lattice parameter n is given explicitly and we assume

$$W \equiv W(n) = \begin{bmatrix} a(n) & b(n) \\ c(n) & -a(n) + n\,\tau/\lambda \end{bmatrix} \quad . \tag{3.26}$$

Inserting this ansatz for W into (3.22) we obtain the conditions

$$\begin{aligned} \lambda Da &= \tau - rT_+b + qc - sDa \quad , \\ \lambda b &= -sb + q(T_+ + 1)a + T_+b - mq\tau\lambda^{-1} \quad , \\ \lambda T_+c &= -sT_+c + r(T_+ + 1)a + c - (m+1)r\tau\lambda^{-1} \quad , \\ Da + rb - qT_+c - \tau\lambda^{-1} &= 0 \quad , \end{aligned} \tag{3.27}$$

with $D := T_+ - 1$. We observe that the last of these equations can be deduced from the first three. In fact, from the first three equations we conclude

$$\lambda(Da + rb - qT_+c - \tau\lambda^{-1}) = -s(Da + rb - qT_+c - \tau\lambda^{-1}) \quad , \tag{3.28}$$

hence $(\lambda+s)(Da+rb-qT_+c-\tau\lambda^{-1}) = 0$ implying the last equation of (3.27). Consequently we need only to solve the first three equations of (3.27) for a, b and c. Expanding a, b, and c into powers of λ, i.e. inserting

$$a = \sum_{i\geq 0} a_i\lambda^{-i} \quad , \quad b = \sum_{i\geq 0} b_i\lambda^{-i} \quad , \quad c = \sum_{i\geq 0} c_i\lambda^{-i} \tag{3.29}$$

into (3.27) we obtain the following recurrence relations

$$\begin{aligned} Da_{i+1} &= \tau\delta_{i0} - rT_+b_i + qc_i - sDa_i \quad , \\ b_{i+1} &= -sb_i + q(T_+ + 1)a_i + T_+b_i - nq\tau\delta_{i1} \quad , \\ T_+c_{i+1} &= -sT_+c_i + r(T_+ + 1)a_i + c_i - (n+1)r\tau\delta_{i1} \quad . \end{aligned} \tag{3.30}$$

Using this recursive scheme we can calculate successively the a_i's, b_i's, and c_i's imposing the starting conditions

$$a_0 = \alpha/2 \equiv const \quad , \quad b_0 = c_0 = 0 \quad . \tag{3.31}$$

The first of the higher quantities following from (3.30) are easily calculated as

$$a_1 = \tau n \quad , \quad b_1 = \alpha q \quad , \quad T_+c_1 = \alpha r \quad ; \tag{3.32}$$

$$\begin{aligned} a_2 &= -\alpha q T_- r - D^{-1}s\tau \quad , \\ b_2 &= \alpha(T_+q - q^2 r) + \tau(n+1)q \quad , \\ T_+c_2 &= \alpha(T_-r - r^2 q) + \tau n r \end{aligned} \tag{3.33}$$

and so on.

We now search for the corresponding hierarchy of dynamical systems admitting the *discrete zero-curvature* form (3.17). According to the procedure outlined above we first rewrite (3.22) as follows

$$\tau U_0\lambda^k - T_+(\lambda^k W)_+ \cdot U + U(\lambda^k W)_+ = T_+(\lambda^k W)_- \cdot U - U(\lambda^k W)_- \quad , \tag{3.34}$$

where

$$(\sum_i f_i\lambda^i)_+ \equiv \sum_{i\geq 0} f_i\lambda^i \quad , \tag{3.35}$$

corresponding to the choice $i_0 = 0$. Since in the present case we have (see (3.10)) $l = 0$ and $h = 1$, both sides of (3.34) contain powers λ^i with $i \in [h + i_0 - 1, l + i_0] = \{0\}$. In other words, both sides of (3.34) are independent of λ. Therefore, for $k > 0$ we obtain

$$\begin{aligned} \tau U_0\lambda^k - T_+(\lambda^k W)_+ \cdot U + U(\lambda^k W)_+ &= \tau U_0\lambda^k - T_+(\lambda^k W)_+ \cdot U + U(\lambda^k W)_+|_{\lambda=0} \\ &= -(T_+W_k)U_1 + U_1(W_k) \\ &\overset{(3.22)}{=} (T_+W_{k+1})U_0 - U_0W_{k+1} \\ &= \begin{bmatrix} (T_+ - 1)a_{k+1} & -b_{k+1} \\ T_+c_{k+1} & 0 \end{bmatrix} . \end{aligned} \tag{3.36}$$

Hence, according to the procedure described above we obtain the following hierarchy of equations

$$U_{1t_k} + (\tau\lambda^k)U_0 - T_+(\lambda^k W)_+ \cdot U + U(\lambda^k W)_+ = 0 \quad , \tag{3.37}$$

or equivalently

$$dq/dt_k = b_{k+1} \quad , \quad dr/dt_k = -T_+c_{k+1} \quad , \quad ds/dt_k + Da_{k+1} - \delta_{k0}\tau = 0 \quad . \tag{3.38}$$

At this point we emphasize that the third equation holds automatically when choosing $s = qr$.

To write out explicitly the first set of equations in this hierarchy (3.38), we make use of (3.33) and obtain (putting $s = qr$):

$$\begin{aligned} dq/dt_1 &= \quad \alpha(T_+q - q^2r) + \tau(n+1)q, \\ dr/dt_1 &= -\alpha(T_-r - r^2q) - \tau nr. \end{aligned} \tag{3.39}$$

Hence, choosing $\tau = 0$, we have identified the simplest iso-spectral equation of the hierarchy of integrable systems associated to the scattering problem (3.20) with $s = qr$

as

$$\frac{d}{dt_1}\begin{bmatrix} q \\ \\ r \end{bmatrix} = \alpha \begin{bmatrix} T_+q - q^2r \\ \\ -T_-r + r^2q \end{bmatrix} \quad . \tag{3.40}$$

The additional non-isospectral term in (3.39)

$$\tau_0 = \begin{bmatrix} (n+1)q \\ \\ -nr \end{bmatrix} \tag{3.41}$$

corresponds to the simplest (scaling) mastersymmetry for (3.40) (see section 4.4). Higher iso-spectral equations are given by (3.38), when inserting the expressions for a_i, b_i, c_i obtained from the recursion (3.30) with $\tau = 0$. For $\tau = 1$ the additional terms correspond to higher mastersymmetries of the iso-spectral hierarchy.

As shown in [22] the iso-spectral hierarchy (3.38) can be written in its Hamiltonian form

$$\frac{d}{dt_k}\begin{bmatrix} q \\ \\ r \end{bmatrix} = P_0 \, \frac{\delta H_{k+1}}{\delta u} \tag{3.42}$$

where

$$P_0 = \begin{bmatrix} 0 & 1 \\ \\ -1 & 0 \end{bmatrix} \quad , \quad H_k = (rb_k - a_k)/k \quad , \tag{3.43}$$

and the corresponding recursion operator Φ: $P_0 \frac{\delta H_n}{\delta u} \to P_0 \frac{\delta H_{n+1}}{\delta u}$ is given by

$$\Phi = \begin{pmatrix} T_+ - 2q(1-T_-)^{-1}r & , & q^2 - 2q(1-T_-)^{-1}q \\ \\ -r^2 + 2r(1-T_-)^{-1}r & , & T_- - 2rq + 2r(1-T_-)^{-1}q \end{pmatrix} \quad . \tag{3.44}$$

Example 2: As second example we take the spectral problem (3.4) with the $(p+1) \times$

$(p+1)$ matrix [23]

$$U = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & u \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \lambda^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} , \tag{3.45}$$

where λ is the spectral parameter and u is the field variable. This is the linear problem underlying a hierarchy of lattice equations considered in [23]. A further discussion will be given in section 4.5.

Applying the analysis described before one obtains the corresponding isospectral hierarchy of equations as

$$du/dt_k = u(T_+ - T_-^p)a_k \quad , \tag{3.46}$$

where $a_k = (W_k)_{p,p}$, the $(p+1) \times (p+1)$-matrix $W = \sum \lambda^{-k} W_k = (W_{ij})$ being a solution of the matrix equation

$$(T_+ W)U - UW = 0 \quad . \tag{3.47}$$

In particular, for the simple cases $p = 1$ and $p = 2$ we obtain the hierarchies

$$du/dt_k = u(T_+ - T_-)a_k \quad , \quad W_k = \begin{bmatrix} a_k & b_k \\ c_k & -a_k \end{bmatrix} \tag{3.48}$$

for $p = 1$ and

$$du/dt_k = u(T_+ - T_-^2)e_k \quad , \quad W_k = \begin{bmatrix} a_k & b_k & c_k \\ d_k & e_k & f_k \\ g_k & h_k & -a_k - e_k \end{bmatrix} \quad . \tag{3.49}$$

for p=2. For these two cases of p the above hierarchies can be written in their Hamiltonian form

$$\frac{d}{dt_k} u = P^{(p)} \ \frac{\delta H_k^{(p)}(u)}{\delta u} \tag{3.50}$$

with Poisson operators

$$P^{(1)} = u(T_+ - T_-)u \quad , \quad P^{(2)} = u(T_+^2 + T_+ - T_- - T_-^2)u \tag{3.51}$$

and Hamiltonian functions

$$H_k^{(1)}(u) = H_k^{(2)}(u) = a_k/u \quad . \tag{3.52}$$

The corresponding recursion operators $\Phi^{(p)}$ are found to be

$$\begin{aligned} \Phi^{(1)} &= u(1 + T_-)(uT_- - T_+u)(1 - T_-)^{-1}u^{-1}; \\ \Phi^{(2)} &= u(1 + T_- + T_-^2)(T_+^2 u - uT_-)(uT_- - T_+u)^{-1}(uT_-^2 - T_+u)(1 - T_-^2)^{-1}u^{-1} \quad . \end{aligned} \tag{3.53}$$

4 Results for some lattice hierarchies

4.1 The mKdV hierarchy

We consider the Hamiltonian lattice equation [19]

$$du/dt = K_1(u) = S(T_+ - T_-)u = P_0 \nabla f_1 \quad , \tag{4.1}$$

where $S = 1 + \epsilon u^2$ with some arbitrary parameter $\epsilon \neq 0$. The Poisson operator P_0 and the Hamiltonian function f_1 are given by

$$P_0(u) = S(T_+ - T_-)S \quad , \quad f_1(u) = \frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} ln(S) \ . \tag{4.2}$$

Applying the Computer Algebra package described before we obtain a symmetry

$$K_2(u) = S(T_+ - T_-)S(T_+ + T_-)u \tag{4.3}$$

and a mastersymmetry

$$\tau_1(u) = [n]K_1(u) + S(T_+ + T_-)u \tag{4.4}$$

satisfying $[\tau_1, K_1] = K_2$. Following the general scheme of mastersymmetries [27], [18], [39] we expect to obtain a hierarchy of higher symmetries for (4.1) by the recursion $K_{i+1} := [\tau_1, K_i]$ starting from K_1. Trying to derive a bi-Hamiltonian formulation for (4.1) using τ_1 [17] we note that this mastersymmetry turns out to be Hamiltonian

$$\tau_1(u) = P_0 \nabla \frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} n \, ln(S) \quad , \tag{4.5}$$

hence $L_{\tau_1} P_0 = 0$ and no second Poisson operator can be derived from τ_1. However, again using the Computer Algebra package a further mastersymmetry

$$\tau_2(u) = [n]K_2 + 2S(T_+ + T_-)S(T_+ + T_-)u - 4Su + 2\epsilon K_1(u)(1 - T_-)^{-1}(uT_-u) \tag{4.6}$$

was found such that

$$K_3 = L_{\tau_2} K_1 - 2K_1 = S(T_+ - T_-)S(\, T_+ST_+u + T_-ST_-u + \epsilon u(u_+ + u_-)^2\,) = \frac{1}{2} L_{\tau_1} K_2 \quad (4.7)$$

(with $u_\pm = T_\pm u$) yields a further symmetry for (4.1). The mastersymmetry τ_2 turns out to be non-Hamiltonian w.r.t. P_0. Hence we obtain a candidate for a second Hamiltonian structure for (4.1) by calculating the operator $P_2 := L_{\tau_2} P_0$. It is of such complicated form that we do not state it explicitly. Nevertheless, using Computer Algebra it was verified that P_2 is indeed a second Poisson operator for (4.1) thus providing a hereditary recursion operator $\Phi = P_2 P_0^{-1}$ for (4.1).

4.2 The Bruschi-Ragnisco hierarchy

We consider the Hamiltonian lattice equation [21]

$$\frac{\partial}{\partial t} \begin{pmatrix} b \\ c \end{pmatrix} = K_1(b,c) = \begin{pmatrix} cT_+b - bT_-c \\ c^2 - cT_-c \end{pmatrix} \quad (4.8)$$

with boundary conditions $b(n) \to 0\,, c(n) \to 1$ as $n \to \pm\infty$. A hereditary recursion operator is given by

$$\Phi(b,c) = \begin{pmatrix} cT_+ & ; & (cT_+b - bT_-c)\,T_-\,(1-T_-)^{-1}c^{-1} \\ 0 & ; & c(1-T_-)cT_-(1-T_-)^{-1}c^{-1} \end{pmatrix} , \quad (4.9)$$

with formal inverse

$$\Phi^{-1}(b,c) = \begin{pmatrix} T_-c^{-1} & ; & (T_-bc^{-1}T_- - bc^{-1})(1-T_-)^{-1}c^{-1} \\ 0 & ; & (cT_+c^{-1} - 1)(1-T_-)^{-1}c^{-1} \end{pmatrix} , \quad (4.10)$$

where $c^{-1}(n) = 1/c(n)$. One finds a symmetry

$$K_{-1}(b,c) \ = \ \begin{pmatrix} T_-(b/c) \ - \ b/c \\ \\ c/(T_+c) - 1 \end{pmatrix} \tag{4.11}$$

of (4.8) lying in the kernel of Φ. The vector field K_1 lies in the kernel of Φ^{-1}. A hierarchy of commuting symmetries K_i , $i \in \mathbb{Z}$, may be defined by $K_{i+1} := \Phi^i K_1$ for $i > 0$, $K_{i-1} := (\Phi^{-1})^{-i} K_{-1}$ for $i < 0$ and $K_0 := 0$. Additional symmetries are found with

$$H_i := \Phi^i H_0 \quad , \quad i \in \mathbb{Z} \quad , \quad H_0(b,c) = \begin{pmatrix} b \\ \\ 0 \end{pmatrix} \quad , \tag{4.12}$$

mastersymmetries are given by

$$\tau_i := \Phi^i \tau_0 \quad , \quad i \in \mathbb{Z} \quad , \quad \tau_0(b,c) = \begin{pmatrix} b \\ \\ c \end{pmatrix} \quad . \tag{4.13}$$

The Lie algebra structure for these vector fields is given by

$$\begin{array}{llllll} [K_i, K_j] = 0 & , & [K_i, H_j] = 0 & , & [H_i, H_j] = 0 & , \\ \\ [\tau_i, K_j] = jK_{i+j} & , & [\tau_i, H_j] = jH_{i+j} & , & [\tau_i, \tau_j] = (j-i)\tau_{i+j} & , \end{array} \tag{4.14}$$

with $i, j \in \mathbb{Z}$. A Poisson operator for (4.8) is given by

$$P_0 \ = \ \begin{pmatrix} cT_+b - bT_-c & , & c(T_+ - 1)c \\ \\ c(1 - T_-)c & , & 0 \end{pmatrix} \quad . \tag{4.15}$$

Using the recursion operator and its inverse a countable family of local Poisson operators is constructed by $P_j := \Phi^j P_0$, $j \in \mathbb{Z}$. One finds

$$L_{K_i} P_j = 0 \quad , \quad L_{H_i} P_j = -P_{i+j} \quad , \quad L_{\tau_i} P_j = (j-i) P_{i+j} \quad . \tag{4.16}$$

Hence all the K_i, $i \in \mathbb{Z}$, are Hamiltonian w.r.t. all the Poisson operators P_j, $j \in \mathbb{Z}$, whereas the H_i and τ_i are non-Hamiltonian. But obviously certain combinations of H_i and τ_i leave the Poisson operators invariant, hence we may introduce Hamiltonian functions f_j and ϕ_j ($j \in \mathbb{Z}$) by

$$K_j = P_0 \nabla f_j \quad , \quad \tau_j - jH_j = P_0 \nabla \phi_j \quad , \tag{4.17}$$

i.e. $K_{i+j} = P_i \nabla f_j \quad , \quad \tau_{i+j} - jH_{i+j} = P_i \nabla \phi_j$. One concludes

$$\begin{array}{llll} < \nabla f_j, K_i >= 0 & , \; < \nabla f_j, H_i >= f_{i+j} & , \; < \nabla f_j, \tau_i >= (i+j) f_{i+j} & , \\ < \nabla \phi_j, K_i >= -i f_{i+j} & , \; < \nabla \phi_j, H_i >= \phi_{i+j} & , \; < \nabla \phi_j, \tau_i >= j \phi_{i+j} & , \end{array} \tag{4.18}$$

with $i, j \in \mathbb{Z}$. From this recursive scheme the functions f_j and ϕ_j may be constructed in a straightforward way from $f_0 = \sum_{n \in \mathbb{Z}} b(n)/c(n)$ and $\phi_0 = \sum_{n \in \mathbb{Z}} n\, b(n)/c(n)$ using H_1 and H_{-1}, say. These functions represent action and angle coordinates for (4.8), their Poisson brackets are calculated as

$$\{f_i, f_j\}_k = 0 \quad , \quad \{f_i, \phi_j\}_k = -(i+k) f_{i+j+k} \quad , \quad \{\phi_i, \phi_j\}_k = (j-i) \phi_{i+j+k} \quad , \quad i, j, k \in \mathbb{Z} \; , \tag{4.19}$$

where $\{f, g\}_k =< \nabla g, P_k \nabla f >$ is the Poisson bracket engendered by the kth Poisson operator P_k.

4.3 The Ablowitz-Ladik hierarchy

We consider the Hamiltonian lattice equation [20]

$$\frac{\partial}{\partial t} \begin{pmatrix} q \\ r \end{pmatrix} = K(q, r) = \begin{pmatrix} \alpha(1-qr)T_+ q - \beta(1-qr)T_- q \\ -\alpha(1-qr)T_- r + \beta(1-qr)T_+ r \end{pmatrix} \quad , \tag{4.20}$$

with arbitrary parameters α and β. The vector field K in (4.20) is a linear combination of two of its symmetries

$$K_1(q,r) = \begin{pmatrix} (1-qr)T_+q \\ -(1-qr)T_-r \end{pmatrix} \quad , \quad K_{-1}(q,r) = \begin{pmatrix} (1-qr)T_-q \\ -(1-qr)T_+r \end{pmatrix} \quad . \tag{4.21}$$

One finds two further symmetries

$$K_2(q,r) = \begin{pmatrix} (1-qr)\,(\;\; T_+(1-qr)T_+q \; - \; rT_+q^2 \; - \; T_-rT_+qT_+q \;) \\ (1-qr)\,(\; -T_-(1-qr)T_-r \; + \; qT_-r^2 \; + \; T_+qT_-rT_-r \;) \end{pmatrix} \quad ,$$

$$K_{-2}(q,r) = \begin{pmatrix} (1-qr)\,(\;\; T_-(1-qr)T_-q \; - \; rT_-q^2 \; - \; T_+rT_-qT_-q \;) \\ (1-qr)\,(-T_+(1-qr)T_+r \; + \; qT_+r^2 \; + \; T_-qT_+rT_+r \;) \end{pmatrix} \quad , \tag{4.22}$$

and two master symmetries

$$\tau_1(q,r) = [n]K_1(q,r) \; + \; \begin{pmatrix} (1-qr)T_+q \; - \; q(1-T_-)^{-1}qT_-r \\ (1-qr)T_-r \; + \; r(1-T_-)^{-1}qT_-r \end{pmatrix} \quad ,$$

$$\tau_{-1}(q,r) = [n]K_{-1}(q,r) \; - \; \begin{pmatrix} (1-qr)T_-q \; + \; q(1-T_-)^{-1}rT_-q \\ (1-qr)T_+r \; - \; r(1-T_-)^{-1}rT_-q \end{pmatrix} \tag{4.23}$$

satisfying $[\tau_1, K_1] = K_2$ and $[\tau_{-1}, K_{-1}] = -K_{-2}$. We apply these mastersymmetries to the first Poisson operator for (4.20) given by

$$P_0(q,r) \; = \; \begin{pmatrix} 0 & (1-qr) \\ -(1-qr) & 0 \end{pmatrix} \tag{4.24}$$

and find two further Poisson operators $P_1 := -L_{\tau_1}P_0$, $P_{-1} := L_{\tau_{-1}}P_0$ for (4.20). The resulting hereditary recursion operators $\Phi := P_1P_0^{-1}$ and $\Phi^{-1} := P_{-1}P_0^{-1}$ turn out to be the formal inverses of one another (observing $\Phi K_{-1} = 0 = \Phi^{-1} K_1$). Putting

$$K_0(q,r) = \begin{pmatrix} q \\ \\ -r \end{pmatrix} \tag{4.25}$$

one defines a set of commuting symmetries $K_i = \Phi^i K_0$, $i \in \mathbb{Z}$, with $K_k := \Phi^k K_0$, $K_{-k} := (\Phi^{-1})^k K_0$, $k \in \mathbb{N}$, and in the same way a set of mastersymmetries $\tau_i := \Phi^i \tau_0$, $i \in \mathbb{Z}$, starting with $\tau_0 := [n] K_0$. The Lie algebra structure generated by these fields turns out to be

$$[K_i, K_j] = 0 \quad , \quad [\tau_i, K_j] = j K_{i+j} \quad , \quad [\tau_i, \tau_j] = (j-i)\tau_{i+j} \quad , \quad i \in \mathbb{Z} . \tag{4.26}$$

The Lie-derivatives of the Poisson operators $P_i := \Phi^i P_0$ into the direction of these vector fields are given by

$$L_{K_i} P_j = 0 \ , \ L_{\tau_i} P_j = (j-i) P_{i+j} \ , \ i,j \in \mathbb{Z} . \tag{4.27}$$

For the Hamiltonian potentials f_j introduced by $K_j = P_0 \nabla f_j$ one finds

$$< \nabla f_j, K_i >= 0 \ , \ \ < \nabla f_j, \tau_i >= (i+j) f_{i+j} \ , \ i,j \in \mathbb{Z} , \tag{4.28}$$

which may also serve as the recursive scheme defining these functions from the first conserved quantities

$$f_{-1} = \sum_{n \in \mathbb{Z}} q(n-1) r(n) \ , \ f_0 = \sum_{n \in \mathbb{Z}} ln(\frac{1}{1 - q(n) r(n)}) \ , \ f_1 = \sum_{n \in \mathbb{Z}} q(n+1) r(n) \ . \tag{4.29}$$

4.4 The Ragnisco-Tu hierarchy

We consider the Hamiltonian lattice equation [22]

$$\frac{\partial}{\partial t} \begin{pmatrix} q \\ \\ r \end{pmatrix} = K_1(q,r) = \begin{pmatrix} T_+ q - q^2 r \\ \\ -T_- r + r^2 q \end{pmatrix} \tag{4.30}$$

derived from the spectral problem (3.20). From the analysis of section 3.2 or using Computer Algebra we obtain a symmetry

$$K_2(q,r) = \begin{pmatrix} T_+^2 q - T_+ q^2 r - q^2 T_- r - 2qrT_+ q + q^3 r^2 \\ -T_-^2 r + T_- r^2 q + r^2 T_+ q + 2rqT_- r - r^3 q^2 \end{pmatrix} \tag{4.31}$$

and a mastersymmetry

$$\tau_1(q,r) = [n]K_1 \; + \begin{pmatrix} 2T_+ q - 2q(1-T_-)^{-1} rq \\ T_- r - r^2 q + 2r(1-T_-)^{-1} rq \end{pmatrix} \tag{4.32}$$

satisfying $[\tau_1, K_1] = K_2$. Using the first Poisson operator

$$P_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.33}$$

derived in (3.43) we obtain a second Poisson operator by applying the mastersymmetry τ_1 to P_0, i.e. defining

$$P_1 := -\frac{1}{2} L_{\tau_1} P_0 = \begin{pmatrix} -q^2 + 2q(1-T_-)^{-1} q & , & T_+ - 2q(1-T_-)^{-1} r \\ -T_- + 2rq - 2r(1-T_-)^{-1} q & , & -r^2 + 2r(1-T_-)^{-1} r \end{pmatrix} . \tag{4.34}$$

The resulting hereditary recursion operator $\Phi = P_1 P_0^{-1}$ coincides with (3.44) derived from the spectral problem (3.20) for (4.30). Starting with

$$K_0(q,r) = \begin{pmatrix} q \\ -r \end{pmatrix} \quad , \quad \tau_0(q,r) = [n]K_0 + \begin{pmatrix} q \\ 0 \end{pmatrix} \tag{4.35}$$

a set of symmetries $K_i := \Phi^i K_0$ and mastersymmetries $\tau_i := \Phi^i \tau_0$ for (4.30) can be constructed satisfying

$$[K_i, K_j] = 0 \quad , \quad [\tau_i, K_j] = jK_{i+j} \quad , \quad [\tau_i, \tau_j] = (j-i)\tau_{i+j} \quad . \tag{4.36}$$

Following [36],[18] the action variables f_j, i.e. the potentials of the Hamiltonian symmetries $K_j = P_0 \nabla f_j$ are obtained by the recursion

$$< \nabla f_j, \tau_i >= (i+j+1) f_{i+j} \quad , \quad i,j \in \mathbb{N}_0 \quad , \quad f_0 = \sum_{n\in\mathbb{Z}} r(n)q(n) \quad . \tag{4.37}$$

4.5 The Bogoyavlensky hierarchy

We consider the lattice equations [23]

$$\begin{aligned}
\frac{\partial}{\partial t} u(n) &= (K_1^{(p)}(u))(n) = u(n)(\sum_{k=1}^{p} u(n+k) - \sum_{k=1}^{p} u(n-k)) \quad , \\
\frac{\partial}{\partial t} v(n) &= (\tilde{K}_1^{(p)}(v))(n) = v(n)(\prod_{k=1}^{p} v(n+k) - \prod_{k=1}^{p} v(n-k)) \quad , \\
\frac{\partial}{\partial t} w(n) &= (\hat{K}_1^{(p)}(w))(n) = w^2(n)(\prod_{k=1}^{p} w(n+k) - \prod_{k=1}^{p} w(n-k)) \quad ,
\end{aligned} \tag{4.38}$$

for some $p \in \mathbb{N}$. Note that for fixed p these 3 equations define the same dynamics, as they are related by the *Bäcklund transformations*

$$u(n) = \prod_{k=0}^{p-1} v(n+k) = \prod_{k=0}^{p} w(n+k) \quad . \tag{4.39}$$

These equations are Hamiltonian with Poisson operators given by

$$\begin{aligned}
P^{(p)}(u) &= u \left(\frac{T_+^{p+1} - 1}{T_+ - 1} - \frac{1 - T_-^{p+1}}{1 - T_-} \right) u = u \, (\sum_{k=1}^{p} T_+^k - \sum_{k=1}^{p} T_-^k \,) u \quad , \\
\tilde{P}^{(p)}(v) &= v \left(\frac{1 - T_-}{1 - T_-^p} \, \frac{T_+^{p+1} - 1}{T_+^p - 1} - \frac{T_+ - 1}{T_+^p - 1} \, \frac{1 - T_-^{p+1}}{1 - T_-^p} \right) v \\
&\qquad = v \, \frac{T_+^{p+1} - T_+^p - T_+ + T_- + T_-^p - T_-^{p-1}}{(T_+^p - 1)(1 - T_-^p)} \, v \quad , \\
\hat{P}^{(p)}(w) &= w \left(\frac{1 - T_-}{1 - T_-^{p+1}} - \frac{T_+ - 1}{T_+^{p+1} - 1} \right) w = w \, \frac{T_+^p - T_+^{p-1} - T_+ + T_- + T_-^{p-1} - T_-^p}{(T_+^p - 1)(1 - T_-^p)} \, w \quad ,
\end{aligned} \tag{4.40}$$

these operators being related via the transformations (4.39). The Hamiltonian function is given by

$$f_1(u) = \tilde{f}_1(v) = \hat{f}_1(w) = \sum_{n\in\mathbb{Z}} u(n) \;=\; \sum_{n\in\mathbb{Z}} \prod_{k=0}^{p-1} v(n+k) \;=\; \sum_{n\in\mathbb{Z}} \prod_{k=0}^{p} w(n+k) \quad . \quad (4.41)$$

Note, that the cases $p = 1$ and $p = 2$ had been discussed in section 3.2 making use of the underlying scattering problem (3.45). Also note, that for $p = 1$ the dynamics of (4.38) coincides with example 4 of [18], the recursive structure of this case was exhibited there by deriving mastersymmetries and a recursion operator.

For $p \geq 2$ one can use the procedure SYM of section 3.1 to find higher symmetries for (4.38). The results for $p = 2$ and $p = 3$ are:

$$\begin{aligned}
K_1^{(2)}(u) &= u(T_+^2 + T_+ - T_- - T_-^2)u; \\
K_2^{(2)}(u) &= SYM(K_1^{(2)}, K_1^{(2)}) \\
&= \quad 2uT_+uT_+u + u^2T_+u + uT_+^2uT_+^2u + uT_+^2uT_+u + uT_+uT_+^2u \\
&\quad -2uT_-uT_-u - u^2T_-u - uT_-^2uT_-^2u - uT_-^2uT_-u - uT_-uT_-^2u \\
&\quad +u^2T_+^2u + uT_+^2u^2 + uT_+u^2 \\
&\quad -u^2T_-^2u - uT_-^2u^2 - uT_-u^2 \ ;
\end{aligned}$$

$$\begin{aligned}
K_1^{(3)}(u) &= u(T_+^3 + T_+^2 + T_+ - T_- - T_-^2 - T_-^3)u \ ; \\
K_2^{(3)}(u) &= SYM(K_1^{(3)}, K_1^{(3)}) \\
&= \quad 2uT_+uT_+u + 2uT_+uT_+^2u + 2uT_+^2uT_+u + uT_+^2uT_+^2u + uT_+^2u^2
\end{aligned}$$

$$-2uT_-uT_-u - 2uT_-uT_-^2u - 2uT_-^2uT_-u - uT_+^2uT_+^2u - uT_-^2u^2$$

$$+u^2T_+^2u + uT_+u^2 + u^2T_+u + u^2T_+^3u + uT_+^3u^2 + uT_+^3uT_+u$$

$$-u^2T_-^2u - uT_-u^2 - u^2T_-u - u^2T_-^3u - uT_-^3u^2 - uT_-^3uT_-u$$

$$+uT_+uT_+^3u + uT_+^3uT_+^2u + uT_+^2uT_+^3u + uT_+^3uT_+^3u$$

$$-uT_-uT_-^3u - uT_-^3uT_-^2u - uT_-^2uT_-^3u - uT_-^3uT_-^3u \quad ;$$

$$\begin{aligned}
\tilde{K}_1^{(2)}(v) &= v(T_+vT_+ - T_-vT_-)v; \\
\tilde{K}_2^{(2)}(v) &= SYM(\tilde{K}_1^{(2)}, \tilde{K}_1^{(2)}) \\
&= vT_+vT_+vT_+v^2 + vT_+vT_+v^2T_+v + v^2T_+v^2T_+v + vT_+v^2T_+v^2 \\
&\quad -vT_-vT_-vT_-v^2 - vT_-vT_-v^2T_-v - vT_-v^2T_-v^2 - v^2T_-v^2T_-v \\
&\quad +v^2T_-vT_+^2vT_+v \\
&\quad -v^2T_+vT_-^2vT_-v^2 \ ;
\end{aligned}$$

$$\begin{aligned}
\tilde{K}_1^{(3)}(v) &= v(T_+vT_+vT_+ - T_-vT_-vT_-)v; \\
\tilde{K}_2^{(3)}(v) &= SYM(\tilde{K}_1^{(3)}, \tilde{K}_1^{(3)}) \\
&= vT_+vT_+vT_+vT_+vT_+v^2 + vT_+vT_+vT_+v^2T_+vT_+v + v^2T_+v^2T_+v^2T_+v \\
&\quad -vT_-vT_-vT_-vT_-vT_-v^2 - vT_-vT_-vT_-v^2T_-vT_-v - v^2T_-v^2T_-v^2T_-v \\
&\quad +vT_+v^2T_+v^2T_+v^2 + v^2T_-vT_+^2v^2T_+vT_+v + v^2T_-vT_-vT_+^3vT_+vT_+v \\
&\quad -vT_-v^2T_-v^2T_-v^2 - v^2T_+vT_-^2v^2T_-vT_-v - v^2T_+vT_+vT_-^3vT_-vT_-v \ .
\end{aligned}$$

$$
\begin{aligned}
\hat{K}_1^{(2)}(w) &= w^2(T_+wT_+ - T_-wT_-)w; \\
\hat{K}_2^{(2)}(w) &= SYM(\hat{K}_1^{(2)}, \hat{K}_1^{(2)}) \\
&= \quad w^3T_+wT_+w^2T_+wT_+w + w^2T_+w^2T_+w^2T_+w + w^3T_+w^2T_+w^2 + w^3T_-wT_+^2w^2T_+w \\
&\quad -w^3T_-wT_-w^2T_-wT_-w - w^2T_-w^2T_-w^2T_-w - w^3T_-w^2T_-w^2 - w^3T_+wT_-^2w^2T_-w \ ;
\end{aligned}
$$

$$
\begin{aligned}
\hat{K}_1^{(3)}(w) &= w^2(T_+wT_+wT_+ - T_-wT_-wT_-)w; \\
\hat{K}_2^{(3)}(w) &= SYM(\hat{K}_1^{(3)}, \hat{K}_1^{(3)}) \\
&= \quad w^2T_+wT_+wT_+w^2T_+wT_+wT_+w + w^2T_+w^2T_+w^2T_+w^2T_+w + w^3T_+w^2T_+w^2T_+w^2 \\
&\quad -w^2T_-wT_-wT_-w^2T_-wT_-wT_-w - w^2T_-w^2T_-w^2T_-w^2T_-w - w^3T_-w^2T_-w^2T_-w^2 \\
&\quad +w^2T_+wT_+w^2T_+w^2T_+wT_+w + w^3T_-wT_-wT_+^3w^2T_+wT_+w + w^3T_-wT_+^2w^2T_+w^2T_+w \\
&\quad -w^2T_-wT_-w^2T_-w^2T_-wT_-w - w^3T_+wT_+wT_-^3w^2T_-wT_-w - w^3T_+wT_-^2w^2T_-w^2T_-w \quad .
\end{aligned}
$$

Higher symmetries for these cases as well as symmetries for the equations (4.38) with $p > 3$ were found using the Computer Algebra package. The resulting vector fields are of such complicated form that we do not exhibit them here.

All the equations (4.38) obviously admit a conformal symmetry generated by the scaling mastersymmetry $\tau_0^{(p)}(u) = u$, and $\tilde{\tau}_0^{(p)}(v) = v$, $\hat{\tau}_0^{(p)}(w) = w$, respectively.

For the case $p = 2$ a recursion operator $\Phi^{(2)}$ given by (3.53) was derived for the vector field $K_1^{(2)}$. Applying this recursion operator to the scaling mastersymmetry $\tau_0^{(2)}(u) = u$

one finds a first non-trivial mastersymmetry

$$\tau_1^{(2)}(u) \;=\; 1/2\ [n]K_1^{(2)}(u) + u^2 + 2uT_+u + 2uT_+^2u + 1/2\ uT_-u + 1/2\ uT_-^2u$$

$$+\, 1/2\ u(T_+ - T_-^2)uT_-(T_+u - uT_-)^{-1}u \tag{4.42}$$

satisfying $[\tau_1^{(2)}, K_1^{(2)}] = K_2^{(2)}$ with $K_2^{(2)}$ as given before. The Lie derivative of the Poisson operator $P^{(2)}$ given by (4.40) along $\tau_1^{(2)}$ yields

$$Q^{(2)} \;:=\; -\ L_{\tau_1^{(2)}}P^{(2)} =$$

$$u(1 + T_- + T_-^2)(T_+^2u - uT_-)(uT_- - T_+u)^{-1}(uT_-^2 - T_+u)(1 + T_+ + T_+^2)u\ \ . \tag{4.43}$$

This operator is checked to be a further Poisson operator for (4.38). Thus, by construction, it is compatible with $P^{(2)}$ and hence $\Phi^{(2)} = Q^{(2)}(P^{(2)})^{-1}$ (coinciding with (3.53)) yields a hereditary recursion operator for (4.38). The bi-Hamiltonian formulation and the hereditary recursion operators for the dynamics (4.38) represented in the coordinates v and w now can easily be calculated from (4.39) using the usual transformation laws [40]. Setting $K_{i+1}^{(2)} := (\Phi^{(2)})^i K_1^{(2)}$, $\tau_i^{(2)} := (\Phi^{(2)})^i \tau_0^{(2)}$, $i = 0, 1, ..$, one finds a hierarchy of Hamiltonian symmetries $K_i^{(2)} = P^{(2)}\nabla f_i^{(2)}$ for (4.38) as well as a hierarchy of non Hamiltonian mastersymmetries $\tau_i^{(2)}$. The element $K_2^{(2)}$ coincides with the above symmetry found by the Computer Algebra package. The conserved Hamiltonians f_i in involution can be found recursively from the scheme

$$< \nabla f_i^{(2)}, \tau_j^{(2)} >= (i + j) f_{i+j}^{(2)}\ \ ,\ \ \ i = 1, 2, ..\ ;\ j = 0, 1, ..\ . \tag{4.44}$$

The first of these conserved quantities are given by

$$
\begin{aligned}
f_1^{(2)} &= \sum_{n\in\mathbb{Z}} u(n) \;\; ; \;\; f_2^{(2)} = \sum_{n\in\mathbb{Z}} u(n) \; (\; \frac{u(n)}{2} + u(n+1) + u(n+2) \;) \;\; ; \\
f_3^{(2)} &= \sum_{n\in\mathbb{Z}} u(n) \; (\; \frac{u(n)^2}{3} + u(n)u(n+1) + u(n)u(n+2) + u(n+1)^2 + u(n+2)^2 \\
&\quad + 2u(n+1)u(n+2) + u(n+1)u(n+3) + u(n+2)u(n+3) \\
&\quad + u(n+2)u(n+4) \;) \; .
\end{aligned}
\tag{4.45}
$$

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