Physics Letters, 86A, 341-345, 1981

THE HIERARCHY OF THE BENJAMIN–ONO EQUATION

A.S. FOKAS [1] and B. FUCHSSTEINER
Universität Paderborn, D 4790 Paderborn, W. Germany

Received 18 August 1981

The Benjamin–Ono (BO) equation is shown to possess two *non-local* linear operators, which generate its infinitely many commuting symmetries and constants of the motion in involution. These symmetries define the hierarchy of the BO equation, each member of which is a hamiltonian system. The above operators are the nonlocal analogues of the Lenard operator and its adjoint for the Korteweg–de Vries equation.

1. Introduction. It is well known that certain nonlinear partial differential equations of evolution type, like the Korteweg–de Vries equation, possess various special features such as inifinitely many commuting symmetries and constants of motion in involution, N-soliton solutions, Bäcklund transformations, etc. [1]. It has turned out that considering a local operator, called in ref. [2] a strong symmetry, or hereditary symmetry, which maps symmetries of a given equation onto symmetries, provides a useful approach to all these features. (For example, the existence and involutionary property of the constants of the motion follows from the symplectic–implectic factorization of the strong symmetry [3–5].) The strong symmetry of the Korteweg–de Vries equation is the well-known Lenard operator (i.e. the squared-eigenfunction operator).

Recently it has been established that another class of nonlinear equations of physical importance possess similar important features. This class consists of certain nonlinear singular integro–differential equations [6–11], the most well known of which is the Benjamin–Ono (BO) [12,13] equation. This equation is interesting both physically and mathematically. Physically, because it describes the long internal gravity waves in a stratified fluid. Mathematically, because not only it possesses the above special features but also:

(i) its linear eigenvalue problem is a differential Riemann–Hilbert problem which results in a novel type of inverse scattering transform [14];

(ii) the squared eigenfunctions, which can be shown to be conserved densities, satisfy a *non-linear* eigenvalue problem [15] (hence, one cannot use either the AKNS [16] approach nor that of refs. [17,18] to obtain a strong symmetry).

In this note we discuss another novel aspect of the BO equation: It possesses two non-local linear operators which generate infinitely many commuting symmetries and constants of the motion in involution. These operators are the analogues of the strong symmetry and its adjoint, respectively. The symmetries of the BO equation define a hierarchy of evolution equations each of which is a hamiltonian system with infinitely many constants of the motion in involution. We do not believe that for the Benjamin–Ono equation there is a strong symmetry of the usual local type. Definitely, there is one of polynomial type (as we shall see in section 4).

The general theory associated with this new symmetry approach is developed elsewhere. Here, in order to make the paper self-contained we give an independent proof of the relevant results, exploiting the galilean and the scaling symmetries of the BO equation.

[1] Permanent address: Department of Mathematics and Computer Science, Clarkson College, Potsdam, NY 13676, USA.

2. Notations and basic notions. For simplicity the solutions of the Benjamin–Ono equation

$$u_t = K(u)\,,\quad K(u) = Hu_{xx} + 2uu_x\,, \tag{1}$$

are considered to be elements of $\mathcal{S}$, i.e. the space of C^∞ functions vanishing rapidly at infinity. H denotes the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x}\,d\xi\,,$$

the principal value of the integral being understood. It is known that $H^2 = -1$ and $H^* = -H$, where the star means transposition with respect to the bilinear form

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\,dx\,. \tag{2}$$

By $C^\infty(\mathcal{S})$ we denote the algebra of C^∞ functions from $\mathcal{S}$ into $\mathcal{S}$, and by $\mathcal{A}(x)$ we denote the algebra generated by $C^\infty(\mathcal{S})$ and the functions x, 1 (being understood as constant functions on $\mathcal{S}$). In $\mathcal{A}(x)$ we introduce the Lie product

$$[G, K] = G'[K] - K'[G]\,, \tag{3}$$

where $G'[K]$ is derivative in the direction K, i.e.

$$(G'[K])(u) = (\partial/\partial\epsilon)G(u + \epsilon K(u))|_{\epsilon=0}\,.$$

An element $\sigma \in C^\infty(\mathcal{S})$ is said to be an infinitesimal generator of a symmetry of (1) iff

$$[K, \sigma] = 0\,. \tag{4}$$

This is equivalent to requiring that the infinitesimal transformation $u(t) \to u(t) + \epsilon\sigma(u(t))$ leaves (1) form invariant, or that $w(t) = \sigma(u(t))$ is a solution of the perturbation equation

$$w_t = K'[w] = Hw_{xx} + 2(uw)_x\,. \tag{5}$$

Let $p\colon \mathcal{S} \to C$ be a C^∞ function on $\mathcal{S}$. The gradient of p is defined by

$$\langle \operatorname{grad} p(u), f\rangle = (\partial/\partial\epsilon)p(u + \epsilon f)|_{\epsilon=0}\,,$$

and p is called the potential of grad p. A function γ: $\mathcal{S} \to \mathcal{S}$ has a potential if and only if $\gamma'^* = \gamma'$.

The potential is then, up to a constant, given by

$$p(u) = \int_0^1 \langle \gamma(\lambda u), u\rangle\,d\lambda\,. \tag{6}$$

The quantity p is said to be an integral (or a constant of the motion) of eq. (1) if $p(u(t))$ is time independent for every solution $u(t)$ of (1). This is equivalent to requiring that $\gamma(t) = \operatorname{grad}(p(u(t)))$ satisfies $\langle\gamma, K\rangle = 0$ or that $\gamma(t)$ is a solution of the adjoint of the perturbation equation,

$$\gamma_t = -K'^*[\gamma] = H\gamma_{xx} + 2u\gamma_x\,. \tag{7}$$

An equation $u_t = K(u)$ is a hamiltonian system if it can be written in the form $u_t = \theta\gamma$, where θ is an implectic operator and γ a gradient function [5]. In this case θ is a Noether operator, that is, if γ is defined by (7) then $\sigma = \theta\gamma$ fulfills (5).

The BO equation can be written in the form

$$u_t = D(Hu_x + u^2)\,.$$

Since D is implectic and $Hu_x + u^2$ is a gradient function, it follows that its symmetries σ and gradients of conserved quantities γ must be related by

$$\sigma = D\gamma\,. \tag{8}$$

3. Principal results. Consider the special $\tau \in \mathcal{A}(x)$ given by

$$\tau(u) = xK(u) + u^2 + \tfrac{3}{2}H(u_x)\,. \tag{9}$$

Define two linear maps Φ and $\tilde{\Phi}$ from $\mathcal{A}(x) \to \mathcal{A}(x)$ by

$$\Phi(\sigma) = [\sigma, \tau]\,,\quad \tilde{\Phi}(\gamma) = \operatorname{grad}\langle\gamma, \tau\rangle\,, \tag{10}$$

and the hierarchies K_n, γ_n, p_n by

$$K_n = \Phi^n(K_0)\,,\quad \gamma_n = \tilde{\Phi}^n(\gamma_0)\,,$$

$$p_n(u) = \int_{-\infty}^{\infty} \gamma_{n+1}(u)(\xi)\,d\xi\,,\quad n = 1, 2, \ldots, \tag{11}$$

where $K_0 = u_x$, $\gamma_0 = u$. Then:

(1) Every K_n, γ_n, p_n is an infinitesimal generator of a symmetry of (1), a gradient of an integral of (1) and an integral of (1), respectively. Furthermore, K_n, γ_n do not depend explicitly on x, i.e. are elements of $C^\infty(\mathcal{S})$.

(2) All K_n commute, all γ_n are orthogonal (with respect to D) and all p_n are in involution (with respect to D), i.e.

$$[K_n, K_m] = 0\,, \quad \langle \gamma_n, D\gamma_m \rangle = 0\,,$$
$$\{p_n, p_m\} = \langle \mathrm{grad}\, p_n, D\, \mathrm{grad}\, p_m \rangle = 0\,, \qquad (12)$$

for all $m, n \in \mathrm{N}_0$.

(3) The BO hierarchy, defined by

$$u_t = K_n\,, \qquad (13)$$

(the BO is given by $n = 1$), is a hierarchy of hamiltonian systems, $u_t = (-1)^n D\gamma_n$. Furthermore, the result (1) above is valid if the BO is replaced by any of eqs. (13).

We shall now prove the above results.

Observation 1. Consider some $\tau \in \mathcal{A}(x)$ and a symmetry generator σ of the equation $u_t = K$. Assume that $[K, \tau]$ is a symmetry generator which commutes with σ. Then $[\sigma, \tau]$ is again a symmetry generator.

The proof of this observation is trivial since the Jacobi identity implies (together with the assumptions)

$$[[\sigma, \tau], K] = [[\sigma, K], \tau] - [\sigma, [K, \tau]] = 0\,.$$

This simple observation results in a far-reaching theory, which will be published elsewhere. One of the amazing consequences is, for example, that whenever the symmetry group of $u_t = K(u)$ is abelian then $\Phi(\sigma) = [\sigma, \tau]$ is a self-map in the space of symmetry generators. This actually happens, as will be shown below, in the case of the Benjamin–Ono equation. Before showing that, we would like to give two simple examples.

Let K be the right-hand side of (1) and consider

$$\tau_0(u) = 1\,, \quad \tau_1(u) = x u_x + u\,. \qquad (14)$$

Then use the following:

$$(H(xf))(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x f(x)}{x - z}\, \mathrm{d}x$$
$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x - z H(f)(z)\,,$$

to obtain

$$[K, \tau_0] = 2K_0\,, \quad [K, \tau_1] = 2K\,. \qquad (15)$$

Certainly, K and K_0 (translation invariance) commute with all symmetries of the Benjamin–Ono equation. Hence

Observation 2. The maps $\sigma \to [\sigma, \tau_0]$ and $\sigma \to [\sigma, \tau_1]$ are self-maps in the space of symmetry generators of the Benjamin–Ono equation.

Now, consider the τ given by (9). One easily verifies

$$[K, \tau] = [2u^3 + 3H(uu_x) + 3uHu_x - 2u_{xx}]_x\,,$$

which is the first nontrivial symmetry generator of the Benjamin–Ono equation [9]. Now, define

$$K_{n+1} = [K_n, \tau]\,, \qquad (16)$$

and observe that $K_1 = K$, hence formula (16) increases the order in u. We say that $\sigma \in \mathcal{A}(x)$ has scaling degree m if

$$[\sigma, \tau_1] = m\sigma\,.$$

Since

$$[K_0, \tau_1] = K_0\,, \quad [\tau, \tau_1] = \tau\,, \qquad (17)$$

the K_n must have scaling degree $(n + 1)$. Furthermore one sees that the scaling degree is additive with respect to commutators. Hence $[K_n, K_m]$ has scaling degree $(n + m + 2)$.

Observation 3. Let σ be a symmetry of the Benjamin–Ono equation which is a polynomial in u, u_x, u_{xx}, etc. Assume that $[\sigma, \tau_0] = 0$. Then σ is of the form $\sigma = \lambda u_x$.

If $f(\alpha n) = \alpha^n f(n)$ for all $\alpha \in \mathrm{C}$ then we say that f has order n. Now, to prove observation 3, look at the highest-order term of σ. Then this must commute with the highest-order term of K, which is $2uu_x$, i.e. it must be a symmetry generator of $u_t = uu_x$. Hence, it must be of the form $\lambda u^n u_x$. On the other hand we have $[\lambda u^n u_x, \tau_0] = 0$, which can only happen for $\lambda = 0$ or $u = 0$.

Now, we show that all the K_n are commuting elements of $\mathrm{C}^\infty(\mathcal{S})$. Assume that for $n \leqslant N$ the K_n are commuting elements of $\mathrm{C}^\infty(\mathcal{S})$. (This is certainly true for $N = 2$. Observation 1 shows that K_{N+1} is a symmetry of u.) Put

$$\sigma_n = [K_{N+1}, K_n]\,.$$

Observe that $[\tau, \tau_0] = 2\tau_1$. This yields

$$[K_{n+1}, \tau_0] = [[K_n, \tau], \tau_0] = [[K_n, \tau_0], \tau] + 2[K_n, \tau_1]\,.$$

Since K_n has scaling degree $(n + 1)$ if follows, because of $[K_0, \tau_0] = 0$, that

$$[K_{n+1}, \tau_0] = n(n+1)K_n .$$

This yields, by induction, that

$$[\sigma_n, \tau_0] = 0 ,$$

since $[K_0, \tau_0] = 0$ and $[K_N, K_n] = 0$. Observe that the symmetry generators are a Lie algebra. Hence σ_n is a symmetry generator and must be equal to λu_x (observation 3). But σ_n has scaling degree $(N+n+2)$ and u_x has scaling degree 1. Thus $\lambda = 0$, and all $K_n, n \leqslant N+1$, do commute. By construction K_{N+1} must be of the form

$$K_{N+1} = xG + H, \quad G, H \in C^\infty(\mathcal{S}) .$$

Since $[K_{N+1}, u_x] = 0$ this implies $G = 0$ and K_{N+1} must be an element of $C^\infty(\mathcal{S})$.

Now, since all the $K_n \in C^\infty(\mathcal{S})$ do commute, any K_n is a symmetry generator of any of eqs. (13).

One may easily verify that

$$\tau' D + D\tau'^* = 0 , \qquad (18)$$

which implies

$$\Phi D - D\tilde{\Phi} = 0 . \qquad (19)$$

Observe that

$$K_n = \Phi^n u_x = \Phi^n Du = (-1)^n D\tilde{\Phi}^n u = (-1)^n D\gamma_n .$$

Since K_n are symmetries and D is a Noether operator for (1), it follows that γ_n are gradients of integrals of (1). Since $K_n = (-1)^n D\gamma_n$ and γ_n are gradients, it follows that $u_t = K_n$ are hamiltonian systems having D as a Noether operator. Since all K_m are symmetries of $u_t = K_n$, all γ_m are gradients of integrals of $u_t = K_n$. Hence the potentials of the γ_{m+1}, namely

$$\int_{-\infty}^{\infty} \gamma_m(u)\tau(u)(\xi)\,d\xi \qquad (20)$$

are integrals of the BO hierarchy. Because all K_n commute, the well-known formula (see, for example ref. [5])

$$[\theta\gamma_m, \theta\gamma_n] = \mathrm{grad}\langle\gamma_m, \theta\gamma_n\rangle ,$$

implies that all the integrals defined by (20) are in involution.

Using the fact that $\tau_0 = 1$ is a self-map in the space of symmetries (and hence in the space of gradients of integrals) we can give a more convenient formula than (20): Observe that

$$\mathrm{grad}\langle\gamma_{n+1}, 1\rangle = \gamma'_{n+1}[1] = D^{-1}[K_{n+1}, \tau_0]$$
$$= n(n+1)D^{-1}K_n = n(n+1)\gamma_n .$$

Hence

$$\gamma_n = \frac{1}{n(n+1)} \mathrm{grad}\langle\gamma_{n+1}, 1\rangle , \qquad (21)$$

which shows that the potential of γ_n is proportional to $\langle\gamma_{n+1}, 1\rangle$. Hence the formula for $p_n(u)$ given by (11) follows, which shows that γ_n are also conserved densities.

4. The Benjamin–Ono equation has no strong symmetry of polynomial type. Consider an operator-valued function $u \in \mathcal{S} \to \Phi(u)$: $C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$. In the case of the popular exactly solvable equations one can find such an operator having the property that $\Phi(u)\sigma(u)$ is a symmetry whenever $\sigma(u)$ is one. For example,

$$\Phi(u) = \alpha D + DuD^{-1} , \qquad (22)$$

has this property for the Burgers equation [2]

$$u_t = \alpha u_x + 2uu_x . \qquad (23)$$

Assume that $\Psi(u)$ is such an operator, for the Benjamin–Ono equation, of polynomial type. Of course, we assume that $\Psi(u)$ is nontrivial (i.e. not a multiple of the one-operator). And assume furthermore that $\Psi(u)$ is of minimal order in u. Now since, $\sigma \to \sigma'[1] = [\sigma, \tau_0]$ maps symmetry generators into symmetry generators, it follows that

$$\Psi'(u)[1] = (\partial/\partial\epsilon)\Psi(u+\epsilon 1)|_{\epsilon=0} ,$$

also must have the property that it is a self-map in the space of symmetry generators. Since its order is less than the order of $\Psi(u)$, it must be a multiple of the one-operator. From this we would like to conclude that $\Psi(u)$ had order one. To see this, we consider the space $\mathcal{S}^+$ of functions which can be extended analytically in the upper complex plane, i.e. $\mathcal{S}^+ = \{v \mid Hv = \mathrm{i}v\}$. The Benjamin–Ono equation leaves this space invariant and clearly reduces to

$$u_t = \mathrm{i}u_{xx} + 2uu_x , \quad u \in \mathcal{S}^+ ,$$

a special form of the Burgers equation. But for the Burgers equation any polynomial strong symmetry must be a polynomial in $\Phi(u)$ (given by (23) for $\alpha = \mathrm{i}$). The only polynomial $P(\Phi(u))$ in Φ having the property that

$$P(\Phi(u))'[1] = \lambda I\,, \quad \lambda \in \mathrm{C}\,,$$

is the one of first order. Hence, if H is replaced by multiplication with i, the operator $\Psi(u)$ must be of the form $\beta_1 I + \beta_2 \Phi(u)$. Checking all possibilities one easily sees that among the operators of first order there is none which is a self-map in the space of symmetry generators.

The authors thank M. Kruskal for posing the problem of finding a strong symmetry for the BO equation and M.J. Ablowitz for many valuable discussions.

References

[1] M.J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, SIAM Monograph (1981).
[2] B. Fuchssteiner, Nonlinear Anal. Theory Methods Appl. 3 (1979) 849; Prog. Theor. Phys. 65 (1981) 861.
[3] F. Magri, J. Math. Phys. 19 (1978) 1156; A geometrical approach to the nonlinear solvable equations, Lecture notes in physics, Vol. 120, eds. M. Boiti, F. Pimpinelli and G. Soliani (Springer) p. 233.
[4] I.M. Gelfand and I.Ya. Dorfman, Funkt. Anal. Prilozh 13 (1979) 13; 14 (1980) 71.
[5] A.S. Fokas and B. Fuchssteiner, Lett. Nuovo Cimento 28 (1980) 299;
B. Fuchssteiner and A.S. Fokas, Physica 4D (1981) 47.
[6] R.I. Joseph, J. Phys. A101 (1977) L225.
[7] T. Kubota, D.R.S. Ko and D. Dobbs, J. Hydronaut. 12 (1978) 157.
[8] H.H. Chen and Y.C. Lee, Phys. Rev. Lett. 43 (1979) 264.
[9] T.L. Bock and M.D. Kruskal, Phys. Lett. 74A (1979) 173.
[10] J. Satsuma, M.J. Ablowitz and Y. Kodama, Phys. Lett. 73A (1979) 283.
[11] Y. Kodama, J. Satsuma and M.J. Ablowitz, Phys. Rev. Lett. (1981), to be published.
[12] T.B. Benjamin, J. Fluid Mech. 29 (1967) 559.
[13] H. Ono, J. Phys. Soc. Japan 39 (1975) 1082.
[14] M.J. Ablowitz and A.S. Fokas, The inverse scattering transform of the Benjamin–Ono equation, preprint.
[15] M.J. Ablowitz, private communication.
[16] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. 53 (1974) 249.
[17] B. Fuchssteiner, Application of spectral-gradient methods to nonlinear soliton equations, preprint.
[18] A.S. Fokas and R.L. Anderson, On the use of isospectral eigenvalue problems for obtaining hereditary symmetries, preprint.