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The bi-Hamiltonian structure of some nonlinear fifth- and seventh-order differential equations and recursion formulas for their symmetries and conserved covariants

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Using a bi-Hamiltonian formulation we give explicit formulas for the conserved quantities and infinitesimal generators of symmetries for some nonlinear fifth- and seventh-order nonlinear partial differential equations; among them, the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and the Kupershmidt equation. We show that the Lie algebras of the symmetry groups of these equations are of a very special form: Among the C^∞ vector fields they are generated from two given commuting vector fields by a recursive application of a single operator. Furthermore, for some higher order equations, those multisoliton solutions, which for $|t|\to\infty$ asymptotically decompose into traveling wave solutions, are characterized as eigenvector decompositions of certain operators.

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I. THE MAIN RESULTS

It has long been known[1–6] that the Caudrey-Dodd-Gibbon-Sawada–Kotera (CDGSK) equation

$$u_t = K_1(u) = u_{xxxxx} + \tfrac{5}{2}\zeta u u_{xxx} + \tfrac{5}{2}\zeta u_x u_{xx} + \tfrac{5}{4}\zeta^2 u^2 u_x \quad (\zeta \text{ arbitrary} \in R) \tag{1}$$

is "completely integrable" in the sense that it admits infinitely many conservation laws and infinitely many symmetries (via a suitable version of Noether's theorem). Reference 1 provides explicit formulas for only four conserved quantities "because of the enormity of the calculation."

Satsuma and Kaup[2] have found a Bäcklund transformation and an inverse scattering method for (1) that enabled them to give an explicit recursion formula for the conserved densities. It turned out that many of those densities are trivial in the sense that the conserved charges vanish. Moreover, conserved densities of certain order in u do not seem to exist at all, e.g., there obviously is no conserved polynomial density with $u^2, u^5, u^8 \ldots$ as the highest power of u.

A. Our main result is that all symmetries and all nontrivial conserved densities of (1) can be obtained by a straightforward recursion scheme. We claim that (J_1, K_1, Θ_1) is a generalized bi-Hamilton system, where Θ_1 and J_1 are given by

$$\Theta_1(u) = D^3 + \zeta(uD + Du), \tag{2}$$

$$J_1(u) = 2D^3 + \zeta(D^2 u D^{-1} + D^{-1} u D^2) + \tfrac{1}{4}\zeta^2(u^2 D^{-1} + D^{-1} u^2). \tag{3}$$

Here D stands for the differential operator and D^{-1} for its inverse (defined on a suitable solution space that is explained later on). As will be shown in Sec. II the bi-Hamiltonian structure means that $\Theta_1(u)$ maps conserved convariants of (1) (especially the gradients of conservation laws) onto infinitesimal generators of one-parameter symmetry groups, and $J_1(u)$ works in the opposite way, i.e., infinitesimal generators of symmetries are mapped onto conserved covariants. Hence, the operator

$$\Phi_1(u) = \Theta_1(u) J_1(u) \tag{4a}$$

is a recursion operator in the sense that it maps infinitesimal generators of symmetries onto infinitesimal generators of symmetries, and

$$\Phi_1^+(u) = J_1(u)\Theta_1(u) \tag{4b}$$

maps conserved covariants of (1) onto conserved covariants.

We review some basic notions.

A vector field $\sigma(u)$ is said to be the *infinitesimal generator of a symmetry* of

$$u_t = K(u) \tag{5a}$$

if the infinitesimal transformation $u(t) \to u(t) + \epsilon\sigma(u(t))$ leaves (5a) form-invariant. In other words $v(t) = \sigma(u(t))$ has to be a solution of

$$v_t = \frac{\partial}{\partial\epsilon} K(u + \epsilon v)|_{\epsilon=0}. \tag{5b}$$

A covector field $G(u)$ is called a conserved covariant if $\langle G(u(t)), v(t)\rangle$ is time independent whenever $u(t)$ is a solution of (5a) and $v(t)$ is a solution of (5b). Special conserved covariants are given by gradients of conservation laws. The quantity $p(u)$ is a conservation law of (5a) if $p(u(t))$ is time independent for every solution $u(t)$ of (5a). The gradient of $p(u)$ is defined by

$$\langle \operatorname{grad} p(u), v\rangle = \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} p(u + \epsilon v).$$

Here $\langle\ ,\ \rangle$ stands for the bilinear form given by

$$\langle f, g\rangle = \int_{\mathbf{R}} f(x) g(x)\, dx.$$

Now, we return to Eq. (1). Picking $\sigma_1(u) = u_x$ and $\sigma_3(u) = K_1(u)$ as the infinitesimal generators of space and time translation, we can construct two sequences of symmetry generators by

$$\sigma_{3n+1}(u) := \phi_1^n(u) u_x \quad \text{and} \quad \sigma_{3n+3}(u) := \phi_1^n(u) K_1(u). \tag{6a}$$

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The subscripts are chosen in such a way that they indicate the order in u (i.e., the maximal power of u) of each generator. Analogously there are 2 hierarchies of conserved covariants that are connected to the symmetry generators via Θ_1 and J_1 and can as well be constructed from simple ancestors. The simplest conserved densities of (1) are u and $[(\zeta/12)u^3 - u_x^2/2)]$ so that we pick their gradients as the starting points of the two hierarchies. Defining

$$G_0(u) = \text{grad} \int_{-\infty}^{\infty} u\, dx = 1$$

and

$$G_2(u) = \text{grad} \int_{-\infty}^{+\infty} \left(\frac{\zeta}{12} u^3 - \frac{u_x^2}{2} \right) dx = \frac{\zeta}{4} u^2 + u_{xx},$$

we obtain the sequences

$$G_{3n} = \phi_1^+(u)^n 1 \quad \text{and} \quad G_{3n+2} = \phi_1^+(u)^n \left(\frac{\zeta}{4} u^2 + u_{xx} \right) \tag{6b}$$

for which the relations

$$\sigma_{k+1}(u) = \Theta_1(u) G_k(u) \quad \text{and} \quad J_1(u)\sigma_k(u) = G_{k+2}(u) \tag{7}$$

hold for $k = 0,2,3,5...$ and $k = 1,3,4,6...$, respectively.

In the particular case under consideration all the quantities $G_k(u)$ are gradients. Hence the corresponding conservation laws can be found by integration, i.e., all the

$$I_{k+1}(u) = \int_0^1 \langle G_k(\lambda u), u \rangle \, d\lambda \tag{8}$$

are conserved densities. The first four of these densities can be found in Ref. 1.

Looking at the orders of the I_k we see that there are the two hierarchies I_{3n+1} and I_{3n+3}, so we have verified the conjecture "that a series of conservation laws exists with every third polynomial conservation law (p.c.l.) missing, i.e., if $\frac{1}{3}(2r-1)$ is an integer then the p.c.l. (of that order in u) does not exist."[1] This conjecture was already proved[6] (by the inverse scattering method and under strong compact-support assumptions). It should be noted that our results do not depend on inverse scattering techniques.

B. We shall see that the same analysis holds for the Kupershmidt equation[5] (among others):

$$u_t = K_2(u) = u_{xxxxx} + \tfrac{5}{2}\zeta u_{xxx} u + \tfrac{25}{4}\zeta u_{xx} u_x + \tfrac{5}{8}\zeta^2 u^2 u_x. \tag{9}$$

Here, the crucial operators $\Theta(u)$ and $J(u)$ have the following form:

$$\Theta_2(u) = D^3 + \tfrac{1}{4}\zeta(uD + Du), \tag{10}$$

$$J_2(u) = 2D^3 + \tfrac{5}{2}\zeta(uD + Du) + \zeta(D^2 u D^{-1} + D^{-1} u D^2) + \zeta^2(u^2 D^{-1} + D^{-1} u^2), \tag{11}$$

and the two gradients of conservation laws we have to start with are in this case

$$G_0(u) = 1, \tag{12a}$$

$$G_1(u) = u_{xx} + \zeta u^2 = \text{grad} \int_{-\infty}^{+\infty} \left(\frac{\zeta}{3} u^3 - \frac{u_x^2}{2} \right) dx. \tag{12b}$$

C. Another fifth-order equation that can be dealt with in the same way is

$$v_t = v_{xxxxx} - 5(v_x v_{xxx} + v_{xx}^2 + v_x^3 + 4 v v_x v_{xx} + v^2 v_{xxx} - v^4 v_x). \tag{13}$$

It was recently discovered[5] that this equation is connected to (1) as well as to (9) by a Bäcklund transformation (modified Miura transformation). Hence we can apply the transformation formulas of Ref. 7 to obtain the corresponding operators $\tilde{\Theta}(v)$, $\tilde{J}(v)$, as well as the conserved covariants to start our two sequences.

Equation (1) goes over into (13) via

$$v_x - v^2 = \tfrac{1}{2}\zeta u. \tag{14}$$

In Sec. III we explain that such a formula, which is called a Bäcklund transformation, yields transformation formulas[7,13] for the corresponding operators Θ and J. By application of these transformation formulas we obtain the corresponding operators $\Theta(v)$ and $J(v)$ for Eq. (13). They are given by

$$\tilde{\Theta}(v) = \frac{4}{\zeta^2}(2v - D)^{-1} \Theta_1\left(\frac{2}{\zeta}(v_x - v^2) \right)(2v + D)^{-1}, \tag{15a}$$

$$\tilde{J}(v) = \frac{\zeta^2}{4}(2v + D) J_1\left(\frac{2}{\zeta}(v_x - v^2) \right)(2v - D), \tag{15b}$$

where $J_1(\cdot)$ and $\Theta_1(\cdot)$ are the operators given by (2) and (3). The conserved covariants to obtain the two sequences of conservation laws from are

$$G_0(v) = 1, \tag{16}$$

$$G_2(v) = (D + 2v)[(v_x - v^2)^2 + 2(v_x - v^2)_{xx}]. \tag{17}$$

D. In addition our method yields recursion formulas for conservation laws and infinitesimal generators of symmetries of some seventh-order equations that were not yet discovered as completely integrable. To be precise, take any pair of operators Θ, J given by either (2), (3) or (10), (11) or (15a), (15b). Then the equations

$$u_t = \Phi(u)^n u_x, \quad \text{where} \quad \Phi(u) = \Theta(u) J(u), \tag{18}$$

are completely integrable and the operators Θ, J play the same role as before.

The infinitesimal generators of one-parameter symmetry groups are given by

$$\sigma_n(u) = \Phi(u)^n u_x, \quad n \in N, \tag{19a}$$

and

$$G_n(u) = \Phi^+(u)^n G_0(u), \quad n \in N, \tag{19b}$$

[where $G_0(u) = 1$ and $\Phi^+(u) = J(u)\Theta(u)$] are gradients of conserved quantities. Furthermore for Eq. (18) we have a very transparent description of the multisoliton solutions. As described in Sec. III (see also Refs. 8 and 9) a solution u of (18) is a pure N-soliton, i.e., a solution that decomposes for $|t| \to \infty$ into N traveling wave solutions with prescribed asymptotic speeds such that the total energy is carried by the asymptotic waves, if and only if u_x can be written as

$$u_x = \sum_{k=1}^{N} \omega_k, \tag{20a}$$

where

$$\Phi(u)\omega_k = -\lambda_k \omega_k, \quad k = 1,...,N, \tag{20b}$$

i.e., the ω_k are eigenvectors of $\Phi(u)$. Hence the manifold of N-soliton solutions of (18) can be described in terms of the solution manifold of a system of ordinary differential equations. The asymptotic speeds are given by the eigenvalues, and are equal to $\lambda_1^n,...,\lambda_N^n$.

II. COMPARISON WITH OTHER WORK

A recursion formula for the conserved quantities (integrals) of the CDGSK equation has been given before (see, for example, Ref. 2). This recursion formula is rather complicated since the nth integral I_n is a polynomial of third order in the I_k, $k < n$. Although this formula is the result of ingenious considerations, it seems absolutely hopeless to find an explicit solution for this recursion scheme. But indeed, formula (6b) constitutes such an explicit solution. To be precise, it constitutes an explicit formula for the gradients of the integrals. But obtaining the integrals from their gradients is only an elementary calculation [formula (8)]. In fact, dealing with the gradients instead of the integrals has a considerable advantage since one immediately works within the framework of the Poisson brackets. In addition, the operator Φ^+ [given by formula (4b)] is a deformation in that Lie-algebra of Poisson brackets having strong linear interpolation properties[14]—and since the kernel of Φ^+ is empty, one can be sure to construct only those integrals for which the corresponding densities do not vanish.

We should like to emphasize that our results are not obtained by inverse scattering techniques. Although we do appreciate inverse scattering as one of the most ingenious contributions in the field, we cannot overlook the fact that, for the equations under consideration, the validity of the results obtained by this method depends, up to now, on very strong compact support assumptions (see Ref. 6, p. 215). In contrast, all the results obtained in this paper can be checked by direct (although cumbersome) calculations.

Our approach yields in addition a complete description of the symmetry group of the CDGSK equation (in terms of infinitesimal generators). Symmetries for this equation have not been considered in detail. Of course, barring some minor complications, one obtains the symmetry group from the conserved quantities via a suitable version of Noether's law. But the point is, that for finding such a version of Noether's law one has to perform, at least partly, those calculations that are presented in this paper (namely, finding Hamiltonian formulations for the equations under consideration). Since the infinitesimal generators of the one-parameter symmetry groups do commute, one has, in addition, found the generalized CDGSK equations. Here "generalized" is meant in exactly the same sense as in the case of the Korteweg–de-Vries (KdV) equation.

III. THE BACKGROUND

We consider a dynamical system

$$u_t = K(u), \tag{21}$$

where u is in some manifold M and where K is a suitable C^∞ vector field. For our purpose we assume throughout this paper that $M = S =$ the Schwartz space of C^∞ functions on the real line vanishing rapidly at $\pm\infty$. Because of this assumption we can identify the manifold with the typical fiber of the tangent bundle, and everything becomes very transparent. In the C^∞ vector fields we introduce the usual Lie-algebra product (see any standard textbook, for example Ref. 10 or 11) given by

$$[H_1,H_2](u) = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} [H_1(u+\epsilon H_2(u)) - H_2(u+\epsilon H_1(u))],$$

where H_1, H_2 are arbitrary C^∞ vector fields.

The importance of this Lie algebra lies in the fact that a flow defined by a vector field H is a one-parameter symmetry group of (21) if and only if $[K, H] = 0$. Therefore if $[K, \sigma] = 0$, then σ is the *infinitesimal generator* of a *symmetry* and has the properties required in Sec. I.

Now, let us turn our attention towards the dual. Let S^* be a space of C^∞ functions on the real line such that the elements $f \in S^*$ define continuous linear functionals on S via

$$\langle f,s \rangle = \int_{-\infty}^{+\infty} f(\xi)\, s(\xi)\, d\xi, \quad s \in S.$$

Let us henceforth restrict all notions of continuity and differentiability to the topology given by the pair (S^*,S) (where S^* will be chosen later on in such a way that the operators we deal with do make sense). A C^∞ covector field G is called a *conserved covariant* if $L_K G = 0$, where L_K denotes the Lie derivative. That is the same as saying that G defines a flow in the cotangent bundle such that $\langle G, K \rangle$ is a conserved quantity for (21). Gradients of conservation laws are standard examples for conserved quantities; in that case we have trivially $\langle G,K \rangle = 0$. But in general a conserved covariant need not be closed.

It is very useful to look for a relationship between symmetries and conserved quantities of the dynamical system given by $K(u)$. Such a connection similar to Noether's theorem can be formulated, if $K(u)$ has a special, i.e., Hamiltonian, structure. To make precise what this means, we first have to establish the notion of symplectic and implectic (inverse-symplectic) operators.

Definition: An operator-valued function $u \to J(u): S \to S^*$ is called *symplectic* if (1) $J(u)$ is antisymmetric with respect to $\langle\ ,\ \rangle$; and (2) the 2-form ω defined by $\omega(s_1,s_2) := \langle J(u)s_1,s_2 \rangle\ \forall s_1,s_2 \in S$ is closed.

A 2-form ω is closed if and only if the following Jacobi identity is satisfied:

$$L_{X_3}(\omega(X_1,X_2)) + L_{X_1}(\omega(X_2,X_3)) + L_{X_2}(\omega(X_3,X_1)) = 0$$

for all vector fields X_1, X_2, X_3. This is equivalent to

$$\langle J'(u)[s_1]s_2,s_3 \rangle + \langle J'(u)[s_2]s_3,s_1 \rangle + \langle J'(u)[s_3]s_1,s_2 \rangle = 0 \quad \text{for all} \quad s_1,s_2,s_3 \in S.$$

Here again L_X is the Lie-derivative with respect to the vector field X and $J'(u)[v]$ stands for the derivative

$$J'(u)[v] = \frac{\partial}{\partial \epsilon} J(u + \epsilon v)_{|\epsilon = 0}.$$

Analogously, an antisymmetric operator $\Theta(u): S^* \to S$ is called implectic if it satisfies the same Jacobi identity as the inverse of a nondegenerate symplectic operator.

Definition: An operator $\Theta(u): S^* \to S$ is called *implectic*, if (1) $\Theta(u)$ is antisymmetric with respect to $\langle\ ,\ \rangle$; and (2) the 2 times contravariant tensor field φ defined by

$$\varphi(X_1^*, X_2^*) := \langle X_1^*, \Theta(u) X_2^* \rangle \quad \forall X_1^*, X_2^* \in S^*$$

satisfies

$$L_{\Theta X_3^*}\varphi(X_1^*, X_2^*) + L_{\Theta X_2^*}\varphi(X_3^*, X_1^*) + L_{\Theta X_1^*}\varphi(X_2^*, X_3^*) = 0$$

for all covector fields X_1^*, X_2^*, X_3^*.

Again, this last identity is equivalent to

$$\langle s_2^*, \Theta'(u)[\Theta(u)s_1^*]s_3^* \rangle + \langle s_1^*, \Theta'(u)[\Theta(u)s_3^*]s_2^* \rangle$$
$$+ \langle s_3^*, \Theta'(u)[\Theta(u)s_2^*]s_1^* \rangle = 0 \quad \text{for all} \quad s_1^*, s_2^*, s_3^* \in S^*.$$

Now we can characterize those dynamical systems $u_t = K(u)$ for which intimate relations between the symmetries and the conserved quantities exist. Let d denote the exterior derivative.

Definition: If there is a symplectic $\Theta(u): S^* \to S$ and a function $H(\cdot): S \to \mathbb{R}$ (i.e., a zero form) such that

$$K(u) = \Theta(u)\, dH(u)$$

then we call (K, Θ) a *generalized Hamiltonian system*.

If there is an implectic $J(u): S \to S^*$ and a function $H(\cdot): S \to \mathbb{R}$ such that

$$J(u) K(u) = dH(u),$$

then we call (J, K) *a generalized inverse Hamiltonian system*. Here "generalized" refers to the fact that we do not assume any nondegeneracy or invertibility conditions for the operators $\Theta(u)$ and $J(u)$.

The nice properties of these systems can be stated in a kind of generalized Noether Theorem:

Theorem:

(i) Let (K, Θ) be generalized *Hamiltonian*. Then Θ maps conserved covariant forms onto infinitesimal generators of symmetries.

(ii) Let (J, K) be generalized inverse Hamiltonian. Then J maps infinitesimal generators of symmetries onto conserved covariant forms. The proof of this theorem can be found in Ref. 13 and a detailed analysis of the Lie-algebraic aspects is contained in Ref. 12.

A most convenient case is a dynamical system that turns out to be generalized *bi-Hamiltonian*, i.e., there is a symplectic Θ and an implectic J such that (K, Θ) is generalized Hamiltonian and (J, K) is generalized inverse Hamiltonian.

For such a system (J, K, Θ) our theorem immediately yields that $\Phi(u) := \Theta(u) \circ J(u)$ and $\Phi^+(u) := J(u) \circ \Theta(u)$ are self-maps in the spaces of infinitesimal generators of symmetries and of conserved covariant forms, respectively. Therefore the bi-Hamiltonian structure of a system gives us a recursion operator in the sense of Olver,[15] which maps one generator of a symmetry to another one and the adjoint of which maps one conserved quantity to the next.

If this recursion process is not cyclic or stops at a certain stage, we can construct an infinite hierarchy of symmetries starting with simple ones such as space or time translation. To the members of this hierarchy correspond conserved covariant forms, which as well can be constructed via Φ^+ from common ancestors. This method goes back to an ingenious idea of Magri,[18] who considered bi-Hamiltonian systems under the additional hypothesis that Φ^+ is a self-map in those closed covector fields commuting (in the Lie algebra of Poisson brackets) with $J(u)K(u)$.

We claim that, for all the cases considered in Sec. I, the system (J, K, Θ) is bi-Hamiltonian; this then proves all assertions of Secs. IA and IB except the statement that the G_n are closed and in involution. The necessary calculations to prove that (J, K, Θ) is bi-Hamiltonian are given in Sec. IV; the reasons for the latter statement, and the claims of Sec. ID are presented in Sec. V.

Let us turn our attention to Sec. IC. Let $F: S \to S$ be a C^∞-diffeomorphism, mapping solutions of $v_t = H(v)$ onto solutions of $u_t = K(u)$. For example

$$u = (2/\zeta)(v_x - v^2).$$

Such a map is said to be a *Bäcklund transformation* between the two evolution equations. It is well known (Ref. 11, p. 132) that the derivative F' of a C^∞-diffeomorphism defines a Lie-algebra isomorphism in the vector fields; its adjoint F'^+ is an isomorphism with respect to the Poisson brackets. Application of these facts yields the formulas (15). (For more examples of such transformations see Refs. 7 and 13.)

IV. THE CALCULATIONS

Consider the vector field K given by

$$K(u) = u_{xxxxx} + \alpha_1 u u_{xxx} + \alpha_2 u_x u_{xx} + \alpha_3 u^2 u_x, \tag{22}$$

where $u \in S$ and $\alpha_1, \alpha_2, \alpha_3$ are numbers.

For $\alpha_3 = \frac{1}{10}(2\alpha_2 - \alpha_1)(3\alpha_1 - \alpha_2)$ this can be written as[13]

$$K(u) = \Theta(u) \operatorname{grad} \int_{-\infty}^{+\infty} \left(-\frac{u_x^2}{2} + \frac{2\alpha_2 - \alpha_1}{30} u^3 \right) dx, \tag{23}$$

where

$$\Theta(u) = D^3 + \tfrac{1}{5}(3\alpha_1 - \alpha_2)(uD + Du). \tag{24}$$

The operator Θ is in fact implectic (already observed in Refs. 7 and 13, an easy calculation). Now, we consider the operator

$$J(u) = 2D^3 + \gamma(uD + Du) + \alpha(D^2 u D^{-1} + D^{-1} u D^2)$$
$$+ \beta(u^2 D^{-1} + D^{-1} u^2) + \delta u D^{-1} u, \quad u \in S \tag{25}$$

and we fix our dual space S^* in such a way that $J(u): S \to S^*$ (which can be done easily). One proves that the corresponding 2-form is closed; hence $J(u)$ is symplectic. Now we want to determine our coefficients in such a way that $J(u)\,K(u)$ is a closed form (i.e., a gradient). This appears to be a simple exercise, but in fact is not (at least if the calculation is done by hand; some of our students are at present writing a computer program for calculations of this kind). Here, we only present the essential steps of that calculation.

One looks at different powers in u of $J(u)K(u)$. The first- and fifth-order terms are easily seen to be gradients.

Second order in u: The gradient of

$$A\int_{-\infty}^{+\infty} u_{xx}^3\,dx + B\int_{-\infty}^{+\infty} uu_{xx}^2\,dx$$

is equal to

$$(6A-B)u_{xxx}u_{xxx} + (6A-6B)u_{xx}u_{xxxx} - 6Bu_x u_{xxxxx} + 2Buu_{xxxxxx}.$$

If one assumes that the second-order term of $J(u)K(u)$ is of that form one obtains the following expressions:

$$\begin{aligned}&\alpha = \tfrac{2}{5}\alpha_1,\\ &\gamma = \tfrac{2}{5}(\alpha_2-\alpha_1),\\ &A = \tfrac{2}{15}(\alpha_1+7\alpha_2),\\ &B = -\alpha_1 - \tfrac{2}{5}\alpha_2.\end{aligned} \tag{26}$$

Third order in u: We assume that the third order term is equal to the gradient of

$$C\int_{-\infty}^{+\infty} u_x^4\,dx + E\int_{-\infty}^{+\infty} u^2u_{xx}^2\,dx.$$

This leads to the following equations for the coefficients:

$$\begin{aligned}&\alpha_3 = \tfrac{1}{10}(3\alpha_1-\alpha_2)(2\alpha_2-\alpha_1),\\ &\beta = \tfrac{1}{25}\alpha_1(2\alpha_2-\alpha_1),\\ &\delta = \tfrac{4}{25}\left[\alpha_2 - \tfrac{5}{2}\alpha_1\right](\alpha_1-\alpha_2),\\ &C = \tfrac{1}{12\cdot 25}(133\alpha_1^2 + 82\alpha_2^2 - 283\alpha_1\alpha_2),\\ &E = \tfrac{1}{50}(-27\alpha_1^2 - 14\alpha_2^2 + 73\alpha_1\alpha_2).\end{aligned} \tag{27}$$

Fourth order in u: We assume that the fourth-order term is equal to the gradient of

$$F\int_{-\infty}^{+\infty} u^3u_x^2\,dx.$$

This yields the following equations:

$$\begin{aligned}&F = \tfrac{13}{50}\alpha_1^3 + \tfrac{2}{25}\alpha_2^3 - \tfrac{43}{150}\alpha_1^2\alpha_2 - \tfrac{14}{75}\alpha_1\alpha_2^2,\\ &F = -\tfrac{7}{50}\alpha_1^3 + \tfrac{4}{25}\alpha_2^3 + \tfrac{71}{150}\alpha_1^2\alpha_2 - \tfrac{47}{75}\alpha_1\alpha_2^2,\\ &0 = (\alpha_2 - \tfrac{5}{2}\alpha_1)(\alpha_2-\alpha_1)(\alpha_2-2\alpha_1).\end{aligned} \tag{28}$$

Combining the first and second equation for F one gets a cubic equation having the following solutions

$$\alpha_2 = \lambda\alpha_1,\ \lambda = 1,\ \tfrac{5}{2},\ 2. \tag{29}$$

Then the third equation is satisfied automatically. Now one can insert the three cases of (29) into (27) and (26) and obtain three sets of coefficients such that $J(u)K(u)$ is a gradient. Before we list these cases we would like to make a remark: The choice of our potentials seems to be rather special, but in fact one can show that all the possible potentials have to be of the form we assumed.

First case:

$\alpha_2 = \alpha_1;\ \alpha_3 = \tfrac{1}{5}\alpha_1^2;\ \alpha = \tfrac{2}{5}\alpha_1;$

$\beta = \tfrac{1}{25}\alpha_1^2;\ \delta = \gamma = 0.$

These coefficients lead to the operators (2) and (3) and Eq. (1) (putting $\alpha_1 = \tfrac{5}{2}\zeta$).

Second case:

$\alpha_2 = \tfrac{5}{2}\alpha_1;\ \alpha_3 = \tfrac{1}{5}\alpha_1^2;\ \alpha = \tfrac{2}{5}\alpha_1;$

$\gamma = \tfrac{3}{5}\alpha_1;\ \beta = \tfrac{4}{25}\alpha_1^2;\ \delta = 0.$

This leads immediately to the case considered in Sec. I B (where $\alpha_1 = \tfrac{5}{2}\zeta$).

Third case:

$\alpha_2 = 2\alpha_1;\ \alpha_3 = \tfrac{3}{10}\alpha_1^2;\ \alpha = \tfrac{2}{5}\alpha_1;\ \gamma = \tfrac{2}{5}\alpha_1;$

$\beta = \tfrac{3}{25}\alpha_1^2;\ \delta = \tfrac{2}{25}\alpha_1^2.$

This case leads to the first generalization of the Korteweg–de-Vries equation, a case which is not at all interesting since the bi-Hamiltonian structure of that equations is already known.

Having proved the assertions of Secs. IA and IB we obtain IC out of the general transformation formulas for bi-Hamiltonian systems presented in Ref. 7. The verification of the bi-Hamiltonian structure of Eq. (18) is, again, not at all difficult (once the coefficients are known). Another approach to Eq. (18) and its properties will be given in Sec. V.

V. HEREDITARINESS

Let us now turn our attention to the question whether or not the conserved covariants in (6b) and (19b) are closed, i.e., gradients of suitable potentials. Checking that directly leads into a rather frustrating adventure. But there is another method, namely, to determine whether or not the corresponding operator $\Phi(u) = \Theta(u)J(u)$ is hereditary. Before we explain what hereditariness can do for us, let us give the following abstract definition: The tensorfield given by Φ is said to be *hereditary* if, with respect to the Lie algebra of vector fields, the following holds[14]:

$$\Phi^2[H_1,H_2] + [\Phi H_1,\Phi H_2] = \Phi\{[H_1,\Phi H_2] + [\Phi H_1,H_2]\}$$

for all C^∞-vector fields H_1 and H_2 (compare with the notion of a Nijenhuis operator in Ref. 16, or a regular operator in Ref. 17).

Now, assume that (J, K, Θ) is bi-Hamiltonian and that K and $\Phi = \Theta J$ are invariant against x-translation. Invariance against x-translation means that

$$D\Phi(u) - \Phi(u)D = \frac{\partial}{\partial\epsilon}\Phi(u+\epsilon u_x)_{|\epsilon=0} \quad \text{for all } u\in S$$

and that the flow $u_t = K(u)$ commutes with x-translation. Under these assumptions the hereditariness of $\Phi = \Theta J$ has the following consequences (see Refs. 8, 13, or 14): The vector fields given by

$$\{\Phi(u)^n K(u) | n\in N_0\}\ U\ \{\Phi(u)^n u_x | n\in N_0\} \tag{30}$$

are contained in an abelian subalgebra of the Lie-algebra given by all vector fields. All the covector fields given by

$$\{J(u)\Phi^n(u)K(u) | n\in N_0\}\ U\ \{J(u)\Phi^n(u)u_x | n\in N_0\} \tag{31}$$

do have potentials, i.e., they are closed covector fields.

Now, all the operators Φ considered in Sec. I are in fact hereditary. To prove this is only necessary for one of the operators, since all the corresponding bi-Hamiltonian systems are related by Bäcklund transformations (which preserve the property of hereditariness). To check the algebraic relation seems to be a rather straightforward calculation, but it is not, the calculation is cumbersome and extremely boring.

However, there are other ways to derive the hereditary property of Φ from. Let us indicate this briefly: For example, one can use the fact that any of the equations is an isospectral flow for some eigenvalue problem[5,6] and that the gradients of the corresponding eigenvalues are eigenvectors for $\Phi^{+}(u)$. From this one can show that Φ is hereditary.

Another procedure is given by the results of Ref. 21. There the authors showed that for certain dynamical systems the two given Hamiltonian structures are compatible. Since the CDGSK equation is the reduction of one of these systems one can derive that the two Hamiltonian structures of this equation are compatible. However, compatible Hamiltonian structures yield hereditary operators.[14]

Now, application of (30) immediately yields that the σ_{3n+1} and σ_{3n+3} (considered in Sec. I) are infinitesimal generators of an abelian symmetry group. Observation (31) yields that the corresponding G_{3n} and G_{3n+2} have potentials. Since J is a Lie-algebra homomorphism from the vector fields into the gradients (endowed with the Poisson brackets) all these conserved quantities must be in involution.

These results can also be obtained by other methods. The fact that the G have potentials can be derived by the ingenious reasoning presented in Ref. 20, and the abelian structure of the symmetry group can be derived from a general result of Tu.[22]

Finally, we would like to point out the connection with the soliton solutions mentioned in I D.

Let Φ be as above and consider the system

$$u_t = \Phi(u)u_x = K(u). \tag{32}$$

Then it can be shown[8,14] that the manifold

$$M = \left\{ u \in S \,|\, u_x = \sum_{k=1}^{N} \omega_k, \quad \omega_k \quad \text{eigenvector of } \Phi(u) \right\}$$

must be invariant under the flow given by (32). Let $u(t)$ be a pure N-soliton solution, i.e.,

$$u(x,t) = \sum_{n=1}^{N} u_n(x - \Theta_n - c_n t) + \Delta(x,t)$$

with $\|\Delta(\cdot,t)\| \to 0$ for $|t| \to \infty$ (in L^2-norm) and with

$$u_{n_x} = c_n K(u_n) \quad \text{for } n = 1,\dots,N. \tag{33}$$

Then, by (33), the u_{n_x} are eigenvectors of $\Phi(u_n)$ and, since $\Phi(u)$ is semilocal, $u(t)$ belongs, for $|t| = \infty$, (i.e., asymptotically) to the manifold M. Hence $u(t)$ belongs for all t to this manifold. By comparison of the dimensions one sees that M is exactly the manifold of N- soliton solutions [with prescribed asymptotic speeds $c_1,\dots,c_N$ depending on the eigenvalues of $\Phi(u)$].

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