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# The action-angle transformation for interacting solitons and the dynamics of eigenfunctions for soliton equations

Benno Fuchssteiner
Boris Konopelchenko*
University of Paderborn
D 4790 Paderborn
Sandra Carillo
Dipartimento di M.M.M. per le Sc. Appl.
Universita La Sapienza
I 00161 Roma

**Abstract**

A method is presented which allows the explicit construction of the gradients of action and angle variables related to the dynamics of soliton eigenfunction equations. The interacting soliton equations, as well as the Lax-pair eigenfunctions, related to a number of known completely integrable systems are taken as examples to illustrate the method. Among these are the Korteweg de Vries, the mKdV, the nonlinear Schrödinger equation and the ZS-AKNS-system.

## 1 Introduction

Completely integrable flows on infinite dimensional manifolds are generally called *soliton equations* because, under suitable boundary conditions at infinity, solutions often decompose asymptotically into traveling waves. These asymptotically emerging traveling waves are termed *solitons* and if a solution decomposes completely into solitons, it is termed a *multisoliton*. Here, by complete decomposition we mean that there is some suitable energy-norm such that all the energy is carried by the asymptotic solitons.

Interaction of these solitons has been widely studied and, indeed, the very name *soliton* has been chosen referring to them since reductions of these systems

*Permanent address: Institute of Nuclear Physics, Novosibirsk -90, 630090, USSR
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to finite dimensional invariant submanifolds (i.e. to multisoliton-manifolds) often behave like interacting single particles.

Thus, in order to motivate the aim of this paper, let us consider, as an example, the analogy between the multisoliton solutions of a well known integrable system, say the Korteweg de Vries equation $u_t = u_{xxx} + 6uu_x$, with a field consisting of several particles. A plot of a typical two soliton solution $u(x,t)$ of the KdV equation is given by fig. 1.[1]

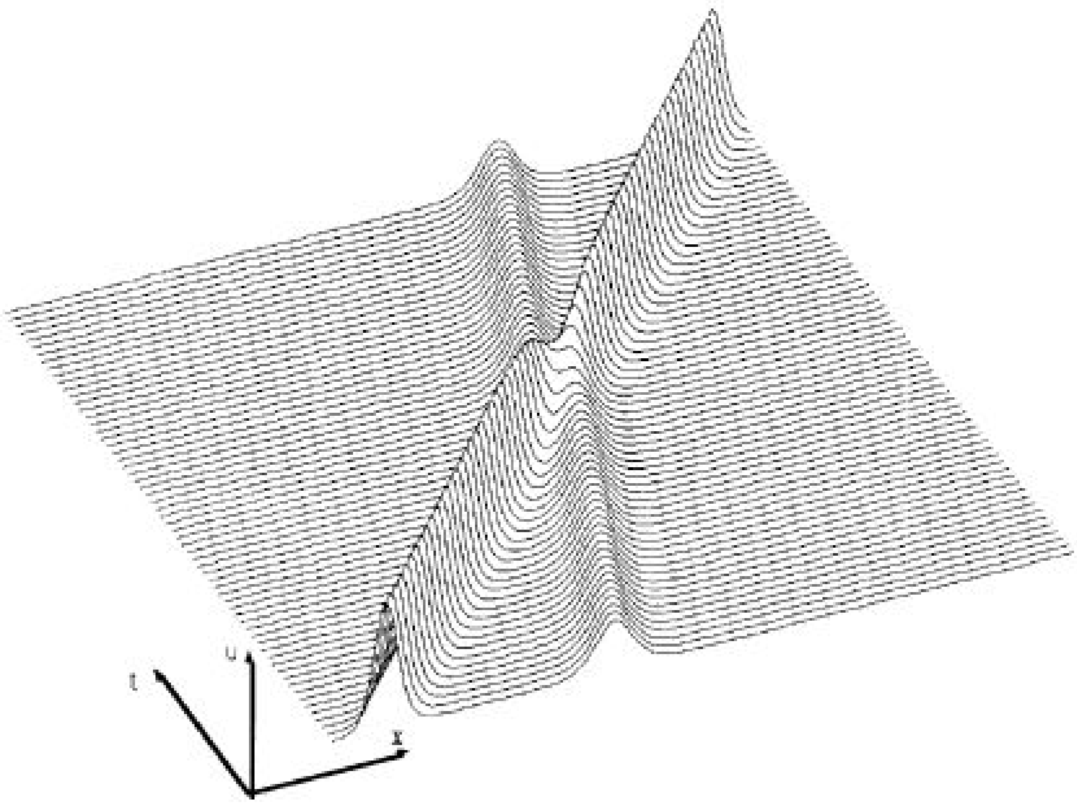


Fig. 1: Two-soliton solution of the KdV equation

Certainly, to study the interaction of these *particles* one would rather like to be able to look at the individual particle during the interaction instead of considering the superposition of all particles. Since the multisoliton solution in itself corresponds to the superposition one easily gets the idea that consideration of the individual soliton should give a better understanding of soliton interaction.

[1] We are indebted to Thorsten Schulze for plotting these figures.

The only problem which remains is how to *individualize* the interacting soliton because a look on the full solution only gives asymptotic solitons, or particles.

Indeed, picking out the individual soliton, even during its interaction, is possible by the use of group theoretical methods (see [7]). The method consists in the identification of the eigenvectors of the so called *recursion operator* with the x-derivatives of the interacting solitons. These eigenvectors are called *interacting solitons* and the group theoretical meaning of the recursion operator induces a nonlinear evolution equation governing the time evolution of these interacting solitons. One of the interacting solitons in the case of the two soliton solution (fig.1) is represented in fig. 2.

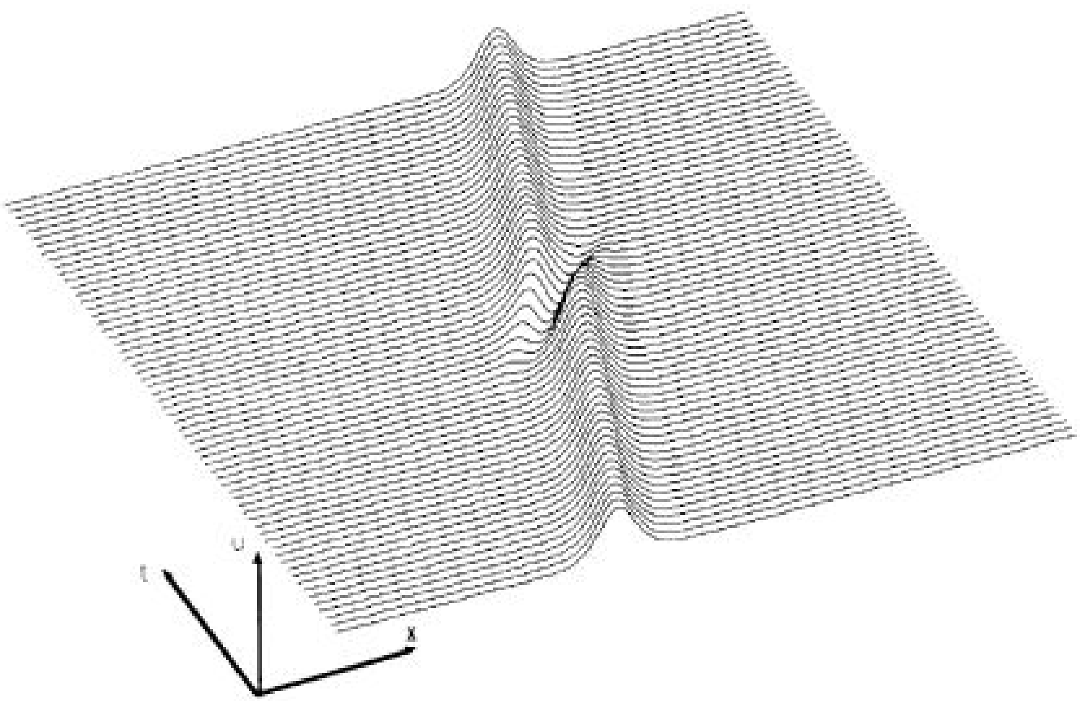


Fig. 2: Interacting soliton of the KdV equation

One should observe that the only field-variable entering in these interacting soliton equations is the soliton itself, not the variable corresponding to the superposition of solitons. The interaction is reflected by the nonlinear terms of this equation. Thus, each interacting soliton is described by a nonlinear flow where only *self-interaction* appears, and the information about the existence of other solitons in the superposed field is hidden in the initial data. Thus small variations of initial data characterize different states of these interacting solitons.

In addition, via this procedure, nonlinear equations governing the evolution of the interacting soliton are obtained in such a way that all parts of the remarkable structure (Hamiltonian formulation, complete integrability, [hereditary] recursion operator, angle variable, auto Bäcklund transformation, etc.) admitted by the original evolution are carried over to the evolution of the interacting soliton. Other advantages of this viewpoint are:

- The analysis of the evolution of the interacting soliton leads to a simple qualitative description of the interaction.
- This analysis gives a simple way to define trajectories described by the moving solitons.
- One is able to find new completely integrable systems and obtains new insight into the complete integrability of flows on infinite dimensional manifolds.

However, an important problem still remains to be investigated, namely the complete construction of the action-angle parametrization related to the flows given by the interacting solitons. In principle, given a finite dimensional flows such an action-angle representation is the most important consequence of complete integrability. Therefore, the complete integrability in the Liouville sense [2] is the most important feature of the dynamics of the multisoliton systems (see e.g. [1], [3], [18], [13]). The inverse spectral transform (IST) method allows to construct the action-angle variables via the corresponding inverse problem data. Another, more algebraic method to construct the action-angle variables (in multi-soliton sector) has been recently proposed ([8], [9], [19]). Within such an algebraic approach the gradients of the action and angle variables are the different eigenvectors of the transposed hereditary [5] recursion operator [20]. However, a direct transfer of these methods to the cases given by the flows of interacting solitons is difficult from the computational viewpoint since the recursion operators are usually of a very complicated nature.

On the other hand, in most cases, it is possible to construct a transformation from gradients of action variables to gradients of angle variables explicitly [10]. This action-angle transformation is an important ingredient to obtain the soliton dynamics as well as for the spectral resolution of the recursion operator. Since the symmetry groups corresponding to angle variables are the prototypes of mastersymmetries this transformation is closely connected with the mastersymmetry approach [6].

In this paper, in the context of the inverse scattering transform (IST), we exploit a method which is based on the fact that the recursion operators delivers an isospectral formulation of the flows under consideration. Thus, a generalization of the problem of constructing the action-angle transformation in case of the interacting soliton is the construction of the action-angle transformation in the case of the dynamics of the eigenvectors of isospectral operators for completely integrable flows. This problem is solved in this paper by considering a sequence of Bäcklund transformations.

Within the framework of the IST method, soliton equations are associated with the system of linear equations

$$
\begin{aligned}
L_1(u;\lambda)\psi &= 0, && (1.1)\\
L_2(u,\lambda)\psi &= 0. && (1.2)
\end{aligned}
$$

where $L_1$ and $L_2$ usually are differential operators and $\lambda$ is a spectral parameter (see e.g. [1], [3], [18], [13]). Elimination of the eigenfunction $\psi$ gives rise to the integrable soliton equation for $u$. Thus a flow on an infinite dimensional manifold, here throughout termed *$u$-manifold* is defined.

Recently it was shown that the eigenfunctions $\psi$ also obey nonlinear equations which are again solvable by the IST method ([14], [15], [16]). Eigenfunction equations have been constructed for many 1+1 and 2+1-dimensional soliton equation ([14], [15], [16]). These equations possess a number of interesting properties, in particular, the analysis of the related Hamiltonian structure is an interesting problem.

In this paper we propose a method for constructing explicitly the action-angle transformation of the $\psi$-dynamic. Such a method is based on the knowledge of the action-angle transformation for the $u$-manifold and uses the explicit transformation from $u$-manifold to $\psi$-manifold. The general scheme can be illustrated by the following figures 3 to 5. Expressing $u$ by $\psi$ and writing the latter as $u = F(\psi)$ we obtain the following transformation scheme

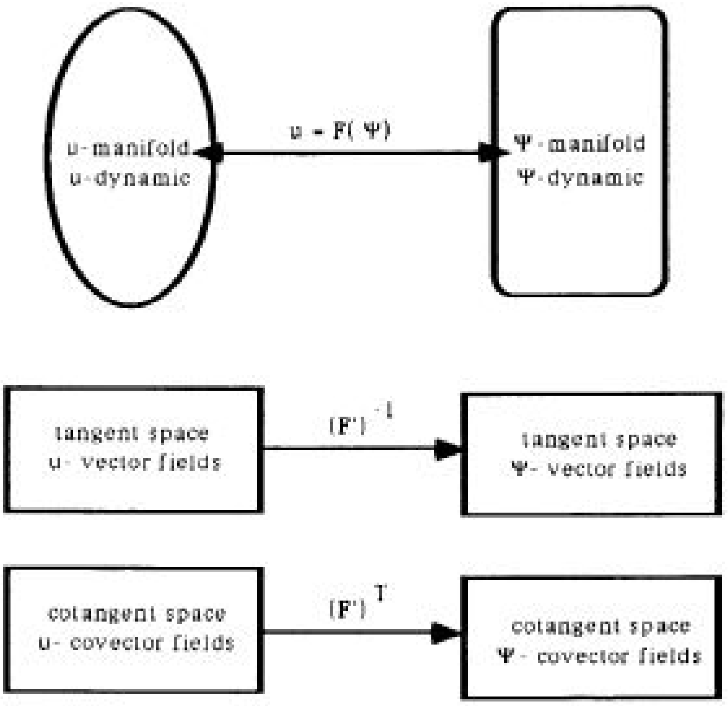


Fig. 3

where $F'$ denotes the variational derivative of $F$ and $F'^T$ its transposed. The map $F'^T$ yields the transformation formula for covector fields. Hence this map transforms gradients of scalar fields on the $u$-manifold into corresponding gradients of scalar fields on the $\psi$-manifold, in particular it transforms $u$-gradients $\nabla_u$ into $\psi$-gradients $\nabla_\psi$.

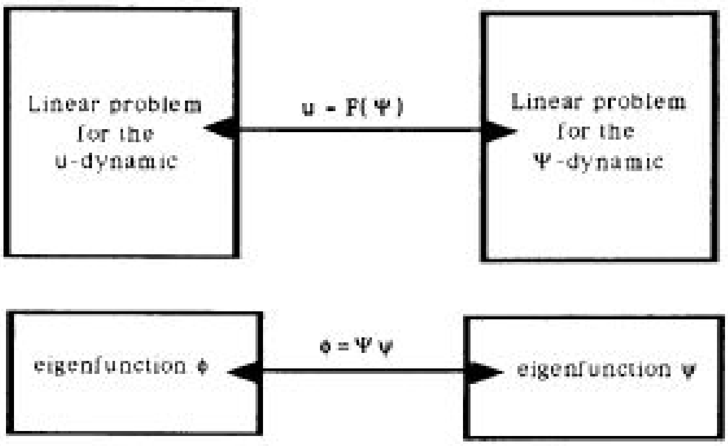


Fig. 4

Figure 4 represents the interrelation between the auxiliary linear problems for the $u$-dynamics and $\psi$-dynamics as it can be deduced from [14], [15] and [16]. Figure 4 is the central one since it gives explicitly the relation between the eigenfunctions for the Lax representations for the flows on the $u$- and $\psi$-manifolds. Let us briefly show how this fundamental relation is explained.

We rewrite (1.1) and (1.2) as

$$L(u)\psi = \lambda\psi \tag{1.3}$$

$$(\partial_t - B(u))\psi = 0 \tag{1.4}$$

thus emphasizing the isospectral operator $L$ and those parts which depend explicitly on the independent variable $t$. These equations represent a set of compatibility conditions; however we can change our viewpoint slightly by considering (1.3) as an implicit relation between $u$ and $\psi$. The latter solved in terms of the potential $u$ instead of the eigenfunction $\psi$ gives the manifold transformation we are looking for

$$u = F(\psi)\ . \tag{1.5}$$

Inserting this in (1.4) produces the dynamics of $\psi$. Now, let us find a Lax representation for that $\psi$-dynamic. Since the change from $u$ to $\psi$ can be understood as just a reparametrization of our original $u$-manifold, the $\psi$-dynamic can be understood as a representation of the original dynamics in different coordinates (on some infinite dimensional manifold, however). Hence the operator $\Lambda(\psi) = L(F(\psi))$ must be an isospectral one for the $\psi$-dynamic. Thus, the following *intermediate Lax representation* follows

$$\Lambda(\psi)\phi = \mu\phi \tag{1.6}$$

$$(\partial_t - B(F(\psi)))\phi = 0 \tag{1.7}$$

where now $\mu$ is some arbitrary spectral parameter (not to be mixed up with the special $\lambda$ used to determine $\psi$). However, this is not yet the *triad representation* for the $\psi$-dynamics introduced in ([15] and [16]); in fact this Lax pair representation is of unnecessarily high order in its derivatives. We can easily reduce the order of derivatives by observing that for the special choice $\mu = \lambda$, one solution of that system is given by $\phi = \psi$. Thus, knowing one solution, the *method of variation of constants* can be applied to reduce the order of derivatives., i.e. the ansatz

$$\phi = \psi\varphi \tag{1.8}$$

leads to a lower order Lax pair representation for the $\psi$-dynamic. This variation of constant method, of course, is the same as the *gauge transformation* introduced in [15] and [16].

Finally, on use of all the transformations we have up to now, and, in addition, of the action-angle transformation on the $u$-manifold we obtain the following scheme for the construction of the action-angle transformation on the $\psi$-manifold.

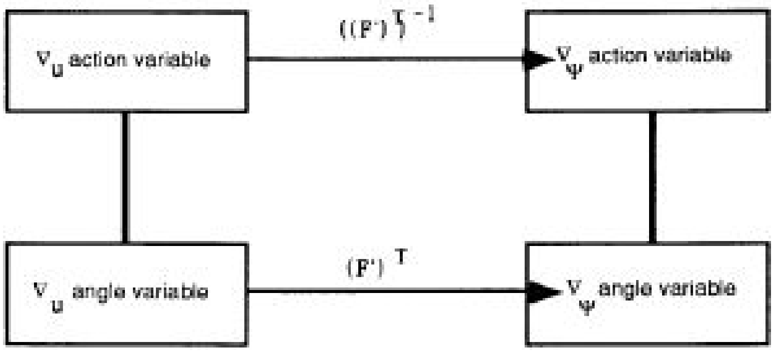


Fig. 5

Thus a canonical map which gives an action-angle representation for the eigenfunction dynamics in terms of the spectral data of the triad representation for these flows is constructed.
To illustrate this method we will consider here the Korteweg-de Vries (KdV), the mKdV and the nonlinear Schrödinger (NLS) equation and, in addition, the ZS-AKNS system.

## 2 The KdV Lax pair eigenfunctions

The KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{2.1}$$

is equivalent to the compatibility condition for the linear system (see e.g. [1], [3], [18], [13]

$$L_1(u)\psi := (\partial^2 + u(x,t) - \lambda)\psi = 0, \tag{2.2}$$

$$L_2(u)\psi := (-\partial_t + 4\partial^3 + 6u\partial + 3u_x)\psi = 0. \tag{2.3}$$

where $\partial \equiv \partial/\partial x$.

An elimination of $\psi$ from (2.2)-(2.3) leads, as usual, to the KdV equation (2.1). Now, $u$ can be expressed in terms of $\psi$ by

$$u = F(\psi) := \lambda - \frac{\psi_{xx}(\lambda)}{\psi(\lambda)} \tag{2.4}$$

which substituted into (2.3) gives the nonlinear equation for the eigenfunction $\psi$ (see [14] and [15] )

$$-\psi_t + 6\lambda\psi_x + \psi_{xxx} - 3\psi^{-1}\psi_x\psi_{xx} = 0 \tag{2.5}$$

The $u$-dynamics of the KdV equation (2.1) and the $\psi$-dynamic, described by the KdV-eigenfunction equation (2.5), can be considered as the two irreducible forms of the mixed $u - \psi$-dynamic, given by the system (2.2) and (2.3).

In the $u$-manifold the gradient of the action variable is the squared eigenfunction $s$ and the gradient of the corresponding angle variable, then, is given by the action-angle transformation $\mathcal{A}(s) = s\partial^{-1}s^{-1}$ [10]. To convert these formulae to the $\psi$-manifold, according to figure 4 it is needed to construct the auxiliary linear system for equation (2.5). This can be done in various ways (see [15] and [16]). Here we will use the well known method of variation of constant as used [7].

We select $\lambda$ and fix by that a particular eigenfunction $\psi$. Furthermore we denote by $\phi$ the eigenfunction appearing in (2.2)-(2.3) for arbitrary spectral $\mu$. Then we substitute the expression (2.4) into the linear system (2.2)-(2.3) for $\phi$ and obtain:

$$L_1(F(\psi))\phi = (\partial^2 - \frac{\psi_{xx}(\lambda)}{\psi(\lambda)} + \lambda - \mu)\phi = 0, \tag{2.6}$$

$$L_2(F(\psi))\phi = (-\partial_t + 4\partial^3 + 6F(\psi)\partial + 3F_x(\psi))\phi = 0\ . \tag{2.7}$$

The highest order derivative $\psi_{xxx}$ can be eliminated from (2.6)-(2.7) by the method of variation of constant, i.e. by the change $\phi = \psi\varphi$. Substituting this expression for $\phi$ into (2.6), we obtain the following system

$$L_1^\psi \varphi := (\psi\partial^2 + 2\psi_x\partial + \lambda\psi - \mu\psi)\varphi = 0 \tag{2.8}$$
$$L_2^\psi \varphi := (-\psi\partial_t + 4\psi\partial^3 + 12\psi_x\partial^2 + 6(\psi_{xx} + \lambda\psi)\partial)\varphi = 0 \ . \tag{2.9}$$

The system (2.8)-(2.9) is exactly the eigenvalue problem we are interested in. Indeed, the compatibility condition for (2.8)-(2.9) or, equivalently, the quartet operator equation $[L_1^\psi, L_2^\psi] = \gamma_1 L_1^\psi + \gamma_2 L_2^\psi$ where $\gamma_1$ and $\gamma_2$ are suitable operators (as described in ([14], [15] and [16] ), is equivalent to the eigenfunction equation (2.5). Here [ , ] denotes the commutator between operators. The important by-product of this derivation is the interrelation

$$\phi = \psi\varphi \tag{2.10}$$

between the eigenfunctions $\phi$ and $\varphi$ of the Lax pair representations for the dynamics of $u$ and $\psi$, respectively. The formulae (2.4) and (2.10) play a central role in our construction. Indeed, the relation (2.10) allows to express the gradients of actions and angles on $\psi$-manifold in terms of the eigenfunctions $\varphi$. Recall that for the KdV equation the eigenfunction $s$ of the recursion operator is a gradient of an action variable and that the corresponding angle variable is given by [10]

$$\mathcal{A}(s) = s\partial^{-1}s^{-1} \ . \tag{2.11}$$

The well known squared eigenfunction relation $s = \phi^2$ between Lax pair representation and recursion operator representation induces

$$\begin{aligned} s &= \psi^2\varphi^2, \\ \mathcal{A}(s) &= \psi^2\varphi^2\partial^{-1}\psi^{-2}\varphi^{-2}. \end{aligned} \tag{2.12}$$

Thus, the action-angle transform $s \to \mathcal{A}(s)$ can be easily expressed in terms of $\psi$ and $\phi$. In order to represent cotangent fields in a suitable way by functions we fix the representation for the duality between covector fields $\gamma$ and vector fields $v$ to be

$$< \gamma, v >:= \int_{-\infty}^{+\infty} \gamma(x)v(x)dx \ .$$

Then formula (2.4) yields for the transposed of the variational derivative of the map from $\psi$ to $u$ the following

$$(F')^T(\psi) = \left(\psi^{-2}(\psi_{xx} - \psi\partial^2)\right)^T = -\psi^{-1}\partial\psi^2\partial\psi^{-2} \tag{2.13}$$

wherein the factorized expression for $F'^T(\psi)$ can be obtained on use of the factorization formula for ordinary differential operators derived in [11].

Finally, combining the formulae (2.11) and (2.12) we obtain according to figure 5, the following expressions for the gradients of action and angle variables of the time-evolution for the function $\psi$ determined by (2.2)-(2.3) and a fixed spectral point $\lambda$

$$\begin{aligned} \nabla_\psi \text{ action} &= -\psi^{-1}\partial\psi^2\partial\varphi^2, && (2.14) \\ \nabla_\psi \text{ angle} &= -\psi^{-1}\partial\psi^2\partial\varphi^2\partial^{-1}\psi^{-2}\varphi^{-2} && (2.15) \end{aligned}$$

Since these variables are parametrized by the spectral parameter $\mu$ (or rather the corresponding eigenfunction $\phi$) this action-angle representation is compatible with the soliton decomposition of the $\psi$-dynamic, i.e. these gradients are eigenfunctions of the transposed of the recursion operator for the $\psi$-evolution. Hence formulae (2.14)-(2.15) give important information about the $\psi$-dynamics and they will be useful for the further analysis of the structure of the KdV-eigenfunction equation.

# 3 The KdV interacting soliton equation

It is well known ([20], [12], [5] and [13]) that the KdV equation possesses also another isospectral representation. This representation is closely connected with the symmetry properties of the system, and it is given by the hereditary recursion operator $\Phi(u) = \partial^2 + 2u + 2\partial u \partial^{-1}$. The recursion property implies that the KdV equation is equivalent to the following dynamic

$$\Phi_t = [K', \Phi] \tag{3.1}$$

where $K(u) = u_{xxx} + 6uu_x$ is the right hand side of the KdV equation. Equation (3.1) is the compatibility condition for the linear system

$$\begin{aligned} (\Phi - 4\lambda)\chi &= 0, && (3.2) \\ (-\partial_t + K')\chi &= 0 \,. && (3.3) \end{aligned}$$

The integro-differential system (3.2)-(3.3) is equivalent to the following pure differential system

$$\begin{aligned} \Phi_1(\psi)s : &= (\partial^3 + 4u\partial + 2u_x - 4\lambda\partial)s = 0, && (3.4) \\ \Phi_2(\psi)s : &= (-\partial_t + \partial^3 + 6u\partial)s = 0 \,. && (3.5) \end{aligned}$$

The compatibility condition for the system (3.4)-(3.5), together with the triad operator representation $[\Phi_1, \Phi_2] = 6u_x\Phi_1$ again is equivalent to the KdV equation (2.1).

So, the $u$-dynamics for the system (3.4)-(3.5) is the same as for linear system (2.2)-(2.3). But the eigenfunction dynamic, of course, is a different one.

One can express $u$ by $s$ (see [7]):

$$u = F(s) := \lambda - \frac{1}{2}\frac{s_{xx}}{s} + \frac{1}{4}\frac{s_x^2}{s^2}. \tag{3.6}$$

Substituting this expression for $u$ into the second equation (3.5), one obtains

$$s_t = 6\lambda s_x + s_{xxx} - 3s^{-1}s_x s_{xx} + \frac{3}{2}\frac{s_x^3}{s^2} \ . \tag{3.7}$$

This equation describes the dynamics of the interacting KdV solitons.

To apply our method to equation (3.7) one should first find the auxiliary linear problem for (3.5) and the interrelation between the KdV-eigenfunction and interacting solitons equation eigenfunction. Thus, let us denote by $s$ the eigenfunction in (3.4)-(3.5) corresponding to fixed $\lambda$ and let $s(\mu)$ denote this eigenfunction for variable $\mu$. The eigenfunction $s(\mu)$, corresponding to $\mu$ , can be obtained on application of the procedure already outlined; on substitution of (3.6) in (3.4)-(3.5) it gives

$$\left\{\partial^3 + (4\lambda - 2\frac{s_{xx}}{s} + \frac{s_x^2}{s^2})\partial - (\frac{s_{xx}}{s} - \frac{1}{2}\frac{s_x^2}{s^2})_x - 4\mu\partial\right\} s(\mu) = 0 \tag{3.8}$$

$$(-\partial_t + \partial^3 + (6\lambda - 3\frac{s_{xx}}{s} + \frac{3}{2}\frac{s_x^2}{s^2})\partial)s(\mu) = 0 \ . \tag{3.9}$$

The highest order derivatives of $s$ are eliminated by the ansatz

$$s(\mu) = s\varphi \tag{3.10}$$

which defines the new eigenfunction $\varphi = \varphi(\mu)$.

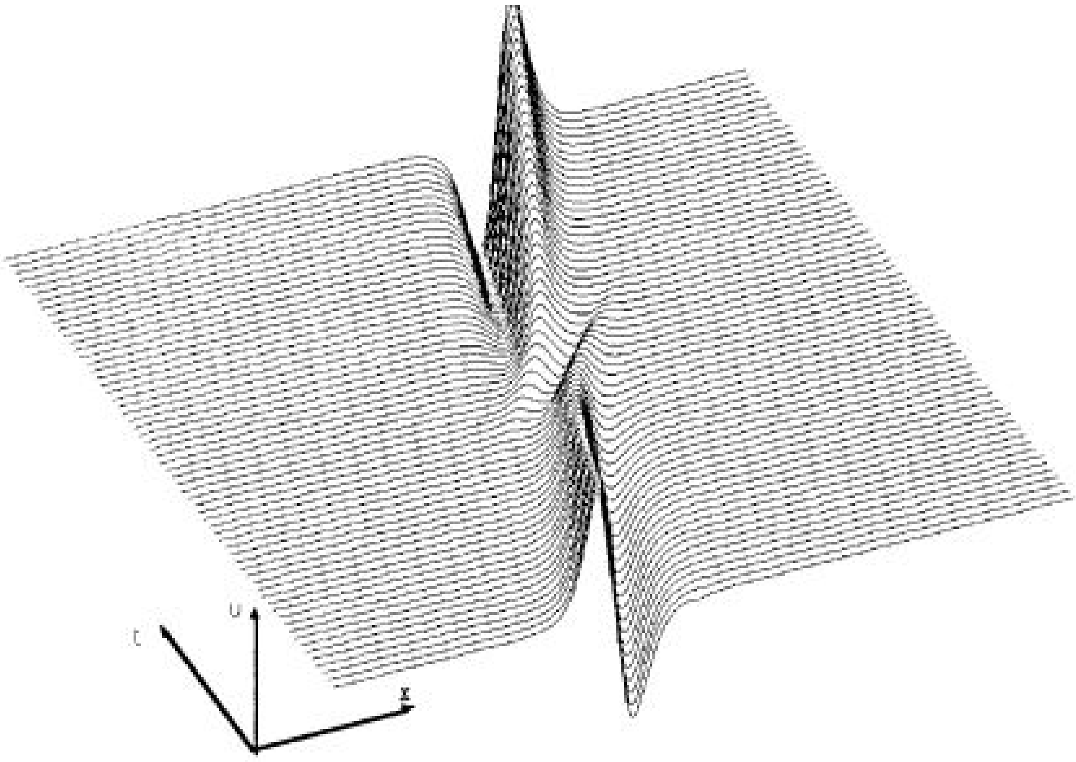


Fig. 6: Derivative of Angle-variable density of the KdV equation

As a result, one gets the following isospectral problem for equation (3.7):

$$\Phi_1(s)\varphi := \left( s\partial^3 + 3s_x\partial^2 + \{4(\lambda-\mu)s + s_{xx} + \frac{s_x^2}{s}\}\partial + 4(\lambda-\mu)s_x \right)\varphi = 0 \tag{3.11}$$

$$\Phi_2(s)\varphi := \left( -s\partial_t + s\partial^3 + 3s_x\partial^2 + \left\{ 6\lambda s + \frac{3}{2}\frac{s_x^2}{s} \right\}\partial \right)\varphi = 0 \ . \tag{3.12}$$

Again this has a quartet operator representation $[\Phi_1, \Phi_2] = \gamma_1\Phi_1 + \gamma_2\Phi_2$ with suitable operators $\gamma_1$ and $\gamma_2$. Relation (3.6) gives (see [7]):

$$F(s)'^T = -\frac{1}{2}s^{-1}\partial s\partial s^{-1} \ . \tag{3.13}$$

Now, taking into account that gradients of action variables on the $u$-manifold are given by the $s(\mu)$ and using the formulae (3.10) and (3.13), one obtains the following expression for action-angle gradients for the $s$-dynamic:

$$\nabla_s \text{action} = -\frac{1}{2}s^{-1}\partial s\partial\varphi, \tag{3.14}$$

$$\nabla_s \text{angle} = -\frac{1}{2}s^{-1}\partial s\partial\varphi\partial^{-1}s^{-1}\varphi^{-1} \ . \tag{3.15}$$

A plot of the $x$-derivative of the density of the angle variable which corresponds to the density of the action variable given by figure 2 is seen in figure 6. In that plot one easily observes that this quantity grows linear with time $t$. The $\psi$-dynamics and $s$-dynamics considered above are closely related to each other via the squared eigenfunction relation. This relation, in particular, implies $\nabla_s = 1/(2\psi)\nabla_\psi$. A simple computation shows that the expressions (2.13)-(2.14) and (3.14)-(3.15) are connected via this relation.

# 4 The mKdV interacting solitons

Concerning the mKdV, as well as other equations, the procedure to be followed remains the same as before. However, some of the formulae change considerably and the necessary computations become more complicated. Thus, we present briefly the mKdV equation as a further example. This equation

$$u_t = K(u) = u_{xxx} + 6u^2 u_x \tag{4.1}$$

admits the recursion operator representation

$$\Phi_t = [K', \Phi] \tag{4.2}$$

where $\Phi = \partial^2 + 4\partial u \partial^{-1} u$ is the well known recursion operator related to that equation. Equivalently, we can consider (4.1) as the compatibility condition of the resulting linear system

$$\begin{aligned} (\Phi - \lambda)\partial s &= 0 && (4.3)\\ (-\partial_t + K')\partial s &= 0\ . && (4.4) \end{aligned}$$

This results in the following map from $s$ to $u$ (see [7])

$$u = F(s) = \frac{\lambda s - s_{xx}}{2\sqrt{\lambda s^2 - {s_x}^2}}\ . \tag{4.5}$$

Inserting this into (4.4) we obtain the dynamics of $s$ as described in [7]

$$s_t = s_{xxx} + \frac{3(\lambda\ s\ -\ s_{xx})^2}{2(\lambda\ s^2\ -\ {s_x}^2)} s_x\ . \tag{4.6}$$

Again, we denote by $s(\mu)$ the eigenfunction of (4.3) for arbitrary spectral parameter $\mu$ (instead od $\lambda$) and we make the ansatz $s(\mu) = s\varphi$ in order to obtain, after some computation, the following Lax representation for (4.6)

$$\left(3s_{xx}\partial + 3s_x\partial^2 + s\partial^3 - (\lambda - \mu)s\partial + 4\partial\frac{\lambda s - s_{xx}}{\sqrt{\lambda\ s^2\ -\ {s_x}^2}}\ \partial^{-1}\frac{{s_x}^2 - s s_{xx}}{\sqrt{\lambda\ s^2\ -\ {s_x}^2}}\partial\right)\varphi = 0 \tag{4.7}$$

$$\left(-s\partial_t + s\partial^3 + 3s_x\partial^2 + 3s_{xx}\partial + \frac{3}{2}\frac{(\lambda s - s_{xx})^2 s}{(\lambda\ s^2\ -\ s_x{}^2)}\ \partial\right)\varphi = 0\ . \tag{4.8}$$

One finds the differential operator $F(s)'^T$ given by (4.5) to be

$$F(s)'^T = -\frac{1}{2s}\partial\frac{s^2}{\sqrt{\lambda s^2 - s_x^2}}\ \partial s^{-1}\ . \tag{4.9}$$

Now, using the known action gradients

$$\nabla_u \text{actions} = s(\mu) \tag{4.10}$$

and the corresponding angle gradients [10]

$$\mathcal{A}(s(\mu)) = s(\mu)\partial^{-1}s(\mu)^{-2}\sqrt{\mu s(\mu)^2 - s(\mu)_x{}^2} \tag{4.11}$$

we find the corresponding gradients of action-angles for the $\psi$-dynamics given by (4.6) to be

$$\nabla_\psi \text{ action} \quad = \quad -\frac{1}{2s}\partial\frac{s^2}{\sqrt{\lambda s^2 - s_x^2}}\ \partial\varphi, \tag{4.12}$$

$$\nabla_\psi \text{ angle} \quad = \quad -\frac{1}{2s}\partial\frac{s^2}{\sqrt{\lambda s^2 - s_x^2}}\ \partial\varphi\partial^{-1}(s\varphi)^{-2}\sqrt{\mu(s\varphi)^2 - (s\varphi)_x^2} \tag{4.13}$$

which again describes a set of action angles for (4.6) in terms of the eigenfunction of the new Lax representation (4.7).

# 5 The NLS interacting soliton equation

The nonlinear Schrödinger equation (see e.g. [1], [3], [18], [13])

$$u_t = -iu_{xx} + 2i|u|^2 u \tag{5.1}$$

admits the recursion operator [5]

$$\Phi(u) = -i\partial + 4iu\partial^{-1}\mathrm{Re}(\bar{u}\cdot) \tag{5.2}$$

where $\mathrm{Re}(\bar{u}\cdot)$ denotes the following operator

$$\omega \longrightarrow \text{real part of}(\bar{u}\omega)\ .$$

Hence the dynamics (5.1) is represented by

$$\Phi_t = [K', \Phi]\ . \tag{5.3}$$

In other words (5.1) is the compatibility condition of

$$(\Phi-\lambda)is \;=\; 0 \tag{5.4}$$

$$(-\partial_t+K')is \;=\; 0 \tag{5.5}$$

where the crucial relation between $u$ and $s$ is [7]

$$u=F(s)=\frac{1}{2}(\lambda s+is_x)|s|^{-1}\ . \tag{5.6}$$

Inserting this into (5.1) we find the well known [7] dynamic

$$i|s|^2 s_t = s_{xx}|s|^2 - s|\lambda s+1/2s_x|^2+(\lambda s+is_x)^2\bar{s}\ . \tag{5.7}$$

As before $s(\mu)$ denotes the general eigenfunction for (5.4) ($\lambda$ again replaced by $\mu$). The ansatz $s(\mu)=s\varphi$ now leads to the following Lax pair representation of (5.7)

$$(\Phi-\mu)i\varphi s-\varphi(\Phi-\lambda)is=0$$
$$(-\partial_t+K')i\varphi s-\varphi(-\partial_t+K')is=0\ .$$

or

$$[\Phi,\varphi]is+(\lambda-\mu)is\varphi=0 \tag{5.8}$$

$$-is\partial_t\varphi+[K',\varphi]is=0\ . \tag{5.9}$$

Of course, here all quantities $u$ occurring in $\Phi$ and $K'$ should be replaced by (5.6). It is well known that the $s(\mu)$ are gradients of action variables of the NLS equation. The gradients of the corresponding angle-variables , obtained in [10] are used to write the gradients of action-angle variables on the $u$-manifold in terms of the new eigenfunction $\varphi$

$$\nabla_u \text{action} = s\varphi \tag{5.10}$$

$$\nabla_u \text{angle} = \mathcal{A}(s) = i\frac{s\varphi}{|s\varphi|}+s\varphi\partial^{-1}\left\{\frac{\left(\text{phase}(e^{-i\mu x}s\varphi)\right)_x}{|s\varphi|}\right\}\ . \tag{5.11}$$

The duality between tangent fields $v$ and cotangent fields $\gamma$ can be fixed, as usual, by

$$<\gamma,v>=\text{real part of}\int_{-\infty}^{+\infty}\bar{\gamma}(x)v(x)dx\ . \tag{5.12}$$

By this choice the transposed of $F'$ becomes

$$F'^T=\frac{1}{2|s|}(\lambda+i\partial)-\frac{1}{2|s|^3}\text{Re}(s\cdot)\ . \tag{5.13}$$

Application of that to (5.11) yields

$$\nabla_s\text{ action} \;=\; F'^T s\varphi \tag{5.14}$$

$$\nabla_s\text{ action} \;=\; F'^T\mathcal{A}(s)\ . \tag{5.15}$$

# 6 The ZS-AKNS-System

In order to illustrate the applicability of the method in the case of systems with several components we consider as final example the well known (see [1], [3],[18]) ZS-AKNS-system

$$i\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} q_{xx} + 2q^2 r \\ -r_{xx} - 2qr^2 \end{pmatrix} \tag{6.1}$$

which contains the systems we considered so far as special reductions (for example the NLS by $r = -\bar{q}$). Introducing the following quantities

$$\vec{\psi}(\mu) = \begin{pmatrix} \psi_1(\mu) \\ \psi_2(\mu) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.2}$$

equation (6.1) can be understood (see [1], [3],[18]) as the compatibility condition of

$$(\sigma_3 \partial - \sigma_3 A)\vec{\psi}(\mu) = \mu \vec{\psi}(\mu) \tag{6.3}$$

$$\sigma_3 \vec{\psi}(\mu)_t = \Omega \vec{\psi}(\mu) \tag{6.4}$$

where

$$\Omega = -2i\mu\sigma_3(A + \mu\sigma_3) + iA^2 - iA_x \; . \tag{6.5}$$

One easily sees that such a compatibility condition leads to

$$A_t = i\sigma_3[2A^3 - A_{xx}] \tag{6.6}$$

which is just an equivalent formulation of (6.1). Now, consider those $\mu$ such that there is a solution $\vec{\psi}(\mu)$ with rapidly vanishing boundary values at infinity [2]. Observe, that in this case that $\mu$ is a conserved quantity for the dynamics of (6.1). Here we denoted by $\langle , \rangle$ the quantity

$$\langle \vec{\gamma}, \vec{v} \rangle := \int_{-\infty}^{+\infty} (\gamma_1 v_1 + \gamma_2 v_2) dx \; . \tag{6.7}$$

This form we also will take as representation for the canonical duality between covector fields $\vec{\gamma}$ and vector fields $\vec{v}$. The gradient of that quantity $\mu$ is easily computed:

Denote $L = (\sigma_3 \partial - \sigma_3 A)$ and observe that $\sigma_1 L$ is symmetric with respect to the scalar product given by $\langle , \rangle$. Of course, $\sigma_1$ is the usual Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

[2] Observe that in the literature ([17], [4]) those solutions with rapidly vanishing boundary values are sometimes denoted by $\vec{\psi}(\mu)^*$, depending on whether $\mu$ is in the upper or lower half of the complex plane.

Then consider the eigenvalue equation (6.3)

$$L\vec{\psi}(\mu) = \mu\vec{\psi}(\mu). \tag{6.8}$$

Thus, by variational derivative in the direction $\delta\vec{u} = (\delta q, \delta r)^T$ we obtain.

$$L'[\delta\vec{u}]\vec{\psi}(\mu) + L\vec{\psi}(\mu)'[\delta\vec{u}] = \mu\vec{\psi}(\mu)'[\delta\vec{u}] - \langle\nabla\mu, \delta\vec{u}\rangle\vec{\psi}(\mu)\ . \tag{6.9}$$

The latter multiplied with $\sigma_1$ and then taking the scalar product with $\langle\vec{\psi}(\mu)$ and observing that $L^T\sigma_1 = \sigma_1 L$ delivers

$$\langle\vec{\psi}(\mu), \sigma_1 L'[\delta\vec{u}]\vec{\psi}(\mu)\rangle = -\langle\nabla\mu, \delta\vec{u}\rangle\langle\vec{\psi}(\mu), \sigma_1\vec{\psi}(\mu)\rangle\ .$$

Here the two remaining terms canceled because of (6.9). We, easily find

$$\sigma_1 L'[\delta\vec{u}] = \begin{pmatrix} \delta r & 0 \\ 0 & \delta q \end{pmatrix} \tag{6.10}$$

hence

$$\nabla\mu = -\langle\vec{\psi}(\mu), \sigma_1\vec{\psi}(\mu)\rangle^{-1}\begin{pmatrix} \psi_2(\mu)^2, \\ \psi_1(\mu)^2 \end{pmatrix} \tag{6.11}$$

Furthermore, we observe that $\langle\vec{\psi}(\mu), \sigma_1\vec{\psi}(\mu)\rangle$ also is a conserved quantity, therefore we can norm it to $-1$ (by using homogeneuity of the eigenvector equation (6.10)). So,

$$\nabla\mu = \begin{pmatrix} \psi_2(\mu)^2 \\ \psi_1(\mu)^2 \end{pmatrix}\ . \tag{6.12}$$

defines a set of gradients of action variables ([17], [4]). Let us first compute the corresponding angle variables, or rather their gradients. By $\vec{\psi}(\mu)^*$ we denote the second solution of (6.3). Of course, this solution does not satisfy the prescribed boundary condition at infinity. By using the variation of constant method (as proposed in [10]) such a $\vec{\psi}(\mu)^*$ can be easily expressed in terms of $\vec{\psi}(\mu)$ .

For further use we do this computation on a more general level. In case of a two-component system

$$\vec{\varphi}_x = B\vec{\varphi} \tag{6.13}$$

we want to express to second solution in terms of the first one. We assume that $B$ has trace $= 0$

$$B = \begin{pmatrix} -a & b \\ c & a \end{pmatrix}\ .$$

Then the ansatz

$$\varphi_1^* = -\frac{1}{2\varphi_2} + \frac{\alpha}{2}\varphi_1, \quad \varphi_2^* = \frac{1}{2\varphi_1} + \frac{\alpha}{2}\varphi_2$$

inserted in (6.13) leads to two linearily dependent equations in $\alpha$. Their solution reads

$$\alpha_x = \frac{b}{\varphi_1^2} - \frac{c}{\varphi_2^2} \ .$$

Hence

$$\varphi_1^* = -\frac{1}{2\varphi_2} + \frac{\varphi_1}{2}\partial^{-1}\{b\varphi_1^{-2} - c\varphi_2^{-2}\} \tag{6.14}$$

$$\varphi_2^* = \frac{1}{2\varphi_1} + \frac{\varphi_2}{2}\partial^{-1}\{b\varphi_1^{-2} - c\varphi_2^{-2}\} \ . \tag{6.15}$$

Thus for our special case (6.3) admits a second solution

$$\psi_1(\mu)^* = -\frac{1}{2\psi_2(\mu)} + \frac{1}{2}\psi_1(\mu)\partial^{-1}\{q\psi_1(\mu)^{*-2} + r\psi_2(\mu)^{*-2}\} \tag{6.16}$$

$$\psi_2(\mu)^* = +\frac{1}{2\psi_1(\mu)} + \frac{1}{2}\psi_2(\mu)\partial^{-1}\{q\psi_1(\mu)^{*-2} + r\psi_2(\mu)^{*-2}\}. \tag{6.17}$$

where the $q, r$ can be expressed in terms of $\vec{\psi}(\mu)$ by use of (6.3)

$$q = \psi_2(\mu)^{-1}(\psi_1(\mu)_x - \lambda\psi_1(\mu)) \tag{6.18}$$

$$r = \psi_1(\mu)^{-1}(-\psi_2(\mu)_x - \lambda\psi_2(\mu)) \ . \tag{6.19}$$

Now, we find the gradient of the corresponding angle variable by using the method which was presented in [10]:

*Consider the linear combination of the Lax-operator eigenfunction $\vec{\psi}(\mu)$ and $\varepsilon$ times the second solution $\vec{\psi}(\mu)^*$ of the same eigenfunction equation. Then insert this new function, instead of $\vec{\psi}(\mu)$, in the expression for the gradient of the corresponding spectral value $\mu$. The linear term in $\varepsilon$ therein represents the gradient of the corresponding angle variable.*

This procedure, in the case here considered, delivers the following angle-gradients

$$\nabla_{\vec{u}} \text{ angle } (\mu) = \begin{pmatrix} \psi_2(\mu)\psi_2(\mu)^* \\ \psi_1(\mu)\psi_1(\mu)^* \end{pmatrix}, \tag{6.20}$$

where $\vec{u} = \binom{q}{r}$.

Now, we concentrate on the dynamics for the eigenfunction itself. We fix a special spectral value, denoted by $\lambda$ and we define

$$\begin{aligned} \vec{\psi} &= \vec{\psi}(\lambda) \\ \vec{\psi}^* &= \vec{\psi}^*(\lambda) \ . \end{aligned} \tag{6.21}$$

Let

$$W(\mu) = \begin{pmatrix} \psi_1(\mu) & \psi_1(\mu)^* \\ \psi_2(\mu) & \psi_2(\mu)^* \end{pmatrix}, \; W^*(\mu) = \begin{pmatrix} \psi_2(\mu)^* & -\psi_1(\mu)^* \\ -\psi_2(\mu) & \psi_1(\mu) \end{pmatrix} \tag{6.22}$$

then

$$(\nabla_{\vec{u}} \text{ action } (\mu), \nabla_{\vec{u}} \text{ angle } (\mu)) = \sigma_1 \begin{pmatrix} \psi_1(\mu) & 0 \\ 0 & \psi_2(\mu) \end{pmatrix} W(\mu) \; . \tag{6.23}$$

where $(\vec{a}, \vec{b})$ denotes the matrix whose columns are given by the vectors $\vec{a}$ and $\vec{b}$. Rewriting the dynamics and the Lax representation for $\vec{\psi}$ we abbreviate

$$W := W(\lambda), \quad W^* = W^*(\lambda). \tag{6.24}$$

The following relations are easily verified by direct computation:

$$\det(W) = 1 \tag{6.25}$$

$$W^* = W^{-1} \tag{6.26}$$

$$(A + \lambda\sigma_3) = W_x W^* \tag{6.27}$$

Relation (6.27) is an immediate consequence of the fact that $W$ is a fundamental solution of (6.3), i.e. $W_x = (A + \lambda\sigma_3)W$. Now, inserting (6.27) into (6.4) and (6.5), respectively we obtain the nonlinear equation describing the $\vec{\psi}$-dynamics (see [14], [15], [16])[3]

$$\sigma_3 \vec{\psi}_t = \{ -2i\lambda\sigma_3 W_x W^* + i(W_x W^* - \lambda\sigma_3)^2 - i(W_x W^*)_x \} \vec{\psi} \; . \tag{6.28}$$

The Lax-representation for $\vec{\psi}$ is obtained from (6.3) by substitution $\vec{\psi}(\mu) = W\vec{\varphi}$ and by use of (6.26) and (6.27) in the following way

$$\begin{aligned} (W\vec{\varphi})_x &= \vec{\psi}(\mu)_x = (A + \sigma_3\mu)\vec{\psi}(\mu) \\ &= (A + \sigma_3\lambda)\vec{\psi}(\mu) + \sigma_3(\mu - \lambda)\vec{\psi}(\mu) \\ &= W_x W^* W\vec{\varphi} + \sigma_3(\mu - \lambda)W\vec{\varphi} \\ &= W_x\vec{\varphi} + \sigma_3(\mu - \lambda)W\vec{\varphi} \end{aligned} \tag{6.29}$$

Hence [16]

$$\vec{\varphi}_x = (\mu - \lambda)W^*\sigma_3 W\vec{\varphi} \tag{6.30}$$

or

$$W\vec{\varphi}_x = (\mu - \lambda)\sigma_3 W\vec{\varphi} \tag{6.31}$$

[3] If one defines $W^*\sigma_3 W = S$ and expands $W^*\sigma_3 W = \sum \sigma_i S_i$ then $\vec{S}$ fulfills ([14], [15]) $\vec{S}_t = \vec{S} \times \vec{S}_{xx}$ which is the dynamics of the Landau-Lifshitz equation. Thus our method also yields the action-angle transformation for that equation.

Analogously, the dynamics of $\vec{\varphi}$ follows

$$iW\vec{\varphi}_t - 2\sigma_3 W\vec{\varphi}_{xx} - 2\sigma_3 W_x\vec{\varphi}_x - 2\lambda W\vec{\varphi}_x = 0\ . \tag{6.32}$$

Now, to obtain the action-angle gradients for (6.28) we proceed as before: First we write these quantities in terms of $\vec{\psi}$ and $\vec{\varphi}$ on the $\vec{u}$-manifold and then we transform them via pullback. For sake of simplicity, let us introduce $\vec{\varphi}^*$ such that $\vec{\psi}(\mu) = W\vec{\varphi}^*$ represents a second solution of (6.30). The latter follows on direct use of (6.14)-(6.15):

$$\varphi_1^* = -\frac{1}{2\varphi_2} + (\mu-\lambda)\varphi_1\partial^{-1}\{\varphi_1^{-2}\psi_1^*\psi_2^* + \varphi_2^{-2}\psi_1\psi_2\} \tag{6.33}$$

$$\varphi_2^* = +\frac{1}{2\varphi_1} + (\mu-\lambda)\varphi_2\partial^{-1}\{\varphi_1^{-2}\psi_1^*\psi_2^* + \varphi_2^{-2}\psi_1\psi_2\}\ . \tag{6.34}$$

Then the action-angle gradients on the $\vec{u}$-manifold are

$$\nabla_{\vec{u}}\ \text{action}\ (\mu) = \begin{pmatrix} (\psi_2\varphi_1 + \psi_2^*\varphi_2)^2 \\ (\psi_1\varphi_1 + \psi_1^*\varphi_2)^2 \end{pmatrix} \tag{6.35}$$

$$\nabla_{\vec{u}}\ \text{angle}\ (\mu) = \begin{pmatrix} (\psi_2\varphi_1 + \psi_2^*\varphi_2)(\psi_2\varphi_1^* + \psi_2^*\varphi_2^*) \\ (\psi_1\varphi_1 + \psi_1^*\varphi_2)(\psi_1\varphi_1^* + \psi_1^*\varphi_2^*) \end{pmatrix} \tag{6.36}$$

They can be transformed into the action-angle gradients on the $\psi$-manifold by means of the map $F'^T$ where $\vec{u} = F(\vec{\psi})$. This follows, fixed the duality between tangent and cotangent space of the $\vec{\psi}$-manifold as beforehand, from the variational derivatives of (6.18) and (6.19). A detailed computation yields

$$F'^T = \begin{pmatrix} -\partial\psi_2^{-1} - \lambda\psi_2^{-1} & \psi_1^{-2}\vec{\psi}_{2x} + \lambda\psi_1^{-2}\psi_2 \\ -\psi_2^{-2}\vec{\psi}_{1x} + \lambda\psi_2^{-2}\psi_1 & \partial\psi_1^{-1} - \lambda\psi_1^{-1} \end{pmatrix} \tag{6.37}$$

This finally gives

$$\nabla_{\vec{\psi}}\ \text{action}\ (\mu) = F'^T\nabla_{\vec{u}}\ \text{action}\ (\mu) \tag{6.38}$$

$$\nabla_{\vec{\psi}}\ \text{angle}\ (\mu) = F'^T\nabla_{\vec{u}}\ \text{angle}\ (\mu). \tag{6.39}$$

These quantities in explicit form can be obtained on substitution of the previous formulae.

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