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ACTION/ANGLE VARIABLES AND ASYMPTOTIC DATA

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Abstract. By use of mastersymmetries we construct the action/angle variables for multi-soliton systems in terms of the field variable u. Furthermore, an interpretation in terms of asymptotic data is given.

On a suitable manifold M we consider hamiltonian evolution equations

$$u_t = K_1(u) \tag{1}$$

in (1+1)-dimensions. Here $u = u(x,t) \in M$ denotes the field variable and $K_1(u)$ is a translation invariant vector field on M. We restrict ourselves to equations (1) which admit a localized hereditary ([3]) recursion operator $\Phi(u) = \Theta_1(u)\,\Theta_0^{-1}(u)$ with an implectic/symplectic factorization ([4]) or equivalent, a compatible hamiltonian pair Θ_1,Θ_0 ([10],[11]). The operator Φ generates a hierarchy of pairwise commuting symmetries ([5])

$$K_n(u) \ := \ \Phi^n(u)K_0(u) \ = \ \Phi^n(u)u_x$$

for the equation (1). If there is a scaling quantity $\tau_0(u)$ and a fixed $\alpha \in \mathbb{R}$ with

$$L_{\tau_0}\Phi := \Phi'[\tau_0] - \tau_0'\Phi + \Phi\tau_0' = \Phi \ \text{ and } \ L_{\tau_0}K_0 := [\tau_0, K_0] = \alpha K_0 \ , \tag{2}$$

then the recursive application of Φ on τ_0 produces a second hierarchy of vector fields $\tau_n = \Phi^n\tau_0$, the so-called mastersymmetries ([2],[6],[15]). Here [,] denotes the usual commutator between vector fields and α is a scalar. For reasons which become obvious later, we use the Lie derivative $L_V\,T$ of a tensor field T into the direction of a vector field V as the standard notation ([16]). Due to the hereditariness of Φ the following commutator relations hold between the symmetries K_n and the mastersymmetries τ_n

$$[K_n, K_m] \ = \ 0 \ \ , \ \ [\tau_n, K_m] \ = \ (m+\alpha)K_{n+m} \ \ , \ \ [\tau_n, \tau_m] \ = \ (m-n)\tau_{n+m} \ \ .$$

Example: For u being an element of the Schwartz space of rapidly decreasing functions we obtain with $K_0(u) = u_x$, $\tau_0(u) = \frac{1}{2}xu_x + u$ and the hereditary recursion operator

$$\Phi(u) \ = \ D^2 + 2DuD^{-1} + 2u \ = \ (D^3 + 2Du + 2uD)D^{-1} \ = \ \Theta_1\Theta_0^{-1}$$

the hierarchy of the Korteweg-deVries equation (KdV)

$$u_t \ = \ u_{xxx} + 6uu_x \ .$$

It is well known that the equations introduced above admit multi-soliton solutions. The N-soliton solutions can be described as elements of the following invariant submanifold M_N of M ([12])

$$M_N = \{ u \mid \text{there exists } \alpha_n \text{ such that } \sum_{n=0}^{N} \alpha_n K_n = 0 \} . \tag{3}$$

Since for vanishing boundary conditions at infinity this manifold can be parametrized by the velocities c_i and the phases q_i of the N-soliton-solutions, it has the dimension $2N$. With the reduction technique introduced in [8], we obtain important properties of M_N (see also [9]). Using these results we can introduce in $T_u M_N$ the following basis of eigenstates of Φ for the discrete eigenvalue c_i

$$A_i := \frac{\partial u}{\partial q_i} \quad \text{and} \quad B_i := \frac{\partial u}{\partial c_i} , \quad i = 1, ..., N . \tag{4}$$

We now are able to give the representation of all quantities on the submanifold M_N w.r.t. the basis A_i and B_i. In this report we will mention only some of them. For convenience we denote these quantities with a bar.

Lemma 1:

(a) W.r.t. the basis A_i and B_i the hamiltonian equation induced by the flow (1) on M_N is of the following linear form (in terms of the q_i and c_i)

$$\frac{\partial}{\partial t} \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_{(-\alpha)} \\ -\Lambda_{(-\alpha)} & 0 \end{pmatrix} \text{grad} \left(\frac{1}{2+\alpha} \sum_{i=1}^{N} c_i^{2+\alpha} \right) ,$$

where Λ_p denotes the diagonal $N \times N$- matrix $\Lambda_p = \begin{pmatrix} c_1^p & 0 & \cdots & 0 \\ 0 & c_2^p & 0 & \cdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & c_N^p \end{pmatrix}$.

(b) The recursion operator Φ is of the following form on M_N

$$\bar{\Phi} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_{(1-\alpha)} \\ -\Lambda_{(1-\alpha)} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\Lambda_{(+\alpha)} \\ \Lambda_{(+\alpha)} & 0 \end{pmatrix} = \bar{\Theta}_1 \bar{\Theta}_0^{-1} .$$

Using this representation our first result is

Lemma 2:

(a) W.r.t. the Poisson bracket given by $\bar{\Theta}_1$ the coordinates c_i^α, q_j fulfill for all $i, j = 1, ..., N$ the following relations

$$\{c_i^\alpha, q_j\}_{\bar{\Theta}_1} = \alpha\, \delta_{ij} , \quad \{c_i^\alpha, c_j^\alpha\}_{\bar{\Theta}_1} = \{q_i, q_j\}_{\bar{\Theta}_1} = 0 . \tag{5}$$

Hence, $\frac{1}{\alpha} c_i^\alpha$, q_i are the canonical coordinates corresponding to $\bar{\Theta}_1$. They are called canonical action/angle variables.

(b) The symmetries $\bar{K}_n = \bar{\Phi}^n(c_1, \ldots, c_N, 0, \ldots, 0)^*$ are bi-hamiltonian vector fields with (* denotes the transposed vector)

$$\bar{K}_n = \bar{\Theta}_1 \text{ grad } \Big(\frac{1}{n+\alpha} \sum_{i=1}^{N} c_i^{n+\alpha}\Big) = \bar{\Theta}_0 \text{ grad } \Big(\frac{1}{n+1+\alpha} \sum_{i=1}^{N} c_i^{n+1+\alpha}\Big) \ .$$

(c) The vector field $\bar{\tau}_0$ is a hamiltonian vector field w.r.t. $\bar{\Theta}_1$ with

$$\bar{\tau}_0 = (-\alpha q_1, \ldots, -\alpha q_N, c_1, \ldots, c_N)^* = \bar{\Theta}_1 \text{ grad } \Big(-\sum_{i=1}^{N} c_i^{\alpha} q_i\Big) \ . \qquad (6)$$

Example : For the KdV equation α is equal to $1/2$. Hence, lemma 2 gives the action/angle coordinates in terms of the asymptotic data q_i, c_i.

Recall that it was our aim to express the action/angle variables in the original field variable u. This is not a trivial task, because the manifold under consideration is not a vector space and only the gradients of these scalar fields are known from [8]. To obtain the final result we need the following **main tools:**

(1) The transformation to a basis of eigenstates of Φ on M_N is related to a change of coordinates Π, which assigns to every N-soliton solution u its asymptotic data. The behaviour of tensor fields under that change of coordinates is used ([1]).

(2) Lie derivatives are invariant under a change of coordinates ([14]).

(3) For the equations under consideration the scaling mastersymmetry $\tau_0 = \Theta \text{grad} F$ has always a unique hamiltonian structure Θ on the whole manifold M. In the reduction, τ_0 remains hamiltonian w.r.t. $\Theta_{|red}$ and the corresponding scalar field on M_N is $F_{|M_N}$. In (q_i, c_i)- coordinates we obtain $\Theta_{|red} = \bar{\Theta}_1$.

Theorem :

(a) The eigenvectors

$$A_i = \frac{\partial u}{\partial q_i} \quad \text{and} \quad c_i^{1-\alpha} B_i = c_i^{1-\alpha} \frac{\partial u}{\partial c_i}$$

of Φ are hamiltonian vector fields w.r.t. the implectic operator $\Theta_{|red}$ (determined by the requirement (3) that $\tau_0 = \Theta$ grad F has to be hamiltonian).

(b) The potentials E_i^{α} and Ω_i of the eigenvectors A_i and $c_i^{1-\alpha} B_i$ are given by the partial derivatives

$$E_i^{\alpha} = -\frac{\partial F}{\partial q_i} \quad , \quad \Omega_i = -\frac{\partial F}{\partial (c_i^{\alpha})} \ . \qquad (7)$$

(c) $\frac{1}{\alpha} E_i^{\alpha}$ and Ω_i are canonical coordinates w.r.t. Θ_1, i.e. for all $i, j = 1, \ldots, N$ it holds

$$\{E_i^{\alpha}, E_j^{\alpha}\}_{\Theta_1} = 0 = \{\Omega_i, \Omega_j\}_{\Theta_1} \ , \ \{E_i^{\alpha}, \Omega_j\}_{\Theta_1} = \alpha\, \delta_{ij} \ .$$

Example : For the KdV equation we have

$$\tau_0 \;=\; (D^3 + 2Du + 2uD)\ \mathrm{grad}\frac{1}{4}\int_{-\infty}^{+\infty} xu\ dx \;=\; \Theta_1(u)\ \mathrm{grad}F(u)\ \ .$$

Hence, for the N-soliton solutions with vanishing boundary conditions at infinity we obtain the action/angle variables w.r.t $\Theta_{1|red}$ explicitly by

$$2\sqrt{E_i} \;=\; -\frac{1}{2}\int_{-\infty}^{+\infty} xu_{q_i}\ dx \quad , \quad \Omega_i \;=\; -\frac{1}{2}\sqrt{c_i}\int_{-\infty}^{+\infty} xu_{c_i}\ dx\ \ . \tag{8}$$

For all proofs and more examples we refer to the exhaustive paper [13].

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