Physics Letters, 88A, 323-327, 1982

EXPLICIT FORMULAS FOR SYMMETRIES AND CONSERVATION LAWS OF THE KADOMTSEV–PETVIASHVILI EQUATION

W. OEVEL and B. FUCHSSTEINER
Universität Paderborn, D-4790 Paderborn, West Germany

Received 22 December 1981

Two nonlocal recursion operators are given, which yield explicit formulas for infinite hierarchies of symmetry generators and conservation laws for the two-dimensional Korteweg–de Vries equation. It is shown that the constants of the motion are in involution and that the symmetries commute.

1. Introduction. For the popular examples of infinite dimensional completely integrable systems (such as the KdV, the sine-Gordon equation, etc.) it has been shown that the infinitely many commuting symmetries are generated by a local operator (cf. ref. [1]), which provides a selfmap in the space of infinitesimal generators of one-parameter symmetry groups and which gives a straightforward approach to other extraordinary features of these systems such as conservation laws and soliton solutions. Such an operator being hereditary in the sense of ref. [2] then gives further information such as the commutativity of the symmetry group.

Recently, in ref. [3] there was given an example (the Benjamin–Ono equation) where such an approach with local operator fails, but where nevertheless explicit formulas for the symmetry generators and the constants of the motion can be found. The recursion map in this case was a nonlocal operator in the space of symmetry generators.

In this paper we derive – by exactly the same method – explicit formulas for the symmetry generators as well as for the constants of the motion of the two-dimensional KdV or Kadomtse–Petviashvili equation. This equation was introduced in ref. [4] in order to describe weak two-dimensional perturbations of shallow water waves governed by the ordinary Korteweg–de Vries equation. This two-dimensional version is known to have infinitely many conservation laws since it admits a Lax-pair formulation [5] as well as an auto-Bäcklund transformation and a Gardner transform.

In addition to the explicit formulas our method immediately yields that the constants of the motion for the two-dimensional KdV are in involution, which corresponds to the fact that the group of symmetries is abelian.

The crucial operator τ_+ was found by the first author, the method goes back to ideas of the second author. The line of arguments is an adaption of ref. [3] (apart from the appendix).

2. Notation. In this paper we will be concerned with the two-dimensional KdV or Kadomtsev–Petviashvili equation

$$\partial(u_t - 6uu_x + u_{xxx})/\partial x + 3\alpha^2 u_{yy} = 0\,, \qquad (1)$$

which we formally rewrite as

$$u_t = K(u) = 6uu_x - u_{xxx} - 3\alpha^2 D^{-1} u_{yy}\,, \qquad (2)$$

$D^{-1} = \int_{-\infty}^{x} d\xi$ being the inverse of $\partial/\partial x$. We restrict ourselves to solutions $u \in S$, S being the space of C^∞ functions on R^2, vanishing rapidly at infinity.

As dual S^* of S we consider the space of all C^∞ functions f on R^2 with at most polynomial growth and with the property that $f(x, y)$ vanishes rapidly at $x = -\infty$ for all fixed y. (This assumption ensures that the operator D^{-1} makes sense.) The evaluation of $f \in S^*$ on S shall be given by

$$\langle f,g\rangle = \int_{\mathrm{R}^2} f(x,y)\, g(x,y)\, \mathrm{d}x\, \mathrm{d}y \tag{3}$$

for all $g \in \mathrm{S}$. The spaces S and S* are endowed with the corresponding weak topologies. The C^∞ functions from S to S and from S to S* are denoted by $C^\infty(\mathrm{S})$ and $C^\infty(\mathrm{S}^*)$, respectively. They play the role of vectorfields and covectorfields.

By $\mathscr{A}$ ($\mathscr{A}^*$) we define the algebras generated by $C^\infty(\mathrm{S})$ [$C^\infty(\mathrm{S}^*)$] and the functions x, y and 1 [interpreted as constant functions on S (S*)]. Denoting the derivative of an element G of $\mathscr{A}$ by a prime, i.e.

$$G'(u)[v] := \partial G(u + \epsilon v)/\partial\epsilon\,|_{\epsilon=0}\ , \tag{4}$$

we obtain the usual Lie bracket on $\mathscr{A}$:

$$[G, H] = G'[H] - H'[G]\ , \tag{5}$$

which in $C^\infty(\mathrm{S})$ induces the usual Lie algebra of vectorfields.

An element $\sigma \in C^\infty(\mathrm{S})$ is the generator of a one-parameter symmetry group of (1), if

$$[K, \sigma] = 0\ . \tag{6}$$

A map γ: S → S* is called a conserved covariant of (1), if it satisfies

$$\gamma_t = -K'^*[\gamma]\ . \tag{7}$$

Here * means transposition with respect to ⟨ , ⟩. If p: S → R is a C^∞ function on S, then its gradient is defined by

$$\langle \mathrm{grad}\, p(u), f\rangle = \partial p(u + \epsilon f)/\partial\epsilon\,|_{\epsilon=0} \tag{8}$$

and p is called the potential of grad p.

A function γ: S → S* has a potential iff $\gamma'^* = \gamma'$, the potential then is given by

$$p(u) = \int_0^1 \langle \gamma(\lambda u), u\rangle\, \mathrm{d}\lambda. \tag{9}$$

If a conserved covariant γ has a potential p, then $p(u(t))$ is time-independent for every solution $u(t)$ of (1), i.e. p is a constant of the motion of (1).

A connection between symmetries and conserved covariants is established for hamiltonian systems by the classical Noether theorem (see also ref. [6]): Let us recall that an equation $u_t = K(u)$ is called hamiltonian, if we have

$$K(u) = \theta(u)\gamma(u)\ , \tag{10}$$

where θ is an implectic operator and γ is a gradient function. An operator $\theta(u)$: S* → S is said to be implectic if the bracket

$$[\![\gamma_1, \gamma_2]\!] = \mathrm{grad}\langle \gamma_1(u), \theta(u)\gamma_2(u)\rangle\ ,$$

$$\gamma_1, \gamma_2 \in C^\infty(\mathrm{S}^*) \tag{11}$$

is a Lie product on the gradient fields and if $\gamma(u) \to \theta(u)\gamma(u)$ is a Lie-algebra homomorphism into the Lie algebra of vectorfields. In case of the hamiltonian system (10) the operator $\theta(u)$ maps conserved covariants onto infinitesimal generators of symmetries, $\theta(u)$ is callled a Noether operator (see ref. [6]). In case $\theta(u)$ is invertible $\theta^{-1}(u)$ works in the opposite way, i.e. it maps generators of symmetries onto conserved covariants.

3. Principal results. With

$$\tau_+(u) = y\, K(u) - 2\alpha^2 x u_y - 4\alpha^2 D^{-1} u_y \tag{12}$$

we define two linear maps ϕ and Ψ from $\mathscr{A}$ to $\mathscr{A}$ and from $\mathscr{A}^*$ to $\mathscr{A}^*$, respectively by

$$\phi(\sigma) = [\sigma, \tau_+]\ ,$$

$$\Psi(\gamma) = \mathrm{grad}\langle \gamma, \tau_+\rangle = \gamma'^*\tau_+ + \tau_+'^*\gamma \tag{13}$$

and construct the hierarchies

$$K_n = \phi^n(K_0)\ ,\quad K_0 = (-1)u_x/2\alpha^2\ ,$$

$$\gamma_n = \Psi^n(\gamma_0)\ ,\quad \gamma_0 = (-1)u/2\alpha^2\ . \tag{14}$$

We find:

(i) All the K_n commute: $[K_n, K_m] = 0$ for all $n, m \in \mathrm{N}_0$.

(ii) All the γ_n are gradients and $K_n = D\gamma_n$ holds. Hence every K_n gives a hamiltonian system with D as Noether operator.

(iii) The potentials p_n of γ_n are given by

$$p_n = \frac{-1}{2\alpha^2(n+1)(n+2)(n+3)} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \gamma_{n+2}(u)\ , \tag{15}$$

they are in involution with regard to the Poisson bracket:

$$\{p_n, p_m\} := \langle \mathrm{grad}\, p_n, D\, \mathrm{grad}\, p_m\rangle\ , \tag{16}$$

i.e. $\{p_n, p_m\} = 0$.

Observing $K_2 = K$ we see, that we have in particular constructed the hierarchy of the two-dimensional KdV, defined by

$$u_t = K_n(u) \, . \tag{17}$$

Every K_m is the generator of a one-parameter symmetry group and every p_m is a constant of the motion for any of the equations (17).

4. The method. We now prove the claims made in the last section. Let us consider

$$\tau_{--} = 1, \quad \tau_{-} = yu_x, \quad \tau_0 = 2yu_y + xu_x + 2u \, ,$$

$$\tau_{+} = y\,K(u) - 2\alpha^2 xu_y - 4\alpha^2 D^{-1}u_y \, , \tag{18}$$

for which we find

$$[\tau_+, \tau_{--}] = 6\tau_-, \quad [\tau_+, \tau_-] = -2\alpha^2\tau_0,$$

$$[\tau_+, \tau_0] = \tau_+ \, . \tag{19}$$

Checking

$$[K_0, \tau_{--}] = 0, \quad [K_0, \tau_-] = 0, \quad [K_0, \tau_0] = K_0 \, , \tag{20}$$

we easily derive via the Jacobi identity:

$$[K_n, \tau_0] = (n+1)K_n \, ,$$

$$[K_n, \tau_-] = -\alpha^2 n(n+1)K_{n-1} \, ,$$

$$[K_n, \tau_{--}] = a_n K_{n-2} \, , \tag{21}$$

with $a_n = -2\alpha^2(n-1)n(n+1)$. From this we obtain

$$[[K_N, K_{N+1}], \tau_0] = (2N+3)\,[K_N, K_{N+1}] \tag{22}$$

and

$$[[K_n, K_{N+1}], \tau_{--}]$$
$$= -(a_N + a_{N+1})[K_{N-1} K_N] + a_N\,[[K_{N-2}, K_N], \tau_+] \, . \tag{23}$$

Furthermore, we have

$$[K_n, K_{N+1}] = -[K_{n+1}, K_N] + [[K_n, K_N], \tau_+] \, . \tag{24}$$

We want to show that all the K_n commute. Let us assume $[K_n, K_m] = 0$ for all $n, m \leqslant N$ (which certainly is the case for $N = 2$). From (24) we see that we only have to show $[K_N, K_{N+1}] = 0$ in order to obtain $[K_n, K_m] = 0$ for all $n, m \leqslant N+1$.

For $N = 2$ we check this directly. An essential step consists in calculating

$$K_3 = [K_2, \tau_+]$$
$$= 12\alpha^2(u_{xxy} - 4uu_y - 2u_x D^{-1}u_y + \alpha^2 D^{-2}u_{yyy}) \, .$$

A straightforward, but lengthy calculation then shows $[K_2, K_3] = 0$.

For $N > 2$ the proof goes as follows: K_N is a generator of a symmetry of K_2, from (24) we find that K_{N+1} is a symmetry generator of K_2 as well, so their commutator $\sigma_N\!: = [K_N, K_{N+1}]$ is another generator. It is clear that σ_N has to be a polynomial in $x, y, u, u_x, u_y, D^{-1}u_y \ldots$. Let us say that a monomial $f(x, y, u, u_x, u_y \ldots)$ has order n if

$$f(x, y, \alpha u, \alpha u_x, \alpha u_y \ldots) = \alpha^n f(x, y, u, u_x, u_y \ldots) \, . \tag{25}$$

If σ_N is to commute with $K_2 = K$, then the highest-order term of σ_N has to commute with $6uu_x$, which is the highest-order term of K.

From (23) we see that the commutator of this term with $\tau_{--} = 1$ vanishes. In the appendix we derive that in the algebra under consideration any symmetry generator of uu_x which commutes with 1 has to be of the form $\lambda u_x + \mu u_y$. Checking $[\lambda u_x + \mu u_y, \tau_0] = \lambda u_x + 2\mu u_y$ and comparing this to (22) we have to conclude $\lambda = \mu = 0$, so the highest order term of σ_N must vanish, i.e. $\sigma_N = 0$. Thus we have proved that the K_n form an abelian hierarchy of flows, so every K_n is the generator of a symmetry to each of the equations

$$u_t = K_m(u) \, . \tag{26}$$

We will now show that all those flows are hamiltonian and have D as Noether operator, i.e.

$$K_n = D\gamma_n \, , \tag{27}$$

where γ_n is a gradient function. To see this we check

$$\tau'_+ D + D\tau'^{*}_+ = 0 \, , \tag{28}$$

which implies

$$\phi D\gamma - D\Psi\gamma = 0 \, , \tag{29}$$

if γ is a gradient, i.e. $\gamma' = \gamma'^{*}$.

Starting with $K_0 = D\gamma_0 = D - u/2\alpha^2$ and using (29) we immediately find $K_n = D\gamma_n$ and all the γ_n are gradients by construction. So every equation (29) has D as a Noether operator and its inverse D^{-1} maps the symmetries K_n to the conserved covariants γ_n. Hence their potentials p_n are constants of the motion for every member of the hierarchy. If the γ_n are calculated, then their potentials p_n can be obtained very easily.

From $[K_{n+2}, \tau_{--}] = [K_{n+2}, 1] = a_{n+2}K_n$ we conclude

$$\begin{aligned}\operatorname{grad}\langle\gamma_{n+2}, 1\rangle &= \gamma'_{n+2}[1] = D^{-1}[K_{n+2}, \tau_{--}] \\ &= a_{n+2}D^{-1}K_n = a_{n+2}\gamma_n\ , \qquad (30)\end{aligned}$$

such that the constants of the motion are given by

$$p_n = a_{n+2}^{-1} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \gamma_{n+2}\ . \qquad (31)$$

For the conservation laws the usual Poisson bracket is defined:

$$\{p_n, p_m\} = \langle \operatorname{grad}p_n, D\operatorname{grad}p_m\rangle = \langle\gamma_n, D\gamma_m\rangle\ . \qquad (32)$$

But p_n is a constant of the motion for the flow $u_t = K_m(u)$, hence

$$\mathrm{d}p_n(u)/\mathrm{d}t = p'_n[K_m] = \langle\gamma_n, K_m\rangle = 0\ , \qquad (33)$$

and all the p_n are in involution.

At last we want to remark that none of the K_n can be zero, as (21_2) would imply the vanishing of the entire hierarchy. Furthermore from (21_1) we can conclude that $K_n \neq K_m$ for $n \neq m$. So we indeed have constructed an infinite hierarchy of symmetry generators and constants of the motion for each of the equations (26).

We are indebted to A.S. Fokas who brought the problem to our attention.

Appendix. We need some information about the symmetry generators of the equation

$$u_t = uu_x\ . \qquad (34)$$

Let A denote the smallest algebra of vector fields containing u and being closed under D and D^{-1}, i.e. being closed under taking the x-derivative and its inverse. We can easily write down those symmetry generators of (34) lying in A: they are exactly of the form

$$\sigma(u) = \sum_{m\geqslant 0} \lambda_m u^m u_x\ , \quad \lambda_m \in \mathrm{R}\ . \qquad (35)$$

But regarding u as a function of the two variables x and y we find more generators than those in A, for example u_y and $2uu_y + u_xD^{-1}u_y$ generate honest symmetries of (34).

Let us define A_y as the smallest algebra of vector fields containing u and 1 (regarded as constant function on S) and being closed under $\partial/\partial x, D^{-1}$ and $\partial/\partial y$, i.e. being closed under taking the x-derivative, its inverse and being closed under taking the y-derivative (but not its inverse). We define

$$\begin{aligned}&y^n\mathrm{A}_y = \{y^n\sigma, \sigma \in \mathrm{A}_y\}\ , \\ &\tilde{\mathrm{A}} = \mathrm{A}_y \oplus y\mathrm{A}_y \oplus y^2\mathrm{A}_y \oplus \dots\ . \qquad (36)\end{aligned}$$

This sum is direct and by P we denote the projection onto A_y. We now prove the following

Remark: If $\sigma \in \mathrm{A}_y$ is a symmetry generator of (34) satisfying

$$[\sigma, 1] = 0\ , \qquad (37)$$

then σ must be of the form $\sigma(u) = \lambda u_x + \mu u_y, \lambda, \mu \in \mathrm{R}$.

In order to show this we have to classify the elements of $\tilde{\mathrm{A}}$ in two different ways. Every $\sigma \in \tilde{\mathrm{A}}$ can be split up into monomials which have a certain order $\mathrm{o}(\sigma)$ defined by (25). By $d(\sigma)$ we define the degree of such a summand as the highest exponent of $1/\lambda$, if y and $\partial/\partial y$ are replaced by λy and $(1/\lambda)\,\partial/\partial y$, respectively. This degree may be negative. For example if $\sigma(u) = yu_{xy}D^{-2}u_y$, then $\mathrm{o}(\sigma) = 2$ and $d(\sigma) = 1$.

Given any symmetry generator of (34) with the property (37) we pick the highest-order term of it, which we call σ. We find that σ has to be a symmetry generator of (34) as well and again $[\sigma, 1] = 0$ holds. Now let us assume $\sigma \in \mathrm{A}_y\backslash\mathrm{A}$ such that we certainly have $d(\sigma) \geqslant 1$. We now construct another symmetry generator $\tilde{\sigma}$ of (34) with degree $d(\tilde{\sigma}) = d(\sigma) - 1$ and order $\mathrm{o}(\tilde{\sigma}) \geqslant \mathrm{o}(\sigma)$, which again satisfies (37). To do this we consider $\tau_m(u) = yu^m u_x$, which commutes with (34). Hence $[\sigma, \tau_m] \in \tilde{\mathrm{A}}$ and $\tilde{\sigma}_m := P[\sigma, \tau_m] \in \mathrm{A}_y$ are symmetry generators. Observe that for $\sigma \in \mathrm{A}_y\backslash\mathrm{A}$ (i.e. σ containing y-derivatives) we cannot have $\tilde{\sigma}_m = 0$ for all $m \in \mathrm{N}$. So pick the smallest $m \geqslant 0$ such that $\tilde{\sigma}_m \neq 0$ and put $\tilde{\sigma} = \tilde{\sigma}_m$. For $\tilde{\sigma}$ we find $d(\tilde{\sigma}) = d(\sigma) - 1$; $\mathrm{o}(\tilde{\sigma}) = \mathrm{o}(\sigma) + m \geqslant \mathrm{o}(\sigma)$ and the Jacobi identity implies:

$$\begin{aligned}[\tilde{\sigma}, 1] &= [\tilde{\sigma}_m, 1] = P[[\sigma, \tau_m], 1] \\ &= P[[\sigma, 1], \tau_m] + P[\sigma, [\tau_m, 1]]\ . \qquad (38)\end{aligned}$$

The first term vanishes since $[\sigma, 1] = 0$ and the second term is equal to $mP[\sigma, \tau_{m-1}]$ (for $m > 0$) and hence has to vanish, too, as m was the smallest number such that $P[\sigma, \tau_m] \neq 0$. For $m = 0$ we find $[\sigma_0, 1] = 0$.

So starting with a generator $\sigma \in \mathrm{A}_y \backslash \mathrm{A}$ of order $\mathrm{o}(\sigma)$ and degree $d(\sigma)$ we have constructed another generator $\tilde{\sigma}$ of lower degree and same or higher order, satisfying (37). Repeating this process finitely many times, we end up with a $\tilde{\sigma}$ of degree $d(\tilde{\sigma}) = 0$ (i.e. $\sigma \in \mathrm{A}$) and order $\mathrm{o}(\tilde{\sigma}) \geqslant \mathrm{o}(\sigma)$, satisfying $[\tilde{\sigma}, 1] = 0$. As $\tilde{\sigma} \in \mathrm{A}$ it must be of the form $\tilde{\sigma} = \lambda u^n u_x$ for some n and from $[\tilde{\sigma}, 1] = 0$ we find $n = 0$, such that $\mathrm{o}(\tilde{\sigma}) = 1$. Hence the order of the generator $\sigma \in \mathrm{A}_y \backslash \mathrm{A}$ we started with must have been 1, too, so we conclude $\sigma = \mu u_y$. So any symmetry generator of (34) in A_y with (37) must be of the form $\lambda u_x + \mu u_y$.

References

[1] P.J. Olver, J. Math. Phys. 18 (1977) 1212;
B. Fuchssteiner, Nonlinear Anal. 3 (1979) 849.

[2] B. Fuchssteiner, Prog. Theor. Phys. 65 (1981) 861.

[3] A.S. Fokas and B. Fuchssteiner, Phys. Lett. 86A (1981) 341.

[4] B.B. Kadomtsev and V.I. Petviashvili, Sov. Phys. Dokl. 15 (1970) 539.

[5] S.V. Manakov, Physica 3D (1981) 420.

[6] A.S. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento 28 (1980) 299;
B. Fuchssteiner and A.S. Fokas, Physica 4D (1981) 47.