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Action-angle representation of Multisolitons by potentials of mastersymmetries

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Abstract. Using the algebra of symmetries/mastersymmetries a purely algebraic construction for the action/angle representation of multisolitons is given. By the same method an explicit construction of the potentials of the eigenstates of the recursion operator is performed in terms of partial derivatives of a fundamental scalar field.

Content :

0 Introduction

Angle variables are scalar fields with an explicit linear dependence in time. Therefore it is intriguing to construct these quantities via mastersymmetries, since the mastersymmetries give rise to symmetry generators having a linear time dependence.

Indeed, this construction can be performed easily in all those cases where all the mastersymmetries are hamiltonian vector fields w.r.t. one common implectic operator (Benjamin-Ono, Kadomtsev-Petviashvili and the like). In all other 1+1-dimensional cases where recursion operators exist the mastersymmetries necessarily do not have this property. Their structure is exactly equivalent to the very existence of a second hamiltonian formulation which then gives rise to the recursion operator.

However, even in this case the authors of [22] were able to prove that the reduction to multisoliton manifolds led to a canonical linear dependence within the mastersymmetries. Using this dependence it was possible to construct explicitly the full set of eigenstates of the recursion operator which belonged to the discrete spectral points. These eigenstates were of such a nature that they could be interpreted as gradients of action/angle variables.

The construction of the potentials of these gradients was left out as a difficult problem in [22]. The reason for that difficulty was the fact that the multisoliton manifold was not a vector space anymore, i.e. explicit integration of the gradients could not be used to find the potentials.

In the present paper we give a different approach to the multisoliton systems. As the main result an explicit construction of the missing potentials is given, i.e. we construct the angle variables in terms of the original field variable. It turns out that the action, as well as the angle variables, can be obtained directly via partial derivatives (with respect to the asymptotic data) of a fundamental scalar field. This quantity is determined by the scaling properties of the system under consideration. Hence, the complete set of action/angle variables is given by one fundamental scalar field.

The considerations which lead to our results are of a very simple nature. First, we reparametrize the multisoliton manifold in terms of asymptotic data, then we linearize the flow completely in terms of these artifical new variables.

In section 1 and 2 we discuss the necessary relations between the asymptotic data and the reparametrization of the multisoliton manifold. Examing the trivial structure of the linear flow on this reparametrized manifold we obtain via pullback the full set of eigenstates of the recursion operator which belong to the discrete part of the spectrum. For the main result in section 3 we need one important property of the scaling mastersymmetry. In all the equations under consideration this quantity has a unique hamiltonian formulation. Together with the properties of the eigenstates of the recursion operator the information about the scaling mastersymmetry is used to obtain the explicit construction of the action/angle variables. In section 4 our method is illustrated and applied to several examples. We finish this paper with some concluding remarks about the hamiltonian structure of the scaling mastersymmetry and about the relationship between the Inverse Scattering Method and our approach.

1 General Notations

On a suitable manifold M we consider evolution equations

$$u_t = K_1(u) \tag{1.1}$$

where $u = u(x,t) \in M$ denotes the field variable and $K_1(u)$ is a vector field on M. We are interested in equations which admit two compatible hamiltonian (implectic or Poisson) ([24],[25],[36]) operators Θ_0 and Θ_1 with

$$K_1(u) = \Theta_0 \ \text{grad} \ H_1(u) = \Theta_1 \ \text{grad} \ H_0(u)$$

where $H_0(u)$ and $H_1(u)$ are scalar fields on the manifold under consideration. If Θ_0 is invertible, then they give rise to a hereditary recursion operator $\Phi = \Theta_1 \Theta_0^{-1}$ which maps symmetries of equation (1.1) again onto symmetries ([14],[15],[16],[32],[35]). Under the condition that $K_1(u)$ does not explicitly depend on x the generator of space translation $K_0(u) := u_x$ is a symmetry for $K_1(u)$, i.e. the two vector fields commute in the Lie-algebra of vector fields

$$L_{K_1(u)} u_x := [K_1(u), u_x] = u_x'[K_1(u)] - K_1(u)'[u_x] = 0.$$

Here the prime denotes the variational derivative

$$A'[B] := \frac{\partial}{\partial \varepsilon}(A + \varepsilon B)_{|\varepsilon=0} \ .$$

As a consequence of the hereditariness of Φ one is able to define a hierarchy of pairwise commuting symmetries $K_n(u) = \Phi^n(u) K_0(u) = \Phi^n(u) u_x$. In addition, all these symmetries are hamiltonian vector fields provided $\Theta_0^{-1} u_x$ is a gradient. There is another recursive way of constructing the hierarchy of commuting symmetries via the so called mastersymmetries ([11],[13],[18],[31],[33],[34],[37]). One example for such an important quantity is the scaling mastersymmetry τ_0, for which we choose the following normalization

$$L_{\tau_0}\Phi := \Phi'[\tau_0] - \tau_0'\Phi + \Phi\tau_0' = \Phi \ \text{ and } \ L_{\tau_0} K_0 = [\tau_0, K_0] = \alpha K_0 \ , \tag{1.2}$$

where $\alpha > 0$ is a scalar. If we define $\tau_n := \Phi^n \tau_0$ we obtain further mastersymmetries and one can show with the help of the product rule for Lie-derivatives

$$\begin{gathered} L_{K_n} K_m = [K_n, K_m] = 0 \ , \ \ L_{\tau_n} K_m = [\tau_n, K_m] = (m + \alpha) K_{n+m} \\ L_{\tau_n} \tau_m = [\tau_n, \tau_m] = (m - n) \tau_{n+m} \ . \end{gathered} \tag{1.3}$$

In this work we are interested in the structural behaviour of N-soliton solutions of (1.1). We first turn our attention to N-soliton solutions which decompose asymptotically for $t \to \pm\infty$ into 1-solitons, i.e.

$$u_N \cong \sum_{i=1}^{N} s_i \ (x + c_i t + q_i) \tag{1.4}$$

If the speeds c_i and the phases q_i are considered as variables then the set of these solutions forms a $2N$-dimensional invariant submanifold M_N of M. The structure of M_N and the embedding of that finite dimensional submanifold into the whole manifold M was examined in [21], [22]. Although the methods used in these papers are quite different to those given in this article, we have to mention one result. By use of the reduction technique introduced in [22] it turns out that the induced flow on the soliton manifold M_N is consistent with the representation we will give in the following chapters.

For the submanifold M_N we are now going to give a new parametrization in the following way. Define a map Π which assigns to each u the set of asymptotic data $(q_1, \ldots, q_N, c_1, \ldots, c_N)$. Of course, this map cannot be written down explicitly. However, the general procedure how to find the new coordinates is very simple in principle: Take an arbitrary $u = u(0)$ out of this manifold as an initial condition for the flow given by (1.1), solve this equation and get the quantities (q_i, c_i) by comparison of the solution with (1.4). Observe that although we use the asymptotic behaviour of the N-solitons this new parametrization is defined for **arbitrary** time. The quantities q_i, c_i are scalar fields on the submanifold M_N and we now define their time dependence by setting

$$\begin{aligned} q_i(t) &= q_i(u(t)) \\ c_i(t) &= c_i(u(t)) \ , \end{aligned}$$

where $u(t)$ is the solution of (1.1) and where now the q_i and c_i are determined by taking this $u(t)$ as the new initial condition for the procedure described above. Thus we have expressed the flow in terms of the new coordinates (q_i, c_i).

Lemma 1 :
For all $i = 1, \ldots, N$ it holds

$$\frac{\partial}{\partial t} \ q_i(t) = c_i \ , \quad \frac{\partial}{\partial t} \ c_i(t) = 0 \ . \tag{1.5}$$

Proof : Let $\bar{u}(t) \cong \sum_{i=1}^{N} s_i(x + c_i(0)t + q_i(0))$ be the solution of (1.1) for the initial condition $\bar{u}(0) = u_0$. For fixed t_0 we consider now $\bar{u}(t_0)$ as a new initial condition. Solving equation (1.1) then gives

$$\bar{u}(t) \cong \sum_{i=1}^{N} s_i(x + c_i(t_0)(t - t_0) + q_i(t_0)).$$

Now comparison of the two solutions yields the desired result

$$\begin{aligned} q_i(t_0) &= c_i t_0 + q_i(0) \ , \\ c_i(t_0) &= c_i(0) \ = \ c_i \ . \end{aligned}$$

Lemma 1 clearly shows that the flow (1.1) is linearized in our new coordinates

$$\frac{\partial}{\partial t}\begin{pmatrix} q_1 \\ \vdots \\ q_N \\ c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix} , \tag{1.6}$$

where I denotes the $N \times N$ unit matrix. Because of this linear structure the dynamics given by (1.1) is completely trivial with respect to this parametrization.

Of course at this stage these considerations don't give any practical informations about the problem with respect to its original coordinates x, t. However, the content of the following chapters is that we are able to construct explicitly canonical coordinates (so called action / angle coordinates) and that we can determine explicitly their corresponding vector fields in terms of the field variable u. Since our arguments will concern three different levels, the manifold M_N itself, the vector fields and the scalar fields on M_N, we would like to give a picture of the mappings which will be important for the next sections. For convenience we call the manifold M_N parametrized in x, t-coordinates the physical space, if we mean M_N endowed with the coordinates (q_i, c_i), linearizing the flow (1.1), we speak about the linear space. Of course, these are only abbreviations, since the manifold is in all cases the same.

2 Linear Aspects

We now restrict ourselves for a while to the linearized system (1.6) to examine its structure. We give the hamiltonian formulations of the system (1.6) and the related action/angle variables in (q_i, c_i)- representation. Furthermore, we define a hereditary recursion operator for the system (1.6) and examine its spectral properties.

Looking at the linearized system (1.6) one observes that this system is given in hamiltonian form. Recall that an equation is hamiltonian if the flow is of the form $\bar{\Theta}\mathrm{grad}\,\bar{H}$, where $\bar{\Theta}$ is an implectic (or Poisson) operator. In our case we have

$$\bar{\Theta} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \bar{H} = \left(\frac{1}{2}\sum_{i=1}^{N} c_i^2\right) \quad . \tag{2.1}$$

Every implectic operator Θ defines a natural Poisson bracket for scalar fields f and g in the following way

$$\{\, f \,,\, g \,\}_\Theta \;:=\; <\ \mathrm{grad}\, g\ ,\ \Theta\, \mathrm{grad}\, f\ > \quad .$$

Equation (1.6) admits many different hamiltonian formulations. For every $p \neq 2$ it holds

$$\begin{pmatrix} c_1 \\ \vdots \\ c_N \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \Lambda_p \\ -\Lambda_p & 0 \end{pmatrix} \ \mathrm{grad}\ \left(\frac{1}{2-p} \sum_{i=1}^{N} c_i^{2-p} \right) \quad . \tag{2.2}$$

Here Λ_p denotes the diagonal $N \times N$- matrix

$$\Lambda_p = \begin{pmatrix} c_1^p & 0 & \cdots & 0 \\ 0 & c_2^p & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & c_N^p \end{pmatrix} \quad . \tag{2.3}$$

For our considerations and for reasons which become obvious later on we choose the hamiltonian operators given by

$$\bar{\Theta}_0 := \begin{pmatrix} 0 & \Lambda_{(-\alpha)} \\ -\Lambda_{(-\alpha)} & 0 \end{pmatrix} \text{ and } \bar{\Theta}_1 := \begin{pmatrix} 0 & \Lambda_{(1-\alpha)} \\ -\Lambda_{(1-\alpha)} & 0 \end{pmatrix} \quad , \tag{2.4}$$

where α is the normalization factor introduced in (1.2). The sum of these two implectic operators again is an implectic operator, hence these operators give rise to the hereditary recursion operator

$$\bar{\Phi} = \bar{\Theta}_1 \bar{\Theta}_0^{-1} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix} \quad . \tag{2.5}$$

Since $\bar{\Phi}$ is in diagonal form the eigenvalues of $\bar{\Phi}$ are $c_1, ..., c_N$; each of them occurs doubly. For every $i = 1, ..., N$ the eigenvectors $\bar{A}_i$ and $\bar{B}_i$ of $\bar{\Phi}$ w.r.t. c_i are given by partial derivatives

$$\bar{A}_i := \begin{pmatrix} 0 \\ \cdot \\ 1 \\ 0 \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \frac{\partial}{\partial q_i} \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_N \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{pmatrix} \quad , \quad \bar{B}_i := \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} = \frac{\partial}{\partial c_i} \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_N \\ c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_N \end{pmatrix} \quad . \tag{2.6}$$

For convenience we introduce $\bar{u} := (q_1, \ldots, q_N, c_1, \ldots, c_N)^*$ (* denotes the transposed vector) and collect all the results of the system with respect to its linear coordinates in the following

Lemma 2 :

(a) W.r.t. the Poisson bracket given by $\bar{\Theta}_1$ the coordinates c_i^α, q_j fulfill for all $i, j = 1, ..., N$ the following relations

$$\{c_i^\alpha, q_j\}_{\bar{\Theta}_1} \;=\; \alpha\, \delta_{ij} \quad , \quad \{c_i^\alpha, c_j^\alpha\}_{\bar{\Theta}_1} \;=\; \{q_i, q_j\}_{\bar{\Theta}_1} \;=\; 0 \ . \tag{2.7}$$

Hence, $\frac{1}{\alpha} c_i^\alpha$, q_i are the canonical coordinates corresponding to $\bar{\Theta}_1$. They are called canonical action/angle variables.

(b) The symmetries $\bar{K}_n := \bar{\Phi}^n (c_1, \ldots, c_N, 0, \ldots, 0)^*$ are bi-hamiltonian vector fields with

$$\bar{K}_n = (c_1^n, c_2^n, \ldots, c_N^n, 0, \ldots, 0)^* \;=\; \bar{\Theta}_1 \operatorname{grad} \Big(\frac{1}{n+\alpha} \sum_{i=1}^{N} c_i^{n+\alpha} \Big) \tag{2.8}$$

$$= \quad \bar{\Theta}_0 \text{ grad } \big(\frac{1}{n+1+\alpha} \sum_{i=1}^{N} c_i^{n+1+\alpha} \big) \ .$$

(c) The vector field

$$\bar{\tau}_0 := (-\alpha q_1, \ldots, -\alpha q_N, c_1, \ldots, c_N)^* = \bar{\Theta}_1 \text{ grad } \big(- \sum_{i=1}^{N} c_i^\alpha q_i \big) \qquad (2.9)$$

is a scaling mastersymmetry for $\bar{K}_n$, i.e.

$$[\bar{\tau}_0, \bar{K}_n] = (n+\alpha)\bar{K}_n \ . \qquad (2.10)$$

It is easily seen that in this normalization $\bar{\tau}_0$ is neither a hamiltonian vector field w.r.t. $\bar{\Theta}_0$ nor for any of the higher implectic operators $\bar{\Theta}_n := \bar{\Phi}^n \bar{\Theta}_0$.

(d) At each point the tangent space is the span of the eigenvectors of $\bar{\Phi}$.

(e) The eigenvectors $\bar{A}_1, \ldots, \bar{A}_N, \bar{B}_1, \ldots, \bar{B}_N$ of $\bar{\Phi}$ are given by the partial derivatives of the "field variable" $\bar{u} = (q_1, \ldots, q_N, c_1, \ldots, c_N)^*$ w.r.t. the coordinates given by the asymptotic data c_i and q_i.

(f) For every $i = 1, ..., N$ it holds

$$\bar{A}_i \ = \ \frac{\partial}{\partial q_i} \, \bar{u} \ = \ \bar{\Theta}_1 \text{ grad } (\frac{1}{\alpha} c_i^\alpha) \ , \qquad (2.11)$$

$$c_i^{1-\alpha} \bar{B}_i \ = \ c_i^{1-\alpha} \frac{\partial}{\partial c_i} \, \bar{u} \ = \ \alpha \, \frac{\partial}{\partial (c_i^\alpha)} \, \bar{u} \ = \ \bar{\Theta}_1 \text{ grad } (-q_i) \ . \qquad (2.12)$$

Observe that the second set of eigenvectors are hamiltonian vector fields only w.r.t. $\bar{\Theta}_1$.

3 The Physical Space

In the previous section we found canonical coordinates for the linear system (1.6). But of course, the natural question arises if one is able to recover the corresponding action/angle variables and their related vector fields on the physical N-soliton manifold in explicit form. By 'explicit' we mean that it must be possible to express these quantities in terms of the original field variable u. This requirement emphasizes the difference between our approach and the approach via the Inverse Scattering Transform, where the action/angles coordinates are expressed in a less direct way, namely in terms of the scattering data.

As we have seen in the last section the structure of the linear system is more or less trivial. Now we are trying to carry over the whole structure to the system represented in physical coordinates. Although we don't know the explicit form of our variable transformation Π, we know how tensor fields behave under a change of coordinates. In other words Π induces the pushforward

$$\Pi' : T_u\, M_N \rightarrow T_{\bar{u}}\, \mathbb{R}^{2N} \quad ,$$

which maps vector fields of the nonlinear space onto vector fields on the linear space. Furthermore we obtain the pullback

$$\Pi^+ := (\Pi')^* : T^*_{\bar{u}}\, \mathbb{R}^{2N} \rightarrow T^*_u\, M_N \quad ,$$

which is the transpose of Π' w.r.t. the duality between the tangent and the cotangent bundle. Then for every implectic operator

$$\bar{\Theta} : T^*_{\bar{u}}\, \mathbb{R}^{2N} \rightarrow T_{\bar{u}}\, \mathbb{R}^{2N}$$

the coordinate transformation Π gives an implectic operator

$$\Theta : T^*_u\, M_N \rightarrow T_u\, M_N \quad \text{by} \quad \Theta = (\Pi')^{-1} \circ \bar{\Theta} \circ (\Pi^+)^{-1} \quad .$$

With the properties of these mappings ([2],[36]) we find in analogy to lemma 2:

Lemma 3 :

(a) $\Phi = (\Pi')^{-1} \circ \bar{\Phi} \circ \Pi' : T_u\, M_N \rightarrow T_u\, M_N$ is a hereditary recursion operator on the physical space.

(b) The eigenvalues of Φ are $c_1, \ldots, c_N$; each of them occurs doubly. At each point $u \in M_N$ the tangent space $T_u M_N$ is the span of the eigenvectors of Φ.

(c) The eigenvectors $A_1, \ldots, A_N, B_1, \ldots, B_N$ of Φ are given by the derivative of the field variable u w.r.t. the coordinates $q_1, \ldots, q_N, c_1, \ldots, c_N$.

(d) For every $i = 1, \ldots, N$ the vector fields $A_i, c_i^{1-\alpha} B_i$ are hamiltonian vector fields w.r.t. $\Theta_1 := (\Pi')^{-1} \circ \bar{\Theta}_1 \circ (\Pi^+)^{-1}$.

Proof:

Of course (a) and (b) are trivial consequences of the definition of Π' and of the fact that Π' is a Lie algebra isomorphism i.e.

$$\Pi'[A,B] = [\Pi' A, \Pi' B] \qquad \forall\ A, B \in T_u\, M_N \quad .$$

(c) By definition of the eigenvectors of Φ we obtain for example that

$$\begin{aligned} B_i &= (\Pi')^{-1} \bar{B}_i = (\Pi')^{-1} \frac{\partial}{\partial c_i} \bar{u} = \\ &= (\Pi')^{-1} \frac{\partial}{\partial c_i} \Pi(u(q_1, \ldots q_N, c_1, \ldots c_N)) = (\Pi')^{-1} \Pi' \frac{\partial u}{\partial c_i} = \frac{\partial}{\partial c_i}\, u \end{aligned}$$

is an eigenvectors of Φ for the eigenvalue c_i.

(d) follows directly from lemma 2 (f).

Lemma 3 is clearly the first step into the direction of the explicit construction of all interesting quantities, since it gives the eigenvectors of the recursion operator as the partial derivatives of a N-soliton-solution w.r.t. the asymptotic data. Of course this is well know for $\frac{\partial u}{\partial q_i}$ ([7],[19]). But it may come as a surprise that the second set of eigenvectors is given simply by $\frac{\partial u}{\partial c_i}$. In another context and with different tools this fact was proven in [22]. But in that paper there was still the task left to identify explicitly the corresponding hamiltonian structures and to find scalar fields E_i, Ω_i which can be interpreted as action/angle variables. This is not a trivial task, because the manifold under consideration is not a vector space anymore and only the gradients of these scalar fields are known.

From the preceeding lemmas we know that on M_N there exists an implectic Θ with

$$A_i = \frac{\partial u}{\partial q_i} = \Theta \text{ grad } (\frac{1}{\alpha} E_i^\alpha) \ , \tag{3.1}$$

$$c_i^{1-\alpha} B_i = c_i^{1-\alpha} \frac{\partial u}{\partial c_i} = \alpha \frac{\partial u}{\partial (c_i^\alpha)} = \Theta \text{ grad } (-\Omega_i) \ , \tag{3.2}$$

where we fixed the normalization of E_i, Ω_i to be the same as for c_i, q_i. At this stage we need two essential ingredients for finding the potentials of these fields: the first is the knowledge that all the mastersymmetries are vector fields tangent to M_N. Secondly, that the scaling mastersymmetry τ_0 is a hamiltonian vector field[1] on M, i.e. on the whole manifold one **uniquely** finds a scalar field F and an implectic operator Θ with

$$\tau_0(u) \quad = \quad \Theta(u) \text{ grad } F(u). \tag{3.3}$$

A nontrivial consequence of equation (3.3) is that with respect to the reduction to M_N the field τ_0 remains hamiltonian w.r.t. $\Theta_{|red}$ ([22]) and the corresponding scalar field is $F(u_N)$ on M_N. This is nontrivial since reductions are in general not compatible with the hamiltonian structure. With this additional information we now easily obtain for the N-soliton manifold the following far reaching result:

Theorem 1 :

(a) The eigenvectors

$$A_i \ = \ \frac{\partial u}{\partial q_i} \quad \text{and} \quad c_i^{1-\alpha} \, B_i \ = \ c_i^{1-\alpha} \, \frac{\partial u}{\partial c_i}$$

of $\Phi(u)$ are hamiltonian vector fields w.r.t. the implectic operator $\Theta_{|red}(u)$ determined by

$$\tau_0 \ = \ \Theta(u) \text{ grad } F(u) \ .$$

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[1] For the existence of this quantity see section 5.

The reduced implectic operator $\Theta_{|red}$ has the following representation on M_N

$$\Theta_{|red} \; = \; \Theta_1 \; = \; (\Pi')^{-1} \circ \bar{\Theta}_1 \circ (\Pi^+)^{-1} \quad .$$

(b) The potentials E_i^α and Ω_i of the eigenvectors A_i and $c_i^{1-\alpha} B_i$ are given by the partial derivatives

$$E_i^\alpha \; = \; -\, \frac{\partial F}{\partial q_i} \quad , \quad \Omega_i \; = \; -\, \frac{1}{\alpha}\, c_i^{1-\alpha}\, \frac{\partial F}{\partial c_i} \; = \; -\, \frac{\partial F}{\partial (c_i^\alpha)} \quad . \tag{3.4}$$

(c) $\frac{1}{\alpha} E_i^\alpha$ and Ω_i are canonical coordinates w.r.t. Θ_1, i.e. for all $i, j = 1, \ldots, N$ it holds

$$\{E_i^\alpha, E_j^\alpha\}_{\Theta_1} \; = \; 0 = \; \{\Omega_i, \Omega_j\}_{\Theta_1} \;\; , \;\; \{E_i^\alpha, \Omega_j\}_{\Theta_1} \; = \; \alpha\, \delta_{ij} \quad .$$

Proof:

(a) One has to keep in mind that the pullback conserves hamiltonian structures. Then (a) is a direct consequence of the fact that in the linear space the corresponding eigenvectors and the scaling mastersymmetry are hamiltonian vector fields only w.r.t. the **same** implectic operator $\bar{\Theta}_1$.

(b) From (a) we know that there are scalar fields E_i and Ω_i with

$$\frac{\partial u}{\partial q_i} = \Theta_1 \text{ grad } (\frac{1}{\alpha}\, E_i^\alpha) \quad \text{and} \quad c_i^{1-\alpha} \frac{\partial u}{\partial c_i} = \Theta_1 \text{ grad } (-\Omega_i) \quad . \tag{3.5}$$

Since these scalar fields are conservation laws it follows

$$\begin{aligned} E_i^\alpha \;\; &= \;\; L_{\tau_0}\, (\frac{1}{\alpha}\, E_i^\alpha) = < \frac{1}{\alpha} \text{ grad } E_i^\alpha, \tau_0 > = < \frac{1}{\alpha} \text{ grad } E_i^\alpha, \Theta_1 \text{ grad } F > \\ &= \;\; - < \text{ grad } F, \Theta_1 \text{ grad } (\frac{1}{\alpha}\, E_i^\alpha) > = - < \text{ grad } F, \frac{\partial u}{\partial q_i} > = - \frac{\partial F}{\partial q_i} \\ -\alpha\, \Omega_i \;\; &= \;\; L_{\tau_0} \Omega_i \;\; = \; < \text{grad } \Omega_i, \Theta_1 \text{ grad } F > \; = \\ &= \;\; < \text{grad } F, \Theta_1 \text{ grad } (-\Omega_i) > \; = \; c_i^{1-\alpha}\, \frac{\partial F}{\partial c_i}. \end{aligned}$$

(c) is a direct consequence of (b) together with lemma 2 (a) and (f).

Thus this theorem gives us the desired result, namely the action/angle coordinates $\frac{1}{\alpha} E_i^\alpha$ and Ω_i of the nonlinear space expressed in terms of the field variable u.

4 Examples

4.1 The Korteweg-deVries Equation

The celebrated Korteweg-de Vries equation (KdV) ([23],[41]) is given by

$$u_t = u_{xxx} + auu_x = K_1(u) \ , \tag{4.1}$$

where u is assumed to be an element of the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$. The hierarchies of commuting symmetries generated by the well known hereditary operator Φ ([15])

$$\begin{aligned} K_n(u) &= \Phi^n(u)K_0(u) = (D^2 + \frac{a}{3}DuD^{-1} + \frac{a}{3}u)^n u_x \\ &= (\Theta_1\Theta_0^{-1})^n u_x = (\ (D^3 + \frac{a}{3}Du + \frac{a}{3}uD)D^{-1})^n u_x \end{aligned} \tag{4.2}$$

and mastersymmetries

$$\tau_n(u) = \Phi^n(u)\tau_0(u) = (D^2 + \frac{a}{3}DuD^{-1} + \frac{a}{3}u)^n \ (\frac{x}{2}u_x + u) \tag{4.3}$$

fulfill the commutator relations ([11])

$$\begin{gathered} [K_n, K_m] = 0 \ , \ [\tau_n, \tau_m] = (m-n)\ \tau_{m+n} \\ [\tau_n, K_m] = (m + \frac{1}{2})\ K_{n+m} \ . \end{gathered} \tag{4.4}$$

Here D denotes the differential operator w.r.t. the variable x and D^{-1} its inverse

$$D = \frac{\partial}{\partial x} \quad , \quad D^{-1} = \int_{-\infty}^{x} ... \ d\xi \ . \tag{4.5}$$

The scaling mastersymmetry τ_0 is a hamiltonian vector field w.r.t. the second implectic structure

$$\tau_0(u) = (D^3 + \frac{a}{3}Du + \frac{a}{3}uD) \text{ grad } \frac{3}{2a}\int_{-\infty}^{+\infty} xu \ dx = \Theta_1(u) \text{ grad } F(u) \ . \tag{4.6}$$

The N-soliton solutions decomposing at ±∞ into 1-soliton solutions are given by ([1])

$$u_N \cong \sum_{i=1}^{N} \frac{3}{a}c_i \ \mathrm{sech}^2 \ [\ \frac{1}{2}\sqrt{c_i}(x + c_i t + q_i)] \ , \tag{4.7}$$

where c_i are the eigenvalues of the recursion operator Φ. Since $\alpha = \frac{1}{2}$ theorem 1 gives the action/angle variables w.r.t. $\Theta_{1|red}$ explicitly as

$$2\sqrt{E_i} = -2\,\frac{\partial F}{\partial q_i} = -\frac{3}{a}\int_{-\infty}^{+\infty} xu_{q_i} \ dx \tag{4.8}$$

$$\Omega_i = -\frac{\partial F}{\partial(\sqrt{c_i})} = -\frac{3}{a}\sqrt{c_i}\int_{-\infty}^{+\infty} xu_{c_i} \ dx \ . \tag{4.9}$$

4.2 The modified Korteweg-deVries Equation

For $u \in S(\mathbb{R})$ we consider the modified Korteweg-deVries equation (mKdV)

$$u_t = u_{xxx} + au^2 u_x =: K_1(u) \ . \tag{4.10}$$

The corresponding recursion operator

$$\begin{aligned} \Phi(u) &= D^2 + \frac{2}{3} aDuD^{-1}u = \\ &= (D^3 + \frac{2}{3} aDuD^{-1}uD)D^{-1} \quad =: \ \Theta_1 \Theta_0^{-1} \end{aligned} \tag{4.11}$$

is hereditary. Here D and D^{-1} again denote the differential operator and its inverse, respectively. The hierarchies of commuting symmetries $K_n(u)$ and mastersymmetries $\tau_n(u)$ are given by

$$\begin{aligned} K_n(u) &= \Phi^n(u)\, K_0(u) \ = \ \Phi^n \ u_x \ , \\ \tau_n(u) &= \Phi^n(u)\, \tau_0(u) \ = \ \Phi^n \ \frac{1}{2}(xu_x + u) \ . \end{aligned}$$

These vector fields fulfill the commutator relations

$$[K_n, K_m] = 0 \ , \ \ [\tau_n, \tau_m] = (m-n)\ \tau_{m+n}$$

$$[\tau_n, K_m] = (m + \frac{1}{2})\ K_{n+m} \ . \tag{4.12}$$

The scaling mastersymmetry $\tau_0(u) = \frac{1}{2}(xu_x + u)$ is a hamiltonian vector field w.r.t. the implectic operator $\Theta_0 = D$

$$\tau_0(u) = D \ \mathrm{grad} \ \frac{1}{4} \int_{-\infty}^{+\infty} xu^2 \ dx \ = \ \Theta_0(u) \ \mathrm{grad} \ F(u) \ . \tag{4.13}$$

The N-soliton solutions decomposing into 1-solitons for $t \to \pm\infty$ are given by ([30])

$$u_N \ \cong \ \sum_{i=1}^{N} \sqrt{\frac{6}{a} c_i} \ \ \mathrm{sech} \ [\ \sqrt{c_i}(x + c_i t + q_i)] \ . \tag{4.14}$$

Since $\alpha = \frac{1}{2}$ theorem 1 determines the action/angle variables w.r.t. $\Theta_{0|red}$

$$2\sqrt{E_i} \ = \ -2 \, \frac{\partial F}{\partial q_i} \ = \ - \int_{-\infty}^{+\infty} xuu_{q_i} \ dx \tag{4.15}$$

$$\Omega_i \ = \ - \, \frac{\partial F}{\partial(\sqrt{c_i})} \ = \ - \sqrt{c_i} \int_{-\infty}^{+\infty} xuu_{c_i} \ dx \ . \tag{4.16}$$

4.3 The 7-th order CDGSK Equation

Our next example is the 7-th order Caudrey-Dodd-Gibbon-Sawada-Kotera equation (CDGSK) ([10],[38])

$$\begin{aligned} u_t &= u_{7x} + \frac{7}{2}\, a\, uu_{5x} + 7\, a\, u_x u_{4x} + \frac{21}{2}\, a\, u_{2x}u_{3x} + \\ &\quad + \frac{7}{2}\, a^2\, u^2 u_{3x} + \frac{21}{2}\, a^2\, u_x u_{2x} u_{3x} + \frac{7}{4}\, a^3\, u_x^3 + \frac{7}{6}\, a^3\, u^3 u_x = \\ &=: K_1(u) \ , \end{aligned} \tag{4.17}$$

where $u \in S(\mathbb{R})$ and u_{nx} denotes the n-th partial derivative of u w.r.t. x. The recursion operator ([17],[28])

$$\Phi(u) := (D^3 + auD + aDu)D^{-1}(D^2 + \frac{1}{2}au)D(D^2 + \frac{1}{2}au)D^{-1}$$

is hereditary ([20]) and generates the hierarchies of commuting symmetries

$$K_n(u) \ := \ \Phi^n K_0 \ = \ \Phi^n u_x \tag{4.18}$$

and the hierarchy of mastersymmetries

$$\tau_n(u) \ := \ \Phi^n \tau_0 \ = \ \Phi^n \ \frac{1}{6}(xu_x + 2u) \ . \tag{4.19}$$

In the first chapter we have restricted ourselves to equations admitting one hierarchy of symmetries, which is recursively generated by application of the recursion operator on the generator of space translation. For that reason we have chosen the 7-th order CDGSK equation as the suitable example in this article. However, since the two hierarchies of the CDGSK belong to the same spectral problem, our considerations can be easily transfered to this situation. Especially it turns out that on the submanifold M_N every root of the recursion operator is again hereditary and can be given explicitly - at least in (q_i, c_i)-representation. Hence, a unified picture of the two hierarchies is obtained on M_N. The same is of course true for other equations admitting two seperated hierarchies of commuting symmetries as the Boussinesq equation (see subsection 4.4) or the Hirota-Satsuma system ([27]) for example.

After this remark we return to the vector fields $K_n(u)$ and $\tau_n(u)$ and calculate their commutator relations

$$[K_n, K_m] = 0 \ , \ \ [\tau_n, \tau_m] = (m-n)\ \tau_{m+n}$$

$$[\tau_n, K_m] = (m + \frac{1}{6})\ K_{n+m} \ . \tag{4.20}$$

The scaling mastersymmetry $\tau_0(u) = \frac{1}{6}(xu_x + 2u)$ is a hamiltonian vector field w.r.t. the implectic operator $\Theta_1(u) = D^3 + auD + aDu$

$$\tau_0(u) = (D^3 + aDu + auD) \text{ grad } \frac{1}{6a}\int_{-\infty}^{+\infty} xu\, dx \ = \ \Theta_1(u) \text{ grad } F(u) \ . \tag{4.21}$$

The N-soliton solutions decomposing into 1-solitons for $t \to \pm\infty$ are given by ([29])

$$u_N \;\cong\; \sum_{i=1}^{N} \frac{3}{a}\, c_i^{1/3}\, \mathrm{sech}^2\, [\, \frac{c_i^{1/6}}{2}(x + c_i t + q_i)]\ . \tag{4.22}$$

Since $\alpha = \frac{1}{6}$ we obtain the action/angle variables by theorem 1 w.r.t. $\Theta_{1|red}$

$$6\, E_i^{1/6} \;=\; -6\, \frac{\partial F}{\partial q_i} \;=\; -\, \frac{1}{a} \int_{-\infty}^{+\infty} x u_{q_i}\, dx \tag{4.23}$$

$$\Omega_i \;=\; -\, \frac{\partial F}{\partial (c_i^{1/6})} \;=\; -\, \frac{1}{a}\, c_i^{5/6} \int_{-\infty}^{+\infty} x u_{c_i}\, dx\ . \tag{4.24}$$

4.4 The 5-th order Boussinesq Equation

All the results above are fully applicable to two-component systems as the 5-th order Boussinesq equation ([39],[40]).

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_t &= \begin{pmatrix} \frac{4}{3}v_{3x} + \frac{2}{3}au_x v + \frac{2}{3}uv_x \\ \frac{4}{3}u_{5x} + \frac{2}{3}avv_x + 4au_x u_{2x} + 2auu_{3x} + \frac{2}{3}a^2u^2u_x \end{pmatrix} = \\ &=: K_1(\begin{pmatrix} u \\ v \end{pmatrix}) \end{aligned} \tag{4.25}$$

for example. For $u, v \in S(\mathbb{R})$ we define the hereditary recursion operator ([9]) in the following form

$$\begin{aligned} \Phi(\begin{pmatrix} u \\ v \end{pmatrix}) &= 6 \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \\ &= \Theta_1 \Theta_0^{-1}\ . \end{aligned}$$

The components of the recursion operator Φ are determined by

$$\begin{aligned} \phi_{11} &:= 3av \;+\; 2av_x D^{-1}\ , \\ \phi_{12} &:= 8D^2 \;+\; 2au \;+\; au_x D^{-1}\ , \\ \phi_{21} &:= 8D^4 + 10auD^2 + 15au_x D + 9au_{2x} + 2a^2u^2 + 2(au_{3x} + a^2uu_x)D^{-1}\ , \\ \phi_{22} &:= 3av \;+\; av_x D^{-1}\ . \end{aligned}$$

The compatible implectic operators Θ_1 and Θ_0 are given by

$$\Theta_1(\begin{pmatrix} u \\ v \end{pmatrix}) \;=\; 6 \begin{pmatrix} \phi_{12}\, D & , & \phi_{11}\, D \\ \phi_{22}\, D & , & \phi_{21}\, D \end{pmatrix}$$

and

$$\Theta_0(\begin{pmatrix} u \\ v \end{pmatrix}) \;=\; \begin{pmatrix} 0 & , & D \\ D & , & 0 \end{pmatrix}\ ,$$

where D is again the differential operator w.r.t. x and D^{-1} its inverse. With the starting symmetry

$$K_0(\begin{pmatrix} u \\ v \end{pmatrix}) \;=\; \begin{pmatrix} u_x \\ v_x \end{pmatrix} \tag{4.26}$$

and the scaling mastersymmetry

$$\tau_0(\begin{pmatrix} u \\ v \end{pmatrix}) \;=\; \frac{1}{3}\begin{pmatrix} xu_x + 2u \\ xv_x + 3v \end{pmatrix} \tag{4.27}$$

we obtain the two hierarchies of symmetries and mastersymmetries

$$K_n(u) \;:=\; \Phi^n K_0 \quad , \quad \tau_n(u) \;:=\; \Phi^n \tau_0 \ . \tag{4.28}$$

of the Boussinesq equation. These vector fields fulfill the commutator relations

$$\begin{gathered} [K_n, K_m] = 0 \quad , \quad [\tau_n, \tau_m] = (m-n)\, \tau_{m+n} \\ [\tau_n, K_m] = (m + \frac{1}{3})\, K_{n+m} \ . \end{gathered} \tag{4.29}$$

Futhermore it holds

$$\tau_0(u) \;=\; \Theta_1 \text{ grad } \frac{2}{a} \int_{-\infty}^{+\infty} xu\, dx \;=:\; \Theta_1 \text{ grad } F(u,v) \ . \tag{4.30}$$

The N-soliton solutions decomposing at $\pm\infty$ into 1-soliton solutions are given by ([26])

$$\begin{aligned} u_N &\cong \sum_{i=1}^{N} \frac{4}{3a} \frac{1}{\gamma^2} c_i^{2/3} \operatorname{sech}^2 [\, \frac{c_i^{1/3}}{\gamma} (x + c_i t + q_i)] \ , \\ v_N &\cong \sum_{i=1}^{N} \frac{9}{4a} c_i \operatorname{sech}^2 [\, \frac{c_i^{1/3}}{\gamma} (x + c_i t + q_i)] \ , \end{aligned}$$

where the constant γ is equal to $\gamma = (32/3)^{1/3}$. In this example α is equal to 1/3 and we again obtain the action/angle variables with the help of theorem 1

$$3\, E_i^{1/3} \;=\; -3 \frac{\partial F}{\partial q_i} \;=\; -\frac{6}{a} \int_{-\infty}^{+\infty} x u_{q_i}\, dx \tag{4.31}$$

$$\Omega_i \;=\; -\frac{\partial F}{\partial (c_i^{1/3})} \;=\; -\frac{6}{a} c_i^{2/3} \int_{-\infty}^{+\infty} x u_{c_i}\, dx \ . \tag{4.32}$$

4.5 The Nonlinear Schrödinger Equation

For the Nonlinear Schrödinger Equation (NLS) ([12],[43])

$$z_t = -iz_{xx} + i|z|^2 z \tag{4.33}$$

with $z : \mathbb{R} \to \mathbb{C}$ a complex valued function we introduce the standard notation as a two-component real system by setting $z = u + iv$ with $u, v \in S(\mathbb{R})$. Then equation (4.33) reads

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v_{xx} - v(u^2+v^2) \\ -u_{xx} + u(u^2+v^2) \end{pmatrix} = K_1(\begin{pmatrix} u \\ v \end{pmatrix}) \tag{4.34}$$

and it is the first member of the hierarchy

$$\begin{aligned} K_n(\begin{pmatrix} u \\ v \end{pmatrix}) &= \Phi^n(\begin{pmatrix} u \\ v \end{pmatrix}) K_0(\begin{pmatrix} u \\ v \end{pmatrix}) = \\ &= \begin{pmatrix} -2vD^{-1}u & , & D - 2vD^{-1}v \\ -D + 2uD^{-1}u & , & 2uD^{-1}v \end{pmatrix}^n \begin{pmatrix} u_x \\ v_x \end{pmatrix} \ . \end{aligned}$$

Here the operators D and D^{-1} are again the differential operator and its inverse. The hereditary recursion operator Φ ([15]) admits the following implectic/symplectic factorization

$$\begin{aligned} \Phi(\begin{pmatrix} u \\ v \end{pmatrix}) &= \begin{pmatrix} D - 2vD^{-1}v & , & 2vD^{-1}u \\ 2uD^{-1}v & , & D - 2uD^{-1}u \end{pmatrix} \begin{pmatrix} 0 & , & 1 \\ -1 & , & 0 \end{pmatrix} = \\ &=: \Theta_1(\begin{pmatrix} u \\ v \end{pmatrix})\, \Theta_0^{-1}(\begin{pmatrix} u \\ v \end{pmatrix}) \ . \end{aligned}$$

The scaling mastersymmetry τ_0 is given by

$$\begin{aligned} \tau_0(\begin{pmatrix} u \\ v \end{pmatrix}) &= \begin{pmatrix} xu_x + u \\ xv_x + v \end{pmatrix} = \\ &= \Theta_1(\begin{pmatrix} u \\ v \end{pmatrix})\, grad \frac{1}{2} \int_{-\infty}^{+\infty} x(u^2+v^2)\, dx = \\ &= \Theta_1(\begin{pmatrix} u \\ v \end{pmatrix})\, grad\, F(\begin{pmatrix} u \\ v \end{pmatrix}) \end{aligned} \tag{4.35}$$

and the higher mastersymmetries $\tau_n = \Phi^n \tau_n$ are obtained by recursive application of the recursion operator. Between the hierarchies of symmetries and mastersymmetries the following commutator relations ([11]) hold

$$[K_n, K_m] = 0 \quad , \quad [\tau_n, \tau_m] = (m-n)\tau_{m+n}$$

$$[\tau_n, K_m] = (m+1)K_{m+n} \ .$$

The pure N-soliton solutions z_N of the NLS have the structure of modulated oscillating waves of which the envelopes behave like KdV-solitons ([42],[43]). Therefore the envelopes are also characterized by $2N$ real parameters, the asymptotic speeds and the asymptotic phases. Again the N-soliton solutions z_N decomposes for $t \to \pm\infty$ into N 1-solitons ([12],[42])

$$z_N \cong \sum_{k=1}^{N} \frac{\delta_k}{\sqrt{2}} \frac{\exp(i\omega_k)}{\cosh(\frac{\delta_k}{2}(x + c_k t + q_k))} \tag{4.36}$$

with

$$\omega_k = \phi_k + \frac{c_k}{2}x - \frac{\delta_k^2 - c_k^2}{4}t - \frac{\pi}{2}$$

for $k = 1, ..., N$. Here c_k and q_k denote the asymptotic speeds (or the discrete eigenvalues of the recursion operator Φ) and the asymptotic phases of the corresponding 1-solitons. δ_k and ϕ_k are additional (real) parameters which determine the oscillating frequencies both in space and time and the amplitudes of the soliton solutions. Splitting $z_N = u_N + iv_N$ into his real and imaginary part one obtains the N-soliton solution

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} \cong \begin{pmatrix} \sum_{k=1}^{N} \frac{\delta_k}{\sqrt{2}} \frac{\cos(\omega_k)}{\cosh(\frac{\delta_k}{2}(x+c_k t+q_k))} \\ \sum_{k=1}^{N} \frac{\delta_k}{\sqrt{2}} \frac{\sin(\omega_k)}{\cosh(\frac{\delta_k}{2}(x+c_k t+q_k))} \end{pmatrix} \tag{4.37}$$

of the two-component system (4.34). Since $\alpha = 1$ we again find the action/angle variables with the help of theorem 1

$$E_k = -\frac{\partial F}{\partial q_k} = -\int_{-\infty}^{+\infty} x(uu_{q_k} + vv_{q_k})\, dx \tag{4.38}$$

$$\Omega_k = -\frac{\partial F}{\partial c_k} = -\int_{-\infty}^{+\infty} x(uu_{c_k} + vv_{c_k})\, dx \quad . \tag{4.39}$$

5 Concluding Remarks

It was our aim to give the action/angle representation of the Multi-soliton solutions in terms of the physical field variable u. Of course, the action/angle coordinates in terms of the asymptotic (scattering) data (here lemma 2) are known for a couple of years ([1],[12]). They are one of the important results of the celebrated Inverse Scattering Method. One should notice that as a by-result of our purely algebraic (reduction) approach we obtain not only the same action/angle coordinates (2.7) but also the identical trace formulas (2.8) without using the Inverse Scattering Method at all. Up to now this only holds for the discrete part of the spectrum, which may not come as a surprise, since the spectral properties of the Lax operator and the recursion operator are related.

However, the Lax operator gives only a spectral interpretation of the action variables, whereas the recursion operator in addition gives such an interpretation of the angle variables. So now, the complete action/angle representation is shown to be of a spectral theoretic nature.

Furthermore,the question arises if we can extend our approach even to the continous part of the spectrum. One hint into this direction might be seen in some papers of Alonso ([3],[4],[5]). For several examples he tried to find an interpretation of the dynamical structures exhibited by the Inverse Scattering Method in terms of canonical realizations of Lie Groups. He already recognized the fundamental role of our hamiltonian so-called scaling mastersymmetry τ_0, the corresponding Hamiltonian F and the admissible implectic operator Θ. In spite to our approach Alonso derived the spectral decomposition of F directly from the Inverse Scattering data. For the discrete spectral set his representation coincides with the one (2.9) found by our approach. However, he also gave the representation of F w.r.t. the continous part of the spectrum. Although this discovery shows one of the possible developments, the full relationship between the Inverse Scattering Method and our approach still is an open question and is left for further investigations.

For our results, we basically used the hamiltonian scaling mastersymmetry τ_0. So the question arises if this quantity has always the desired features. However, all our constructions remain valid, if we take any other mastersymmetry τ_k exhibiting one and only one hamiltonian formulation instead. The important condition we need is the existence of one and only one global master element $F(u)$ in the scalar fields, i.e. a scalar field such that for some $k \in \mathbb{N}_0$

$$\Theta_k(u) \text{ grad } F(u)$$

gives one of the mastersymmetries, say τ_r. Then of course, all the higher mastersymmetries are hamiltonian vector fields with the same scalar field $F(u)$ but w.r.t. different implectic structures

$$\tau_{r+m} = \Phi^m \Theta_k \text{ grad } F = \Theta_{k+m} \text{ grad } F \quad .$$

This situation is true for all hamiltonian soliton systems in $1+1$-dimension with recursion operator and has already stated by several authors (see for example [37]). Hence, our results hold for all these equations.

We would like to finish this paper with another remark about scalar fields and vector fields on the multi-soliton manifold M_N. By construction we know that the Poisson manifold P_{Θ_1} of scalar fields over the physical space and the corresponding one $P_{\bar{\Theta}_1}$ of the linear space are isomorphic. However, we are now able to give this isomorphism explicitly. We define a map $P : P_{\Theta_1} \to P_{\bar{\Theta}_1}$, which simply assigns to every scalar field

$$f(u_N) = \int_{-\infty}^{+\infty} ... \, dx \quad \in P_{\Theta_1}$$

the evaluation of the integral on M_N. For all our examples of section 4 as well as for other ones we find the following relations

$$P(E_i^\alpha) \;=\; \beta\, c_i^\alpha \quad , \quad P(\Omega_i) \;=\; \beta\, q_i \;\; ,$$

where β is some constant depending on the considered equation. Hence, the map P/β is the desired Lie-algebra isomorphism. This map P was already introduced in [6] by one of the authors (M. B.) and was used to give an interpretation of interacting solitons as field representatives of Galilean point particles ([8],[9]). At the level of vector fields one is now able to define the whole Lie algebra of interesting vector fields in (q_i, c_i)-representation. It turns out that some important subalgebras are related to the well known quantities in (x, t)-coordinates. However, some subalgebras are new for the equations under consideration. This aspect will be elaborated in a forthcoming paper by one of the authors (M.B.).

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