Research reports in Physics -Nonlinear Dynamics-
Springer Verlag Berlin-Heidelberg-New York
S. Carillo and O.Ragnisco, eds.
pages 127-130, 1990

The Action-Angle Transformation for the Korteweg-deVries equation

Sandra Carillo
Dipartimento di M. M. M. per le Sc. Appl., Universtitá *La Sapienza*, Roma, Italy

Benno Fuchssteiner
Department of Mathematics, University of Paderborn, D 4790 Paderborn, Germany

Abstract: For the multi-soliton solutions of the KdV (Korteweg-de Vries equation) a map from the action variables to the angle variables is constructed. The analysis presented here is valid for nonlinear evolution equations admitting a recursion operator as well as a Lax operator. The method is based on the nonlinear link between the eigenvectors of these two operators. Since the action-angle map is recognized to be an infinitesimal symmetry generator of the corresponding interacting soliton equation the result also follows directly from the structural properties of the underlying dynamics. In case of the KdV this symmetry group generator can be found from the fact that it must generate a group in the kernel of the Miura transformation.

As an example, to illustrate a more general method, we consider the KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{1}$$

whose hereditary recursion operator [5,8] is given by the Lenard operator [1]

$$\Phi(u) = D^2 + 2u_x D^{-1} + 4u \tag{2}$$

where, as usual, D^{-1} denotes integration from $-\infty$ to x. For this equation we consider only solutions vanishing rapidly at infinity. Further examples and detailed computations concerning the construction of action-angle transformations in general are comprised in [7].

As usual [4] we characterize the N-soliton manifold by

$$M_N = \{u \mid \textit{there are } c_n \textit{ s. t. } \sum_{n=1}^{N} c_n K_n(u) = 0\} \ . \tag{3}$$

According to the general theory [9,10,12], the recursion operator of the given nonlinear system follows to be doubly degenerated when restricted to M_N. Furthermore, when the gradients of canonical action-angle variables are mapped by the hamiltonian formulation onto vector fields, then the eigenvectors of the recursion operator are obtained. Keeping in mind that the hamiltonian formulation of the KdV is given by the differential operator

D, we find that for every eigenvector, which corresponds to a symmetry generator of the system, the corresponding eigenvector (with the same eigenvalue) must be the derivative of the gradient of an angle variable.

According to the Lax formulation, the the KdV represents an isospectral flow, not only for its recursion operator, but also for the Schrödinger operator: $L = D^2 + u$. Now explicit computation easily shows that whenever

$$L\omega = \frac{\lambda}{4}\omega \tag{4}$$

then

$$\Phi(u)(\omega^2)_x = \lambda(\omega^2)_x \tag{5}$$

and this relation is independent on whether or not ω is a genuine eigenvector of L. Hence we have a nonlinear link between the eigenvectors of these isospectral problems. Since the eigenvector of $D^{-1}\Phi D$ given by the square of the eigenvector of L it can be understood as the gradient of the conservation law given by λ, hence this eigenvector corresponds to the gradient of an action variable. Thus, the gradient of the corresponding angle variable must be given by the second eigenvector of $D^{-1}\Phi D$. The latter can be retrieved by use of the fact that the relation between the eigenvectors of L and Φ is nonlinear. If we take a linear superposition of two different solutions ω_1 and ω_2 of (4) then we obtain by linear superposition for solutions of (5) the following:

Lemma: If ω_1 and ω_2 are functions such that

$$L\omega_i = \frac{\lambda}{4}\omega_i, \quad i = 1, 2 \tag{6}$$

then

$$\sigma_1 = ({\omega_1}^2)_x, \ \sigma_2 = ({\omega_2}^2)_x, \ \sigma_3 = (\omega_1\omega_2)_x \tag{7}$$

are solutions of $\Phi(u)\sigma = \lambda\sigma$.

Among these three formal solutions, those with vanishing boundary conditions at infinity, are precisely those which correspond to the doubly degenerate spectrum of Φ. Obviously σ_1 has the required boundary behavior since it decreases exponentially at infinity; conversely, σ_2 cannot fulfill such boundary conditions since it grows exponentially at infinity. However, σ_{3x}, represented by the derivative of a product of two functions growing and decaying exponentially with the same rate, goes to zero at infinity.

Thus, σ_{3x} is the second eigenvector of Φ and σ itself must be the gradient of an angle variable of the system. Correspondingly, a map from the action to the angle variable is given by the map from σ_1 to σ_3. This map follows by combination of

- the construction of the second eigenvector ω_2 when ω_1 is given.
- the nonlinear link between ω and σ.

Application of the method of variation of constants shows that, given one eigenvector ω of the Schrödinger operator $(D^2 + u)\omega = \lambda'\omega$ then another such eigenvector $\tilde{\omega}$, corresponding to the same eigenvalue, is given by $\tilde{\omega} = \omega D^{-1}\omega^{-2}$. Combination of these results gives:

Theorem [7]: On the multisoliton manifold of the KdV the action-angle map

$$\nabla action\ variables \rightarrow \nabla angle\ variables$$

is given by $\sigma \rightarrow \sigma D^{-1}\sigma^{-1}$. Here σ stands for the gradients of the canonical action variables corresponding to the eigenvectors of the recursion operator $\Phi(u)$.

A similar result, implied by the nonlinearity of the relation between the two isospectral problems, can be obtained for most of the well known integrable nonlinear evolution equations which admit both a recursion operator and a Lax Pair. Differences among various cases only depend on technical details (see [7]).

However, a characterization of the action-angle map in terms of symmetry group generators of the interacting soliton equation can be given by recalling that the dynamics of those eigenvectors of the recursion operator, which correspond to action variables, follows itself a nonlinear evolution equation which is again completely integrable [6]. If σ_x is such an eigenvector then the dynamics for σ is the *interacting soliton equation* which, in case of the KdV equation, reads

$$\sigma^2\sigma_t \;=\; \sigma^2\sigma_{xxx} - 3\sigma\sigma_x\sigma_{xx} + \frac{3}{2}{\sigma_x}^3 + \frac{3}{2}\lambda\, \sigma^2\sigma_x \tag{8}$$

and is related, via a Bäcklund transformation, to the KdV equation [2].

Analyzing the method we just applied, we find that it consisted in taking a one-parameter family (with respect to a linear parameter) in the eigenspace of the Schrödinger operator, and then, after application of the nonlinear link, picking out the linear term of the corresponding group action in the eigenspace of the recursion operator. Hence, the result of the above procedure gives nothing else than an infinitesimal generator of a one-parameter symmetry group in the manifold of generalized eigenvectors of the recursion operator, or rather its pull-back to the cotangent bundle given by the hamiltonian formulation. That is, the action-angle map follows to coincide with a symmetry group generator of the interacting soliton equation.

Theorem [7]: The action-angle map $\sigma \rightarrow S(\sigma)$ is always a symmetry group generator for the corresponding interacting soliton equation.

Remarkably, $S(\sigma)$ can neither be the group generator of (8) given by $G(s) = s$ nor can it be a member of the hierarchy for the interacting soliton equation since all of these correspond to action variables of the system. However, the map required in the last theorem can be explicitly obtained by use of the well known links between KdV, mKdV, their singularity equations and their interacting soliton equations (see [2]). One finds that, up to addition of a multiple of the generator for x-translation, the interacting soliton for the KdV equation can be obtained in two ways. Apart from the way described above one can go first, via the Miura transformation, to the mKdV and then by the Cole-Hopf transformation to the interacting soliton equation. Since the Miura transform $u = iv_x + v^2$ is a differential equation between solutions u of the KdV and solutions v of the mKdV, one can change the v's without changing the u's by going over to different solutions of this ordinary differential equation (given by the Miura link). Hence there must be an additional symmetry group for the v's which nevertheless does not change the form of the interacting soliton equation for KdV. Transfer of this change in v via the Cole-Hopf transformation then leads to an additional invariance of the interacting soliton equation.

Explicit computation [7] shows that indeed:

$$2isD^{-1}\frac{1}{s}$$

is the additional symmetry which comes out of this procedure, hence represents the action-angle map.

Similar arguments go through in cases where a canonical Miura-like link between different integrable equations exist, especially for the case of the Caudrey-Dodd-Gibbon equation [3] which plays the role of the KdV equation in the corresponding Bäcklund Chart [11].

References

[1] Calogero F. and A.Degasperis: Spectral Transform and Solitons I, Studies in Mathematics and its Application 13, North Holland, Amsterdam, 1980.

[2] S. Carillo and Fuchssteiner B.: Soliton Structure versus Singularity Analysis: Third-order Completely Integrable Non-Linear Differential Equations in 1+1-dimensions, Physica vol. 154A, pp. 467-510, 1989.

[3] Caudrey, P. J., R. K. Dodd and J. D. Gibbon: A New Hierarchy of Korteweg-de Vries Equations, Proc.R.Soc. London, vol. A 351 pp. 407-422, 1976

[4] Dubrovin B.A. and S.P. Novikov: Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-De Vries equation, Sov. Phys.-JETP, vol. 40,pp.1058-1063, 1975.

[5] Fuchssteiner B.: Application of Hereditary Symmetries to Non-linear Evolution Equations, Non-Linear Analysis TMA, vol. 3, pp. 849-862, 1979.

[6] Fuchssteiner B.: Solitons in Interaction, Progr. Theor. Phys. vol.78, pp.1022-1050, 1987.

[7] Fuchssteiner B. and S. Carillo: The Action-Angle Transformation for Soliton Equations preprint 1989.

[8] Fuchssteiner B. and A.S. Fokas: Symplectic Structures, Their Bäcklund Transformations and Hereditary Symmetries, Physica vol. 4D, pp. 47-66, 1981.

[9] Fuchssteiner B. and Gudrun Oevel: Geometry and Action-Angle Variables for Multi Soliton Systems, preprint, 1989.

[10] Gudrun Oevel, Fuchssteiner B. and M. Blaszak: this volume

[11] Rogers C. and S. Carillo: Bäcklund Charts for the Caudrey-Dodd-Gibbon and Kaup-Kupershmidt Hierarchies, in Non-Linear Evolutions ,ed.J.J.P.Leon, World Scientific Publ., Singapore, 1988.

[12] TenEikelder H.M.M.: On the local structure of Recursion Operators for Symmetries, Indag. Math. vol.89, pp.386-403, 1986