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BÄCKLUND TRANSFORMATIONS FOR HEREDITARY SYMMETRIES

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1. INTRODUCTION

A HEREDITARY symmetry $\Psi(u)$ is an operator-valued function of u (u is an element of a suitable vector space) having the property that it generates a hierarchy of evolution equations for which $\Psi(u)$ is a strong symmetry (or recursion operator in the terminology of [1]). Since a strong symmetry describes more or less completely the symmetries, the conservation laws and the soliton solutions (if they exist) of an evolution equation a method for finding hereditary symmetries is very desirable.

The aim of this paper is to show that suitable implicit functions defined by $B(u, s) = 0$ give rise to transformations between hereditary symmetries, thus (in principle) generating out of a given hereditary symmetry a class of others. Further, a Bäcklund transformation $B(u, s) = 0$ between two evolution equations defines such a transformation between the corresponding hereditary symmetries possessed by those equations. This then explains the fact that in general the whole hierarchy of evolution equations has the same Bäcklund transformation.

For a general discussion and for a convenient characterization of Bäcklund transformations see [2] and [3] respectively. Hereditary symmetries are discussed in [4].

2. REVIEW OF BASIC NOTIONS

In this paper we only deal with differentiable functions $F(u): u \in E_1 \to E_2$ between suitable vector spaces E_1 and E_2. By differentiable we always mean a notion of differentiability such that the derivative $F_u(u)$ is a linear map between E_1 and E_2 and such that the chain rule holds. In other words, we are dealing with functions which are Hadamard-differentiable [5] (with respect to a suitable topology). If $v \in E_1$ the derivative of $F(u)$ in the direction v is denoted by $F_u(u)[v]$ and may be calculated by

$$F_u(u)[v] = \frac{\partial}{\partial \varepsilon} F(u + \varepsilon v)\Big|_{\varepsilon = 0}.$$

Let $B(u, s)$ be a function in the two arguments $u \in E_1$ and $s \in E_2$ with values in some vector space E_3. Then by B_u, B_s we denote the partial derivatives with respect to the arguments indicated. This function is called *admissible* if $B(u, s) = 0$ gives rise to a one-to-one map between the tangent

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spaces, i.e., we require that, for $B = 0$, B_u and B_s are invertible linear maps $E_1 \to E_3$ and $E_2 \to E_3$ respectively.

Consider two evolution equations

$$u_t = K(u), \qquad u(t) \in E_1, \tag{1}$$

$$s_t = G(s), \qquad s(t) \in E_2. \tag{2}$$

An admissible function $B(u, s)$ is said to be a *Bäcklund transformation* between (1) and (2) if, for all t.

$$B(u(t), s(t)) = 0 \quad \text{whenever } B(u(0), s(0)) = 0.$$

Taking the t-derivative of the above and using the evolution equations one sees that this is equivalent to

$$B_u(u, s)\,[K(u)] + B_s(u, s)\,[G(s)] = 0 \quad \text{if } B(u, s) = 0. \tag{3'}$$

We write this in short

$$B_u[K] + B_s[G] = 0 \quad \text{if } B = 0. \tag{3}$$

Since B_u, B_s are invertible one sees that any of the evolution equations (1) or (2) is uniquely determined by the other evolution equation and the admissible function $B(u, s)$. Thus a Bäcklund transformation may be regarded as a transformation of variables (through a suitable implicit function) which transforms (1) into (2) (and vice versa).

Consider now, say, the evolution equation (1). An operator-valued function $\Phi(u)$, $u \in E_1$, is called a strong symmetry [4] (or recursion operator in the terminology of [1], [6]) for (1) if

$$\Phi_u(u)[K(u)] = [K_u(u), \Phi(u)], \tag{4'}$$

where [,] denotes, as usual, the commutator of the operators under consideration. For simplicity we use the shorter notation

$$\Phi_u[K] = [K_u, \Phi]. \tag{4}$$

The knowledge of a (nontrivial) strong symmetry for a given evolution equation contains a large amount of information. The reason is that the strong symmetry commutes with the tangential flow of the given evolution equation. As a consequence of this one finds that

(i) *$\Phi(u)$ maps symmetries of* (1) *onto symmetries of* (1),

(ii) *the transposed $\Phi(u)^T$ (with respect to the algebraic dual) maps conserved covariants onto conserved covariants,*

and the solutions of (1) have a *structural stability* which says that

(iii) *whenever, for a fixed time t_0, a symmetry of* (1) *can de decomposed into a sum of eigenvectors of $\Phi(u(t_0))$ then this decomposition remains valid for all time t. For suitable symmetries this decomposition turns out to be the soliton decomposition* [7].

For details the reader is referred to [4] (or to [8], where a brief outline can be found). The above clearly shows the importance of strong symmetries; one way of finding strong symmetries is using the idea of hereditary symmetries which we now define.

An operator-valued function $\Phi(u)$, $u \in E_1$, is called a *hereditary symmetry* [4] if

$$[\Phi_u(u), \Phi(u)] \text{ is a } \textit{symmetric} \text{ bilinear operator for all } u, \tag{5}$$

i.e.,

$$[\Phi_u, \Phi][v]\, w = [\Phi_u, \Phi][w]\, v, \quad \text{for all } v \in E_1,\ w \in E_1,$$

where

$$[\Phi_u, \Phi][v]\, w \doteqdot \Phi_u[\Phi v]\, w - \Phi\, \Phi_u[v]\, w.$$

For the correct understanding of the commutator appearing above it is necessary to mention that for a bilinear $A\colon E_1 \times E_1 \to E_1$ and a linear $B\colon E_1 \to E_1$ the product AB is defined to be the map $(u_1, u_2) \to A(Bu_1, u_2)$, and that $\Phi_u(u)$ is understood to be the bilinear operator

$$(v_1, v_2) \to \frac{\partial}{\partial \varepsilon}(\Phi(u + \varepsilon v_1))\, v_2\big|_{\varepsilon = 0}.$$

It was proved in [4] that if a hereditary symmetry $\Phi(u)$ is a strong symmetry for (1) then it is also a strong symmetry for the equation

$$u_t = \Phi(u)\, K(u). \tag{6}$$

Thus, (6) inherits its strong symmetry from (1). This simplifies considerably in the case that E_1 is a space of functions $u(x)$, $x \in \mathbb{R}$, and the hereditary symmetry $\Phi(.)$ is translation invariant. In this case $\Phi(u)$ is a strong symmetry for

$$u_t = u_x. \tag{7}$$

Hence $\Phi(u)$ must be a strong symmetry for the following hierarchy of evolution equations

$$u_t = K_n(u), \tag{8}$$

where $K_n(u) \doteqdot (\Phi(u))^n u_x$, $n \in \mathbb{N}$ (or $n \in \mathbb{Z}$ if this makes sense). At this point we would like to remark that all the well-known soliton evolution equations are of this type.

3. MAIN RESULTS

Consider the evolution equations

$$u_t = K(u), \qquad u \in E_1, \tag{1}$$

$$s_t = G(s), \qquad s \in E_2, \tag{2}$$

and a Bäcklund transformation between them, given by the admissible implicit function $B(u, s) = 0$.

THEOREM 1. The operator valued function $\Phi(u)$ is a strong symmetry for (1) if and only if

$$\Psi(s) = B_s^{-1} B_u\, \Phi(u)\, B_u^{-1} B_s, \quad \text{where } B = 0, \tag{9}$$

is a strong symmetry of (2).

Before presenting the proof of this theorem let us first remark on notation and then prove a lemma which is essential for the proofs of Theorems 1 and 2 (see below). Since, when $B(u, s) = 0$ u and s are related by an implicit function one has to distinguish between partial and total derivatives of functions $\Gamma(u, s)$. Subscripts denote, as before, the partial derivatives and by

$d_u\Gamma(u, s)$, $d_s\Gamma(u, s)$ (or $d_u\Gamma$, $d_s\Gamma$ for short) we denote the total derivatives. As before, if we take the derivative of an operator-valued function, we put the variable which indicates in which direction the derivative is taken in the bracket [ ], i.e.,

$$d_u T(u)[v]\, w = \frac{\partial}{\partial \varepsilon}(T(u + \varepsilon v)\, w)\big|_{\varepsilon = 0}.$$

LEMMA. For $B(u, s) = 0$, where B is some admissible function consider the operator $T\colon E_1 \to E_2$ given by

$$T = B_s^{-1} B_u. \tag{10}$$

Then

$$d_s T[v]\, w = d_s T[Tw]\, T^{-1} v, \quad \text{for all } v \in E_1,\ w \in E_2. \tag{11}$$

Proof. If one uses the symmetry of second derivatives, i.e.,

$$B_{ss}[v]\, w = B_{ss}[w]\, v, \quad \text{etc.},$$

and formulas such as

$$(B_s^{-1})_s = -B_s^{-1} B_{ss} B_s^{-1},$$

the proof is straightforward:

$$\begin{aligned} d_s T[v]\, w &= T_s[v]\, w - T_u[T^{-1} v]\, w \\ &= -B_s^{-1} B_{ss}[v]\, B_s^{-1} B_u w + B_s^{-1} B_{us}[v]\, w + B_s^{-1} B_{su}[T^{-1} v]\, B_s^{-1} B_u w - B_s^{-1} B_{uu}\, [T^{-1} v]\, w \\ &= -B_s^{-1} B_{ss}[v]\, Tw + B_s^{-1} B_{us}[v]\, w + B_s^{-1} B_{su}[T^{-1} v]\, Tw - B_s^{-1} B_{uu}[T^{-1} v]\, w \\ &= -B_s^{-1} B_{ss}[Tw]\, v + B_s^{-1} B_{us}[w]\, v + B_s^{-1} B_{us}[Tw]\, T^{-1} v - B_s^{-1} B_{uu}[w]\, T^{-1} v \\ &= d_s T[Tw]\, T^{-1} v. \quad \blacksquare \end{aligned}$$

Proof of Theorem 1. Since B_u and B_s are invertible it suffices to prove the theorem in one direction. We shall use the obvious identities

$$d_s \Phi[v] = -\Phi_u[T^{-1} v], \tag{12}$$

$$d_s K = -K_u T^{-1}. \tag{13}$$

Using the necessary and sufficient condition (3) we can write

$$G = -TK. \tag{14}$$

Further we abbreviate equation (9) by

$$\Psi = T \Phi T^{-1}. \tag{15}$$

This notation is practical and unambiguous if one remembers which variables one has to insert in the functions. The total derivative of Ψ is calculated to be:

$$d_s \Psi[\ ] = (d_s T)[\ ]\, \Phi T^{-1} - T \Phi T^{-1} (d_s T)[\ ]\, T^{-1} + T (d_s \Phi)[\ ]\, T^{-1}. \tag{16}$$

Using (11), (12), (13) we obtain

$$d_s\Psi[G]\,v = (d_sT)[T\Phi T^{-1}v]\,T^{-1}G - T\Phi T^{-1}(d_sT)[v]\,T^{-1}G - T\Phi_u[T^{-1}G]\,T^{-1}v$$
$$= -(d_sT)[\Psi v]\,K + \Psi(d_sT)[v]\,K + T\Phi_u[K]\,T^{-1}v.$$

However, since $\Phi(u)$ is a recursion operator for (1) we can use (4) to get:

$$d_s\Psi[G]\,v = -(d_sT)[\Psi v]\,K + \Psi(d_sT)[v]\,K + TK_u[T^{-1}\Psi v] - \Psi TK_u[T^{-1}v]. \tag{17}$$

In order to establish (4) for Ψ we calculate (with the help of (13) and (14)) the commutator of $G_s = d_sG$ and Ψ:

$$[G_s, \Psi]\,v = (d_sG)[\Psi v] - \Psi d_sG[v]$$
$$= -(d_sT)[\Psi v]\,K - T(d_sK)[\Psi v]\,K + \Psi(d_sT)[v]\,K + \Psi T(d_sK)[v]$$
$$= -(d_sT)[\Psi v]\,K + TK_u[T^{-1}\Psi v] + \Psi(d_sT)[v]\,K - \Psi TK_u[T^{-1}v].$$

Using (17) we conclude that

$$\Psi_s[G] = [G_s, \Psi],$$

i.e., $\Psi(s)$ must be a strong symmetry for the evolution equation (2). ∎

THEOREM 2. Let $B(u, s) = 0$, where B is an admissible implicit function of $u \in E_1$ and $s \in E_2$. Then $\Phi(u)$ is a hereditary symmetry if and only if $\Psi(s)$,

$$\Psi(s) = B_s^{-1}B_u\Phi(u)B_u^{-1}B_s, \tag{18}$$

is hereditary.

Proof. Recall (equation (16)) that

$$d_s\Psi[v]w = (d_sT)[v]T^{-1}\Psi w - \Psi(d_sT)[v]T^{-1}w + T(d_s\Phi)[v]T^{-1}w. \tag{19}$$

So, using the lemma and equation (12) we find

$$[d_s\Psi, \Psi]\,[v]w = (d_s\Psi)\,[\Psi v]w - \Psi(d_s\Psi)\,[v]w$$
$$= (d_sT)\,[\Psi v]T^{-1}\Psi w - \Psi(d_sT)[\Psi v]T^{-1}w + T(d_s\Phi)\,[\Psi v]T^{-1}w$$
$$- \Psi(d_sT)\,[v]T^{-1}\Psi w + \Psi^2(d_sT)\,[v]T^{-1}w - \Psi T(d_s\Phi)\,[v]T^{-1}w$$
$$= (d_sT)\,[\Psi w]T^{-1}\Psi v - \Psi(d_sT)\,[w]T^{-1}\Psi v - \Psi(d_sT)\,[\Psi w]T^{-1}v$$
$$+ \Psi^2(d_sT)\,[w]T^{-1}v - T\{\Phi_u[T^{-1}\Psi v]T^{-1}w - T^{-1}\Psi T\Phi_u[T^{-1}v]T^{-1}w\}.$$

The first and the fourth term in this expression are symmetric in v and w. Furthermore the sum of the second and third terms is also symmetric and the last term is

$$-T[\Phi_u, \Phi]\,[T^{-1}v]T^{-1}w.$$

Hence $[\Psi_s, \Psi]$ is symmetric if $[\Phi_u, \Phi]$ is. And by interchanging the role of s and u we can conclude in the same way that the symmetry of $[\Phi_u, \Phi]$ follows from that of $[\Psi_s, \Psi]$. ∎

COROLLARY. All the members of a hierarchy generated by a hereditary symmetry possess the same Bäcklund transformation. To be more explicit: If $B(u, s) = 0$ is a Bäcklund transformation between (1) and (2) and if $\Phi(u)$ is a hereditary symmetry and a strong symmetry for (1), then $B(u, s) = 0$ defines, for every $n \in \mathbb{N}$, a Bäcklund transformation between

$$u_t = K_n(u) \doteqdot (\Phi(u))^n K(u), \tag{20}$$

and

$$s_t = G_n(s) \doteqdot (\Psi(s))^n G(s), \tag{21}$$

where $\Psi(s)$ is the operator given by (18).

The proof of this is an immediate consequence of the necessary and sufficient condition expressed by (3) and the observation that

$$(\Psi(s))^n = B_s^{-1} B_u (\Phi(u))^n B_u^{-1} B_s \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

4. EXAMPLES AND APPLICATIONS

In order to be rigorous it seems necessary to specify the vector spaces in which the evolution equations are considered. However, we are mainly interested in strong and hereditary symmetries given by integrodifferential operators. And as one can easily see the properties required in the definitions of these symmetries (equations (4) and (5)) are of a purely algebraic nature and rather simple although the operators themselves may be quite complex. Furthermore the validity of (4) and (5) is mainly a consequence of the product rule (Leibniz rule) for differentiation. Thus, if a given operator is a strong or hereditary symmetry in one function space, then it usually has the same property in all other function spaces (provided it makes sense in those spaces).

For the above reasons it seems (at least at this stage) a waste of time to care too much about the specific nature of the function spaces where our operators are acting. Hence we feel free to proceed with our calculations in a rather formal way.

4.1. *The Miura transformation*

The Korteweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x \tag{23}$$

is related to the modified KdV equation

$$s_t = s_{xxx} + 6s^2 s_x \tag{24}$$

through the following Bäcklund transformation [9]

$$B(u, s) = u - s^2 - is_x = 0, \quad i = \sqrt{-1}. \tag{25}$$

A strong symmetry of the KdV is the well known Lenard operator [1]

$$\Phi(u) = D^2 + 4u + 2u_x D^{-1}, \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{26}$$

which is known to be hereditary [4]. Using (9) (or (18)) we can calculate a strong, hereditary symmetry $\Psi(s)$ of the modified KdV (Theorems 1 and 2). Furthermore equation (25) must define a Bäcklund transformation for the hierarchies of the KdV and of the modified KdV (corollary).

Let us obtain $\Psi(s)$. Inserting (25) in (18) we find

$$(2s + iD)\,\Psi(s) = \Phi(u)\,(2s + iD). \tag{27}$$

An easy calculation shows that

$$\Phi(u)\,(2s + iD) = (2s + iD)\,(D^2 + 4s_x D^{-1}\, s + 4s^2).$$

Hence

$$\Psi(s) = D^2 + 4s^2 + 4s_x\, D^{-1}\, s \tag{28}$$

must be a strong, hereditary symmetry of equation (24) (cf. [1, 4]).

Let us remark that our method does not depend on whether or not an evolution equation admits soliton solutions. For example

$$B(u, s) = u - s^2 - s_x \tag{29}$$

is a Bäcklund transformation between the KdV and the soliton-free version of the modified KdV (replace 6 by -6 in equation (24)). Using (29) we easily find $\Psi(s)$ for this equation.

4.2. *Burgers equation*

It is well known [10, 11] that Burgers equation

$$s_t = s_{xx} + 2ss_x \tag{30}$$

can be linearized through the transformation

$$B(u, s) = u_x - us, \tag{31}$$

where u satisfies the heat equation

$$u_t = u_{xx}. \tag{32}$$

Clearly

$$\Phi(u) = D \tag{33}$$

is a hereditary, strong symmetry of the heat equation. Hence using equation (18) we can obtain a strong, hereditary symmetry of Burgers equation. Actually

$$\Psi(s)u^{-1}\,(D - s) = (D - s)u^{-1}\, D, \tag{34}$$

which easily yields

$$\Psi(s) = D + u + u_x D^{-1} \tag{35}$$

(cf. [1, 4]).

4.3. *An auto-Bäcklund transformation*

An auto-Bäcklund transformation (i.e., the case where $G(v) = K(v)$ in (2)) must yield an invariance formula for the corresponding strong symmetry. Let us demonstrate that for Burgers equation. It is found in [12] that

$$B(u, s) = s_x + s^2 - su + \lambda, \quad \lambda \in \mathbb{R}, \tag{36}$$

is an auto-Bäcklund transformation for equation (30). Hence $\Psi(.)$ as defined by (35) must satisfy

$$\Psi(s)(D + 2s - u)s^{-1} = (D + 2s - u)s^{-1}\Psi(u), \quad B = 0. \tag{37}$$

This formula may be easily verified directly.

4.4. *The case when* $B(u, s)$: $R^2 \to R$

Let $\tau(\ )$ be a (suitably often differentiable) function $R \to R$ with nonvanishing first derivative $\tau'(\ )$. Then one may consider the following simple admissible function

$$B(u, s) = \tau(s) - u = 0. \tag{38}$$

Thus equation (18) yields

$$\Psi(s) = (\tau'(s))^{-1}\,\Phi(\tau(s))\tau'(s). \tag{39}$$

The above formula is quite useful for obtaining a strong or hereditary symmetry for an equation obtained from a given one by a simple change of variables (see Example 6). Clearly the above analysis goes through when $B(u, s)$ is a suitable function $R^2 \to R$.

4.5. *The case when* $B(u, s)$ *involves only constant operators.*

Let P be some invertible linear operator and consider the Bäcklund transformation

$$B(u, s) = P(s) - u = 0. \tag{40}$$

Then for a given hereditary $\Phi(u)$ the operator

$$\Psi(s) = P^{-1}\Phi(P(s))P \tag{41}$$

defines again a hereditary symmetry. From the above formula also follows that if $\Phi(u)$ is a strong symmetry of some equation then

$$\Psi(s) = D^{-1}\,\Phi(Ds)D \tag{42}$$

is a strong symmetry of its potential version. For example the strong, hereditary symmetry of the potential KdV is given by (using $\Phi(u)$ as in (26))

$$\Psi(s) = D^2 + 4s_x - 2D^{-1}s_{xx}. \tag{43}$$

4.6. *A whole class of equations*

It is shown in [13] that the class of equations

$$s_t = \phi s_{xx} + \left(\frac{\phi\phi''}{\phi'} - \frac{\phi'}{2}\right)s_x^2 + \alpha\phi s_x, \qquad \phi' = \frac{d\phi}{ds}, \tag{44}$$

where $\phi(s)$ is any suitable smooth function of s, and α is a constant parameter, can be linearized. This class of equations is interesting because it contains the physically meaningful (see [13]) equation

$$s_t = [(\beta s + \gamma)^{-2}s_x + \alpha(\beta s + \gamma)^{-1}]_x, \tag{45}$$

where α, β, γ are arbitrary parameters. Furthermore the class defined by (44) remains invariant under the transformation $s \to \eta(s)$, η a function. Actually, by this transformation one can regain the whole class by applying the transformation to any of its members. Thus it suffices to investigate a single equation of this form. For simplicity we take $\phi(s) = s^2$, i.e., we consider

$$s_t = s^2 s_{xx} + \alpha s^2 s_x. \tag{46}$$

This equation is formally mapped into Burgers equation

$$u_{\bar{t}} = u_{\bar{x}\bar{x}} + 2uu_{\bar{x}} \tag{47}$$

by the transformation [13]

$$t = \bar{t},$$
$$x = \int_{-\infty}^{x} s^{-1}(\xi, t)\, d\xi, \tag{48}$$
$$u(\bar{x}, \bar{t}) = s(x, t).$$

Hence

$$B(u, s)(x) = s(x) - u((D^{-1} s^{-1})(x)) = 0 \tag{49}$$

defines a Bäcklund transformation between (46) and (47). Using the known strong, hereditary symmetry of equation (47) (see Example 2) we shall obtain a strong, hereditary symmetry for (46): Equation (18) becomes

$$(1 + ss_x D^{-1} s^{-2}) \Psi = \Phi(1 + ss_x D^{-1} s^{-2}). \tag{50}$$

Using

$$\Phi(u) = \bar{D} + \alpha u + \alpha uu_x \bar{D}^{-1} = sD + \alpha s + \alpha ss_x D^{-1} s^{-1}, \quad \bar{D} = \frac{d}{d\bar{x}},$$

and equation (50) we obtain

$$\Psi(s) = sD + s^2 s_{xx} D^{-1} s^{-2} + \alpha s + \alpha s^2 s_x D^{-1} s^{-2}. \tag{51}$$

Equation (46) becomes equation (44) under the map $s \to (\phi(s))^{1/2}$. Thus one can easily find (see example 4) a strong, hereditary symmetry of equation (44).

4.7. *The sine–Gordon and the inverse KdV*

It is shown in [4] that the sine–Gordon equation in the form

$$s_t = \tfrac{1}{2} \sin(D^{-1} 2s), \tag{52}$$

admits the strong symmetry $\Psi(s)$ defined by (28). The reason is that equation (52) may be written as

$$s_t = (\Psi(s))^{-1} s_x \tag{53}$$

(where $\Psi(s)$ is given by (28)). Usibg the corollary (which is clearly true even when $n \in \mathbb{Z}$ provided that Φ^n makes sense) we deduce that the Bäcklund transformation defined by equation (25) maps equation (53) to the inverse KdV [14]

$$u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t - u_{xx} = 0, \quad \text{or } u_t = (\phi(u))^{-1} u_x \tag{54}$$

where $\Phi(u)$ is defined by (26). It is further clear that if one knew that equation (52) is related to (54) through equation (25), one could deduce that (52) has the strong symmetry $\Psi(s)$.

Using formula (42) we deduce that a strong, hereditary symmetry for equation

$$s_{tx} = \frac{1}{2} \sin 2s \tag{55}$$

is given by

$$\Psi(s) = D^2 + 4s_x^2 - 4s_x D^{-1} s_{xx}. \tag{56}$$

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