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The Soliton-Singularity Transform

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ABSTRACT

A common approach to singularity analysis and soliton structure is introduced. The Painlev'e analysis is modified in such a sense that it carries over to cases where usually truncation at the lower end causes difficulties. A method to recover the Painlev'e series from its constant level term is exhibited. The soliton-singularity transform is introduced and recognized to be connected to the Möbius group. This gives rise to a Darboux-like result for the spectral properties of the recursion operator. These connections explain why poles of soliton equations move like trajectories of interacting solitons. Furthermore it is explicitly computed how solutions of singularity equations behave under the effect of this soliton-singularity transform. This then leads to the result that only for scaling degrees $\alpha = -1$ and $\alpha = -2$ the usual Painlev'e analysis can be carried

out. A new invariance principle, connected to kernels of differential operators is discovered. This new invariance, for example, connects the explicit solutions of the Liouville equation with the Miura transform.

1. Introduction

In order to motivate the considerations which are to follow we consider, for example, the Painleve expansion for the Korteweg-de Vries equation [14]

$$u \;=\; -2\frac{{\phi_x}^2}{\phi^2} + 2\frac{\phi_{xx}}{\phi} + (-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\frac{{{\phi_{xx}}^2}^2}{\phi_x}) \tag{1.1}$$

and we pay special attention to its constant level term

$$\tilde{u}(\phi) \;=\; (-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\frac{{{\phi_{xx}}^2}^2}{\phi_x}). \tag{1.2}$$

Using the substitution

$$s = \phi_x \tag{1.3}$$

we rewrite this term as

$$\tilde{u} \;=\; -\frac{1}{2}\frac{s_{xx}}{s} + \frac{1}{4}\frac{{s_x}^2}{s^2} \tag{1.4}$$

There are a number of remarkable links between these representations. Let us list some of them:

i) We find

$$\tilde{u}(\frac{\phi_x}{\phi^2}) \;=\; u.$$

Thus a replacement of ϕ in (1.2) leads to a recovery of the Painlev'e series (1.1) from its constant level term.

ii) It is well known that the constant level term $\tilde{u}(\phi) = \hat{u}(s)$ is again a solution of the KdV if u is such a solution. Thus there must be some well defined dynamical behavior of the quantity s given by (1.3). Remarkably, this dynamical behavior does not involve any mixed derivatives in x and t since one easily obtains

$$s_t \;=\; s_{xx} \;-\; 2s_x s_x/s.$$

Now, since (1.4) constitutes a descending sequence in negative powers of a quantity s with well defined dynamical behavior, one may ask why this series is not considered to be the singularity expansion.

There are other surprising properties for the quantities s and ϕ on which we would like to draw the attention of the reader. For example:

iii) For equation (1.4) there are far more invariances than one might expect by considering only the known invariances for the time evolution of the singularity manifold (Möbius group).

iv) Some of the invariances mentioned under iii) are indeed connected to solutions of the Liouville equation

$$h_{xt} \;=\; e^{-2h}. \tag{1.5}$$

v) A simple analysis shows that the quantity s related to u by (1.4) is annihilated by the recursion operator of the KdV, i.e.

$$\Phi(u)s \;=\; (D^2+2u+2DuD^{-1})s=0. \tag{1.6}$$

What we claim in this paper is that all these remarkable properties are intimately connected and can be easily understood, and generalized, by extending the Painlev'e test. In addition it turns out that the structural results leading to these properties indeed show that the usual Painlev'e test can only be carried out for scaling degrees $\alpha = -1$ and $\alpha = -2$. An extended version of this paper will appear elsewhere [7], also there can be found an abundance of examples for the interrelations demonstrated here.

2. Painlev'e test versus Expansion test

We consider solutions of the evolution equation

$$u_t \;=\; K(u) \tag{2.1}$$

where $K(u)$ is a polynomial in u and its derivatives. Eventually we will also allow infinite polynomials (Taylor series). We make the ansatz

$$u \;=\; \sum_{n=0}^{\infty} \psi^{-n} F_{(n)}(\psi_x,\psi_{xx},\cdots) \tag{2.2}$$

in negative descending powers of a quantity ψ. We say that equation (2.1) passes the expansion test if this ansatz is always compatible with a time evolution of ψ given by

$$\psi_t = \sigma(\psi) \tag{2.3}$$

where σ is only depending on ψ and its spatial derivatives and where it is assumed that in the Laurent expansion of σ with respect to its dependence on ψ only negative powers occur. These requirements of course are severe restrictions since in general an ansatz like (2.2) when introduced into (2.1) yields the equation

$$K(\sum_{n=0}^{\infty} \psi^{-n} F_{(n)}) \;=\; \sum_{n=0}^{\infty} \psi^{-n}\{-n\psi^{-1}F_{(n)} + F_{(n)}{\prime}\}\sigma(\psi) \tag{2.4}$$

where very many mixed derivatives occur. In this formula the prime in $F{\prime}$ stands for the variational derivative of F with respect to ψ. Formula (2.4) gives, by comparison of powers in ψ, rise to an algorithmic determination of $F_{(n)}$ once the starting point $F_{(0)}$ has been chosen. There are two important cases for this algorithmic procedure.

Case 1 (*Painlev'e test*):

We assume that $\sigma(\psi)$ is only depending on the spatial derivatives of ψ, i.e. no ψ without x-derivative occurs. This corresponds to say that the t-derivatives affect the singularities given by the zeros of ψ in the same way as x-derivatives. This, of course, is the requirement one usually tacitly assumes in the Painlev'e test. Hence this assumption singles out the Painlev'e expansion. Looking at lowest order in ψ^{-1} in the expression (2.4) one discovers in this case as a consequence

$$F_{(0)_t} = K(F_{(0)}), \tag{2.5}$$

which proves that

$$\tilde{u} = F_{(0)} \tag{2.6}$$

again has to be a solution of (2.1). Looking at the next order one finds the well known fact that

$$K\prime(\tilde{u})[F_{(1)}] \ = \ F_{(1)}(\psi)\prime[\sigma] = F_{(1)_t} \tag{2.7}$$

which implies that $F_{(1)}$ is a symmetry generator for (2.1) at the manifold point $\tilde{u}$.

Case 2 (*constant highest order case, soliton test):*

In order to distinguish the field variable in this case from the previous one we use s instead of ψ. If $u = \lambda$ (λ a constant) does satisfy equation (2.1) then we put $\sigma(s) = D^{-1}K(u)D$ and $F_{(0)} = \lambda$ where we let vary the value of the constant λ. This is possible since any constant obviously satisfies the condition on the highest order term because the variational derivatives of $F_{(0)}$ disappears. In this case it does not matter at all whether or not negative powers of ψ appear explicitly in (2.3).

Remark:
The series which results for $\lambda = 0$ out of the case 2 expansion test, we call the **special expansion series**. However it has to be observed that the coefficients $F_{(n)}$ of this series may have a certain analytical structure with respect to the variable λ such that the choice of $\lambda = 0$ results in a value where the $F_{(n)}$ are singular with respect to λ. In general this means that the special expansion series has to be the analytical continuation of (2.4) with respect to λ at $\lambda = 0$. An example where this happens is provided by the members of the mKdV hierarchy (see [7]).

For reasons which become obvious in the next section, we say that, if the Case 2 expansion-test is successful, then equation (2.1) passes the **soliton-test**. In principle, both test could be run through by starting with $F_{(0)}$ (or $F_{(1)}$ in the special expansion test) as initial data. However, in case 1 this is not a realistic alternative since usually $\tilde{u} = F_{(0)}$ is related to u by a complicated Bäcklund transformation. Therefore in this case, for computational reasons, one has to start with the highest order term which is only possible if the series truncates also on the lower end. This, of course, is not necessary for case 2 since there the constant-level term is known in advance. For this reason we do not need in that situation a truncation at the lower end. So in principle these expansion series can be infinite series. From the computational viewpoint it seems remarkable that case 2 reverses the order of case 1. Therefore it is a surprise that nevertheless both cases almost give the same information.

3. Interacting Solitons.

We assume that $\Phi(u)$ is a hereditary recursion operator [4] for (2.1). Then the potentials s of its eigenvectors

$$\Phi(u)s_x \;=\; \lambda s_x \tag{3.1}$$

are called interacting solitons [5], [6]. Their dynamics is given by

$$s_{xt} \;=\; K\prime(u)[s_x] \;=\; \frac{\partial}{\partial\epsilon}_{\,|\,\epsilon\,=\,0} K(u+\epsilon s_x). \tag{3.2}$$

Usually it is not difficult to express u in terms of the quantity s (see [5])

$$u = \Gamma(s). \tag{3.3}$$

We assume that our equations are of such a form that Γ only depends on s and its spatial derivatives. For example for the KdV this representation of u in terms of s is

$$u \;=\; \frac{1}{4s^2}(\lambda s^2 \;-\; 2ss_{xx} \;+\; {s_x}^2). \tag{3.4}$$

It should be remarked that (1.4) is just the special case of (3.4) for $\lambda = 0$. Insertion of the relation (3.3) into (3.2) leads to the dynamics for s. In case of the KdV this yields

$$s^2 s_t \;=\; s^2 s_{xxx} \;-\; 3ss_x s_{xx} \;+\; \frac{3}{2}{s_x}^3 \;+\; \frac{3}{2}\lambda\; s^2 s_x \tag{3.5}$$

of which the dynamics (1.4) again is a special case.

In the following we only consider equations with negative scaling $\alpha < 0$. Scaling α means that the replacement

$$x \to mx\;, \quad u \to m^{\alpha}u\;, \quad t \to m^{\beta}t \;\;(\textit{some } \beta) \tag{3.6}$$

leaves the equation invariant. As a consequence of negative scaling we obtain that an expansion of (3.3) in terms of powers of s must be of the form

$$u \;=\; c + \frac{1}{s}\Gamma_{(1)} + \frac{1}{s^2}\Gamma_{(2)}\cdots, \qquad c \;=\; \textit{constant} \tag{3.7}$$

with $\Gamma_{(i)}$ polynomials in s_x, s_{xx} etc. This is the form which was required in case 2 of the expansion test. Since the members of the expansion test are uniquely and algorithmically determined we know that this expansion is the unique representation for case 2. This justifies that we gave this expansion the name "soliton expansion".

It should be observed that now all equations for which the dynamics of the interacting soliton is known can serve as examples for case 2 of the expansion test (see [5], [6]). Among these examples one finds several where the series does not truncate at the lower end (mKdV and sine-Gordon for $\lambda \neq 0$).

Now, we compare the Painlev'e series for (2.1) with the soliton expansion. We look at the constant level term $\tilde{u} = F_{(0)}(\phi)$ which is a homogeneous rational function in the derivatives of ϕ. We consider an expansion in negative powers of the lowest derivative $\phi_{(N)}$ in the denominator of that expression

$$\tilde{u} \;=\; \sum_{n=1}^{\infty} {\phi_{(N)}}^{-n} \, G^{(n)}(\phi_{(N+1)}, \phi_{(N+2)}, \cdots). \tag{3.8}$$

The zero order term of that has to disappear because it only could be a constant (independent of ϕ) which would be in contradiction to negative scaling degree. Obviously, that expansion again is a case 2-expansion for $\tilde{u}$ (instead of u), since only ϕ_N has to be renamed by s. Since the case 2 expansions were unique we know that (3.8) must coincide with the special expansion for $\tilde{u}$. It remains to compute the N (order of derivative). For this we look at the next order term in the Painlev'e series. This term is of the form

$$F_{(1)}(\phi_x, \phi_{xx}, \cdots)/\phi. \tag{3.9}$$

Since any rescaling $\phi \rightarrow a\phi$ leaves the whole Painlev'e series invariant we know that F_1 is of first order, and because of the scaling degree it has to bear $-\alpha$ derivatives. Hence we obtain

$$F_{(1)} \;=\; C\phi_{(-\alpha)} \tag{3.10}$$

where, without loss of generality, $C = 1$ can be chosen. Since we know that $F_{(1)}$ is a symmetry around $\tilde{u}$ and that the eigenvector $\tilde{s}_x$ of $\Phi(\tilde{u})$ has the same dynamic as a symmetry generator around $\tilde{u}$ we find the crucial identity

$$\phi_{(-\alpha)} \;=\; \tilde{s}_x. \tag{3.11}$$

Hence we found the N to be

$$N = -\alpha - 1. \tag{3.12}$$

4. Soliton-Singularity Transform

Let us resume the results of the preceding sections in a more systematic way. Again we consider

$$u_t \;=\; K(u) \tag{2.1}$$

with negative scaling $\alpha < 0$. We write the Painlev'e series and the special expansion series for u respectively as

$$u \;=\; PE(\phi) \quad (Painlev'e\ expansion) \tag{4.1}$$

$$u \;=\; SE(s) \quad (special\ expansion) \tag{4.2}$$

If $\tilde{u}$ denotes the constant level term in the Painlev'e expansion for u then we also consider the expansions for this solution $\tilde{u}$ of (2.1).

$$\tilde{u} \;=\; PE(\tilde{\phi}) \tag{4.3}$$

$$\tilde{u} \;=\; SE(\tilde{s}) \tag{4.4}$$

The map going from u, s, ϕ to $\tilde{u}, \tilde{s}, \tilde{\phi}$ we denote by SoSi( ), i.e.

$$\begin{aligned} \tilde{u} &= SoSi(u) \\ \tilde{s} &= SoSi(s) \\ \tilde{\phi} &= SoSi(\phi) \end{aligned} \tag{4.5}$$

The fundamental result of the last section was the

DUALITY:

$$\tilde{s}_x \;=\; \phi_{(-\alpha)} \;, \qquad s_x \;=\; \tilde{\phi}_{(-\alpha)}. \tag{4.6}$$

So SoSi changes, up to derivatives solitons in singularities and vice versa. This explains the name, since SoSi is meant to be an abbreviation for Soliton-Singularity transform.

Interchanging the role between u and $\tilde{u}$ in the respective series shows that SoSi is an involution i.e.

$$SoSi^2 \;=\; I \qquad (identity). \tag{4.7}$$

Now we want to compute SoSi explicitly, at least the effect it has on s and ϕ. Then its effect on u can be computed by considering the series obtained from one of the tests. In order to compute SoSi() we make the ansatz

$$s \;=\; \frac{v}{\phi^k} \tag{4.8}$$

where v is of the form

$$v \;=\; v_0(\phi_x, \phi_{xx}, ...) + \phi v_1(\phi_x, \phi_{xx}, ...) + \phi^2 \cdots . \tag{4.9}$$

Insertion in $u = SE(s)$ yields

$$u \;=\; SE(v/\phi^k). \tag{4.10}$$

Since the special expansion series was homogeneous in s we see that only those parts of the derivatives of s contribute to powers of zero order in ϕ where the $\phi' s$ are left uneffected by the derivatives. Since for those terms the powers of ϕ cancel, we easily obtain the constant level term $\tilde{u}$ as

$$\tilde{u} \;=\; SE(v). \tag{4.11}$$

Comparison with the special expansion for $\tilde{u}$ leads to

$$v \;=\; \beta\tilde{s} \;, \quad \beta \; constant \tag{4.12}$$

or

$$s \;=\; SoSi(\tilde{s}) \;=\; \beta\frac{\tilde{s}}{\phi^k} \;=\; \beta\frac{\tilde{s}}{(D^{\alpha+1}(\tilde{s}))^k}. \tag{4.13}$$

Since $SoSi(\cdot)$ has to be reciprocal we find that only the values $\alpha = -2$ and $\alpha = -1$ are possible, because for other values this transformation never can be an involution. In both cases we find for ϕ that

$$\tilde{\phi} \;=\; SoSi(\phi) \;=\; 1/\phi \tag{4.14}$$

Hence the substitution $\phi \rightarrow 1/\phi$ is a generic invariance for the equation of the singularity manifold. Since only derivatives occur in this equation another invariance must be

$$\phi \rightarrow \phi + c, \quad c\ constant.$$

These two invariances combined yield the Möbius group

$$\phi \rightarrow \frac{a\phi + b}{c\phi + d}, \quad ad - bc \;\neq\; 0.$$

A consequence of this is that the full Painlev'e series is always obtained from its constant level term. This is seen from

$$u \;=\; SE(s) \;=\; SE(SoSI(\tilde{s})) \;=\; SE(SoSi(D^{-1-\alpha}\phi)) \;=\; PE(\phi). \tag{4.15}$$

So substitution of

$$s \;=\; SoSi(D^{-1-\alpha}\phi) \tag{4.16}$$

into the special expansion $u = SE(s)$ yields the Painlev'e expansion $u = PE(\phi)$ for u.

The effect of the SoSi-transform onto the equations under consideration is exhibited in Fig.1. The notation is self explanatory. For example KdV(u) means the KdV equation for the field variable u , and so on. And in order to avoid confusion when arrows have to cross each other then instead of drawing bridges we denote that by circles. If a Since, by a reciprocal transformation, the singularity equation for KdV is related to the Harry Dym equation ([7],[13]) SoSi also must effect this equation as it can be transferred via the Bäcklund transformations shown in this figure.

Strictly speaking, the Painlev'e analysis amounts to a representation of u in terms of its zero soliton. Therefore we believe that the information of the Painlev'e test is somewhat restricted since it corresponds to only one point of the possible spectrum of the recursion operator. If one desires to transfer SoSi to other points of the spectrum of the recursion operator one has to proceed differently: Replace (2.1) by its translated equation

$$u_t = K(u) - \lambda u_x \tag{4.17}$$

and assign to the variable parameter λ a scaling such that the scaling of the equation does not change. Then the special expansion corresponds to the eigenvectors of $\Phi(u)$ with eigenvalues λ. Application of SoSi then leads to the following Darboux-like result:

Theorem:
Let s be an eigenvector of the recursion operator $\Phi(u)$ such that

$$u \;=\; SE(s) \tag{4.18}$$

can be expressed by some homogeneous expression in s. Then

$$\tilde{s} \;=\; SoSi(s) \tag{4.19}$$

is an eigenvector of

$$\tilde{u} \;=\; SE(\tilde{s}) \tag{4.20}$$

with the same eigenvalue.

One should observe that the procedure for going over to other points of the spectrum of the recursion operator sometimes leads to a dramatic change in the behavior of the expansions under consideration. For example, for the mKdV due to the analytical structure with respect to λ, it turns out that the series for $\lambda = 0$ are finite whereas those for $\lambda \neq 0$ have to be infinite (details see [7])

5. Kernel transformations

Looking at the last result in section 4 one is reminded of the classical Darboux method ([1],[2],[3]) for generating eigenvectors of the Schrödinger equation. This especially since there is a connection between the recursion operator of the KdV and the Schrödinger operator. However, any attempt to use SoSi() in order to prove this classical result fails. So we must have overlooked some invariances and indeed that is the case. Let us demonstrate that for the KdV.

Recall from the interacting soliton theory [5] that if u evolves according to the KdV and if

$$u \;=\; SE(s) \tag{5.1}$$

then s evolves in the following way

$$s_{xt} \;=\; K(u)\prime[s_x] \;=\; K\prime(SE(s))[s_x]. \tag{5.2}$$

But obviously (5.1) is a differential equation for s of second order, having a second solution. Changing from s to this second solution, does not change u , therefore the evolution (5.2) is not changed. Hence going from s to the other solutions of (5.1) must yield an invariance of (6.2). These transformations within the solution space of (5.1) we call **kernel-transformations**, since they are transformations within the kernel of $(\Phi(u) - \lambda I)$, the recursion operator of the translated equation. A general transformation group may now be obtained by combining SoSi and the kernel transformations. This group then also comprises the classical Darboux theorem. Therefore we call it the **Darboux-Möbius group**. Let us demonstrate that again in the KdV case:

Given s then the second solution usually can be found by variation of constants. In the KdV-case this procedure is easily carried out. Equation (5.1) may be rewritten in the form

$$u \;=\; SE(s) \;=\; \frac{\lambda}{4} - \frac{(\sqrt{s})_{xx}}{\sqrt{s}}. \tag{5.3}$$

Hence, as it is well known, $\sqrt{s}$ fulfills a linear differential equation of second order. For this the second solution is easily obtained from the given one by variation of constants. This second solution has the general form

$$\sqrt{s_2} \;=\; a\sqrt{s} + \sqrt{s}\;(D^{-1}(1/s)). \tag{5.4}$$

Thus for the solutions of (5.2) we have found the invariance

$$s \rightarrow a^2 s + 2asD^{-1}(1/s) + s(D^{-1}1/s)^2. \tag{5.5}$$

A special case of this is

$$s \rightarrow s(D^{-1}(1/s))^2 \;=\; ke(s) \tag{5.6}$$

which must leave (5.2) invariant. Hence for the singularity equation we have found the new invariance

$$\phi \rightarrow a^2\phi + 2aD^{-1}(\phi_x D^{-1}(1/\phi_x)) + D^{-1}(\phi_x(D^{-1}(1/\phi_x))^2). \tag{5.7}$$

We like to remark that for the translated mKdV the kernel transformation and SoSi coincide. So the question arises what has happened to the kernel transformation by going from KdV to mKdV. The answer is simple. We recall that the Miura transformation relates the KdV to the mKdV. Obviously that is a differential equation for the field quantity v of the mKdV in terms of u having not only one solution but a one-parameter family of solutions. So there must be a kernel transformation for the mKdV which does not effect the KdV. The determination of the kernel transformations of the Miura transformation is not that simple, but actually this it not necessary any more, because we know already the result. Going from the KdV to its zero soliton solution via the Miura transformation we find the following transformation from mKdV to the zero soliton of KdV which can be seen from the transformation table:

Because of the commutativity of the diagram the kernel transformation of the zero-soliton solution induces then a transformation for mKdV which is annihilated by the Miura transform. Hence, this yields the kernel transformation of the Miura transformation. Then

$$v \;=\; \frac{1}{2}\frac{s_x}{s} \quad <===> \quad s \;=\; e^{-2D^{-1}(v)} \tag{5.8}$$

implies (by using (5.6)) that

$$v \;\rightarrow\; \frac{1}{2}\frac{d}{dx}\;\ln(a^2 e^{-2D^{-1}(v)} + 2ae^{-2D^{-1}(v)}D^{-1}(e^{2D^{-1}(v)}) + e^{-2D^{-1}(v)}(D^{-1}(e^{2D^{-1}(v)}))^2) \tag{5.9}$$

leaves the mKdV invariant, and that this transformation substituted into the Miura transform leads to the same u as before.

This last statement seems interesting from another viewpoint: If one tries to find the kernel of the Miura transform directly, one would proceed in the following way: At first one looks for infinitesimal transformations which leave this transform invariant, i.e. one looks for $v \to v + \epsilon\tilde{v}$ such that this leads, up to ϵ^2 -terms, to the same u. This gives for $\tilde{v}$ the condition

$$-\tilde{v}_x + 2\tilde{v}v \;=\; 0 \tag{5.10}$$

or

$$\tilde{v} \;=\; e^{-D^{-1}2v}. \tag{5.11}$$

Hence the evolution equation

$$v_\tau \;=\; e^{-2D^{-1}v} \tag{5.12}$$

is the equation which describes the kernel of the Miura transform. By $h_x \;=\; v$ we rewrite that as

$$h_{xt} \;=\; e^{-2h} \tag{5.13}$$

which is the well known Liouville equation. Although the solution of that looks difficult we know it already, it is given by formula (5.9) which describes a one-parameter manifold invariant under that equation. The solutions obtained this way are the ones usually found in the literature ([10, p254], [8, p 71]).

An interesting feature of the transformation SoSi, and also the kernel transformation, is that they often make out of zeros of interacting solitons poles of solutions for the interacting soliton equation. This explains why usually poles behave like trajectories of solitons (a detailed analysis of this will appear elsewhere).

6. Comparison with other work

On the structural level the new results of this paper are all connected to the newly introduced expansion test, a generalization of the fundamental Painlev'e test for integrability proposed by Weiss, Tabor and Carnevale [14] which gave a new insight in the study of nonlinear partial differential equations. Since its introduction the Painlev'e test has been applied by many authors for a large number of nonlinear systems recognizing they possess the Painlev'e property. Now there are excellent surveys on this subject (for example [12] or [11]) so we do not have to go into a detailed analysis of these very many results.

The expansion test shares with the Painlev'e test its two most important aspects: namely its wide applicability and its algorithmic structure. The generalized test we propose, besides giving meaningful results whenever the Painlev'e test does, also is applicable in the study of those nonlinear systems for which the usual test can say nothing. In fact, by definition, a partial differential equation possesses the Painlev'e property whenever it admits solutions expressed in the form of a finite truncated expansion in terms of its singularity manifold function ϕ . The requirement that the expansion truncates after a finite number of terms is both restrictive and not needed for integrability. Furthermore it is the most difficult task to achieve in performing the test. For instance to prove that the sine-Gordon equation does possess the Painlev'e property "ad hoc" considerations are needed. Such is also the case for other equations. On the contrary, as far as our test is concerned, it does not make any difference whether the expansion has a finite or infinite number of terms [7]. However, since the possibility of infinite series prevents us from starting at the bottom, we have to start at the top of the series thus needing a simple solution of the equation to start with. This gives that the two tests cannot coincide and that the algorithmic procedure has been reversed.

This new algorithmic strategy allows then to reveal the intimate connection between the structure of the singularity equation and the corresponding interacting soliton ([5], [6]). In fact, a particular case of the Expansion test, we addressed to as the special expansion, is represented by the study of the interacting soliton for $\lambda = 0$, so that the singularity equation and the corresponding zero-soliton equation turn out to be linked by an involutive Bäcklund transformation. For the KdV and the AKNS hierarchy such an interpretation of the singularity function in terms of squared eigenfunctions of course is well known (see for example [11]). However, new is that this gives systematically rise to an involutory SoSi-transformation for all equations under consideration and, in addition, that the effect of SoSi on the singularity function indeed can be explicitly computed independently of the equation under consideration. This observation then leads to a wide variety of results (impossibility of the Painlev'e test for certain scaling degrees, recovery of the Painlev'e series from its constant level term, Darboux-Möbius transform, etc.). Another point also is clarified which was missing so far in the literature, namely that contrary to the KdV case, in general a relation between the singularity function and the squared eigenfunctions with nonzero eigenvalue is not possible. Insofar the expansion test really goes beyond the Painlev'e test. It has therefore to be expected that there is a wide variety of nonintegrable equations passing the Painlev'e test where the general expansion test fails.

Anyway the results show a close relation between the recursion operator and the expansion test. An immediate consequence is that a successful expansion test delivers the recursion operator whenever it does exist and that given the recursion operator both the singularity and the zero-soliton equation can be easily obtained without going through the whole expansion test.

The involutivity of the SoSi transformation between the zero-soliton and the singularity equation imply that the special expansion and the Painlev'e expansion can be obtained one from the other in a straight-forward way. This provides a structural reason why no nonlinear system with α different from - 1 or 2 - has been proved to possess the Painlev'e property. This is in accordance with the examples which can be found in the literature. To the same conclusion by different arguments, recently, came R.B. King [9] who was interested in finding necessary conditions a nonlinear partial differential equation must satisfy in order to possess the Painlev'e property.

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