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On the Structure of Symplectic Operators and Hereditary Symmetries.

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In the last fifteen years, there has been a remarkable development in the exact analysis of certain nonlinear evolution equations, like the Korteweg-de Vries equation. It is well known that among the surprising features of these so-called *exactly solvable* equations is the possession of infinitely many symmetries and conservation laws, of N-soliton solutions and Bäcklund transformations. It has turned out that considering an operator which maps symmetries (1) onto symmetries of a given equation yields a useful approach to all these features. This operator is called a *strong symmetry* (2) (or recursion operator (3)). It is particularly useful because its transpose generates conserved covariants from given ones and because its eigenfunctions are also symmetries (which actually characterize the N-soliton solutions). One way of finding strong symmetries is to use the fact that any translation-invariant operator $\Phi(u)$, possessing the property defined by v below is a strong symmetry for the hierarchy of equations $u_t = (\Phi(u))^n u_x$, $n = 0, 1, 2, \ldots$. These operators are called *hereditary symmetries*. The strong symmetries of all well-known exactly solvable equations are hereditary (2).

Recently there has also been progress in understanding the Hamiltonian structure of these evolution equations (4). An evolution equations is said to be a Hamiltonian system if it can be written in the form $u_t = \theta(u)f(u)$, where $\theta(u)$ is impletic (which is, roughly speaking, the same as saying that $\theta^{-1}(u)$ is sympletic) and where $f(u)$ is the gradient of a suitable potential. For these systems the operator-valued function $\theta(u)$ is of particular interest because it is a Noether operator, *i.e.* it maps conserved covariants onto symmetries.

Our paper is related to Magri's work who considered bi-Hamiltonion systems $u_t = \theta_1(u)f_1(u) = \theta_2(u)f_2(u)$ and who showed that these equations have $\Phi(u) = \theta_1(u)\theta_2^{-1}(u)$ as strong symmetries.

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(1) Throughout the paper by symmetries we really mean infinitesimal generators of symmetries.

(2) B. FUCHSSTEINER: *Nonlinear Analysis, Theory, Methods and Appl.*, **3**, 849 (1979).

(3) P. J. OLVER: *J. Math. Phys.* (*N. Y.*), **18**, 1212 (1977).

(4) I. M. GEL'FAND anh L. A. DIKII: *Funct. Anal. and its Appl.*, **10**, 16, 259 (1976); P. J. OLVER: *On the Hamiltonian structure of evolution equation* (preprint); F. MAGRI: *J. Math. Phys.* (*N. Y.*), **19**, 1156 (1978).

In this note we

i) give an explicit relation between the strong symmetries, and also between the Noether operators of two evolution equations related by a Bäcklund transformation;

ii) show how hereditary symmetries as well as impletic operators transform under Bäcklund transformations;

iii) demonstrate that compatible implectic operators yield hereditary symmetries, which then can be used to generate a whole hierarchy of implectic structures.

The above results are not only of theoretical interest but also provide constructive approaches to exactly solvable equations. This is shown for one example, but many more applications are possible (⁵).

First some notation. We consider a linear space S and a suitable dual S^* and by $\langle a, u\rangle$ we denote the application of a linear functional a to $u \in S$. We deal with functions φ (attaining values in S, S^* or in a space of operators) which are assumed to be differentiable in the sense that the chain rule holds. By $\varphi'(u)[v]$ we denote the directional derivative of φ at the point u in the direction v. If no confusion can arise, we write φ and $\varphi'[\;]$ instead of $\varphi(u)$ and $\varphi'(u)[v]$. We recall that a function $\varphi: S \to S^*$ is said to be a gradient function if it has a potential $p: S \to \mathbf{R}$, which means that $p'(u) = \varphi(u)$ for all $u \in S$. A necessary and sufficient condition for the existence of a potential is the symmetry of the second derivative, then the potential can be chosen to be

$$p(u) = \int_0^1 \langle \varphi(\lambda u), u\rangle \, d\lambda \, .$$

This note is concerned with the evolution equation

$$u_t = K(u)\,, \qquad u(t) \in S \tag{1}$$

where K is some function $S \to S$. The following notions are intimately connected with this evolution equation.

i) A function $\sigma: S \to S$ is called a *symmetry* of (1), if we have always

$$\sigma'[K] - K'[\sigma] = 0\,. \tag{2.a}$$

ii) $\gamma: S \to S^*$ is said to be a *conserved covariant* of (1), if we have always

$$\gamma'[K] + K'^{+}[\gamma] = 0 \qquad (K'^{+} \text{ transpose of } K'). \tag{2.b}$$

iii) An operator-valued function $\Phi(u): S \to S$, $u \in S$, is called a *strong symmetry* of (1), if we have always

$$\Phi'[K] - [K', \Phi] = 0\,. \tag{2.c}$$

(⁵) B. FUCHSSTEINER and A. S. FOKAS: *Symplectic structures, their Bäcklund transformations and hereditary symmetries* (to appear).

iv) An operator-valued function $\theta(u): S^* \to S$, $u \in S$, is called a *Noether operator* of (1), if we have always

$$\theta'[K] - \theta K'^{+} - K'\theta = 0 . \tag{2.d}$$

One should keep in mind that all these terms depend on the variable $u \in S$. So, for example, (2.a) should be read as follows: $\sigma'(u)[K(u)] - K'(u)[\sigma(u)] = 0$. And, of course, by « always » we mean that eqs. (2.a)-(2.d) must hold identically in $u \in S$.

The following properties, although independant of the particular evolution equation (1), are very useful in analyzing the structure of this evolution equation.

v) An operator-valued function $\Phi(u): S \to S$, $u \in S$, is called a *hereditary symmetry* if $[\Phi'(u), \Phi(u)]$ is a symmetric bilinear operator for all $u \in S$, *i.e.* if

$$(v, w) \to \Phi'(u)[\Phi(u)v]w - \Phi(u)\,\Phi'(u)[v]w \tag{2.e}$$

is symmetric in v, $w \in S$.

vi) An operator-valued function $\theta(u): S^* \to S$, $u \in S$, which is skew-symmetric is called *implectic* if the bracket defined by

$$\{a, b, c\} = \langle b, \theta'(u)[\theta(u)c]\,b\rangle$$

satisfies the Jacobi identity

$$\{a, b, c\} + \{b, c, a\} + \{c, a, b\} = 0 .$$

Implectic is—by abuse of language—a short form for inverse-symplectic. The reason for this name is, that if $J(u) = \theta^{-1}(u)$ exists, then $J(u)$ is symplectic in the usual sense, *i.e.* the bracket

$$[[a, b, c]] = \langle J'(u)[a]\,b, c\rangle$$

fulfills the Jacobi identity.

A straightforward (but cumbersome) application of the chain rule of differential calculus yields that, if Φ and θ are a strong symmetry of (1) and a Noether operator of (1), respectively, then

(3.a) Φ maps symmetries of (1) into symmetries of (1),

(3.b) $\Phi\theta$ is a Noether operator.

By the same kind of straighforward calculation one verifies (²) that, if Φ is hereditary, then:

(4) If Φ is a strong symmetry of (1), then it is also a strong symmetry for any of the following equations: $u_t = \Phi(u)^n K(u)$, $n = 1, 2, \ldots$. Hence, since every translation-invariant operator is a strong symmetry for $u_t = u_x$, it must be a strong symmetry for any of the equations $u_t = \Phi(u)^n u_x$, $n = 1, 2, 3, \ldots$.

The last statement has very many immediate applications in the theory of soliton equations (2).

Let us consider a function $B(u, s)$ in two variables $u \in S_1$, $s \in S_2$ (S_1, S_2 suitable vector spaces) going into a third vector space S_3.

We assume that $B(u, s) = 0$ defines an honest implicit function between u and s, that is we assume that the corresponding *partial derivatives* B_u and B_s are isomorphisms between the tangent spaces. This function is a *Bäcklund transformation* (6) between the evolution equations

$$(5.a) \qquad u_t = K(u) \,, \qquad u(t) \in S_1 \,,$$

$$(5.b) \qquad s_t = G(s) \,, \qquad s(t) \in S_2 \,,$$

if, for all t, $B\big(u(t), s(t)\big) = 0$, whenever $B(u(0), s(0)) = 0$.

I) An important role is played by the linear transformation $T = B_s^{-1} B_u$, where u and s are related by $B(u, s) = 0$. Again by a rather cumbersome, but nevertheless straighforward, application of the chain role one proves

(6*a*) $\Phi(u)$ is a strong symmetry of (5*a*) if and only if

$\Psi(s) = T\Phi(u)\,T^{-1}$, where $B(u, s) = 0$ is a strong symmetry of (5*b*) (7).

(6*b*) $\theta(u)$ is a Noether operator of (5*a*) if and only if

$\Omega(s) = T\theta(u)\,T^{+}$, where $B(u, s) = 0$ is a Noether operator of (5*b*) (5).

II) The same transformation formulae do hold for hereditary symmetries and implectic operators. To be more precise we have

(6*c*) $\Phi(u)$ is a hereditary symmetry if and only if

$\Psi(s) = T\Phi(u)\,T^{-1}$, where $B(u, s) = 0$ is hereditary (7).

(6*d*) $\Theta(u)$ is an implectic operator if and only if

$\Omega(s) = T\theta(u)\,T^{+}$, where $B(u, s) = 0$ is an implectic operator (5).

III) For describing the connection between implectic operators and special hereditary symmetries we need the notion of compatibility. Two implectic operators θ_1, θ_2 are said to be *compatible* if their sum is again implectic. This is the case if and only if the mixed bracket

$$|a, b, c| = \langle b, \theta_1'[\theta_2 a]c\rangle + \langle b, \theta_2'[\theta_1 a]c\rangle$$

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(6) A. S. Fokas and R. L. Anderson: *Lett. Math. Phys.*, **3**, 117 (1979); F. A. E. Pirani, D. C. Robinson and W. F. Shadwick: *Local Jet Bundle Formulation of Bäcklund Transformations*, *Mathematical Physics Studies*, Vol. **1** (Amsterdam, 1979).

(7) A. S. Fokas and B. Fuchssteiner: *Bäcklund transformations for hereditary symmetries* (to appear).

satisfies the Jacobi identity.

(6e) If θ_1, θ_2 are compatible implectic operators and if θ_1 is invertible, then $\Phi = \theta_2\theta_1^{-1}$ is hereditary. Furthermore, all the operators $\Phi^n\theta_1$, $n = 1, 2, \ldots$ are again implectic. Hence, for invertible Φ, the equation $u_t = \Phi(u)^n u_x$ must be N-Hamiltonian (N arbitrary).

An example. – Let S be the space of C^∞-functions vanishing at $\pm\infty$, D the differential operator and D^{-1} its inverse given by

$$(D^{-1}f)(x) = \int\limits_{-\infty}^{x} f(\xi)\,\mathrm{d}\xi\,.$$

For S^* we take $\{D^{-1}s|s \in S\}$ and we define the bilinear functional as usual:

$$\langle s_1, s_2\rangle = \int\limits_{-\infty}^{+\infty} s_1(x)\,s_2(x)\,\mathrm{d}x\,.$$

Obviously, the constant operator D is implectic. And considering the implicit function

$$B(u, s) = s + \tilde{\alpha}u + \tilde{\beta}u^2 + u_x = 0\,, \qquad \tilde{\alpha}, \tilde{\beta} \text{ arbitrary,}$$

we obtain from (6.4) that

$$\theta(u) = \alpha D + \beta D^3 + \gamma(Du + uD)\,, \qquad \alpha, \beta, \gamma \text{ arbitrary,} \tag{7}$$

are again implectic. Because of the arbitrariness of the scalars α, β, γ the corresponding parts of (7) constitute compatible pairs of implectic operators. Hence (by 6.5) we obtain hereditary symmetries by suitable factorization. Among the hereditary symmetries obtained by this procedure is, for example, the one which completely describes the special features of the KdV (2). Another one is

$$\Phi(u) = \{\alpha + \beta D^2 + \gamma(DuD^{-1} + u)\}\,(1 - D^2)^{-1} = \theta(u)\,\theta_1^{-1}\,, \tag{8}$$

where $\theta(u)$ is given by (7) and $\theta_1 = (D - D^3)$.

The operator $\Phi(u)$ is translation invariant, hence all the equations

$$u_t = K_m(u) = \Phi(u)^m u_x\,, \qquad m = 0, 1, 2, \ldots \tag{9}$$

are exactly solvable. For each of these equations an infinite sequence of symmetries is given by the K_n, $n = 0, 1, 2, \ldots$ (consequence of (3*a*) or (4), and a sequence of conserved covariants is given by the $G_n = \theta_1^{-1}K_n = \Phi^{+n}G_0$, where $G_0 = \theta_1^{-1}u_x$.

This is a consequence of either (6*b*) or (3*b*).

Finally, the soliton solutions of (9) are given by those $u(x, t)$ which can be written as

$$u_x(x, t) = \sum_{n=1}^{N} w_n(x, t)\,,$$

where the $w_n(x, t)$ are eigenvectors (with time-independent eigenvalues) of the operator $\Phi\big(u(x, t)\big)$ (2).