APPLICATION OF SPECTRAL-GRADIENT METHODS TO NONLINEAR SOLITON EQUATIONS

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Abstract: For certain operator-valued functions it is shown that the Hadamard-derivatives of the discrete spectrum are solutions of canonically determined ordinary differential equations. These equations then immediately yield a full description of the soliton solutions and they give recursion formulas for symmetries and conservation laws of many nonlinear evolution equations (among them the popular ones, like the KdV, sine-Gordon and the Sakharov-Shabat equation).

I. Introduction

In a suitable real-linear space, for example $S = S_{\mathbb{R}}$ or $S_{\mathbb{C}}$ the space of ∞-often differentiable rapidly decreasing real or complex functions, we study the evolution equation

$$(1) \quad u_t = K(u) \quad ,$$

where $K(u)$ is assumed to be differentiable in u and to have a certain kind of local behaviour. We assume that we know another local operator $L(u)$ being differentiable in u, acting in a vector space E, being symmetric with respect to a given bilinear functional $\langle \cdot,\cdot \rangle$ on E and having the property that the eigenvalues of $L(u(t))$ do not depend on t if $u(t)$ evolves according to (1).

In this paper we give a straightforward method to determine, based on the information given above, a strong symmetry for (1), thus achieving (as described in [4]) a way to find countably many symmetries and conservation laws and to describe completely the (eventually occurring) soliton solutions of (1).

Before recalling the basic notions we like to mention that by differentiable we always mean Hadamard-differentiable [11] (i.e. the chain rule holds) and that we tacidly assume that for the evolution equations under consideration the initial value problem is properly posed in the sense that for every time t_0 and every initial condition $u(t_0) = u_0 \in S$ there is a unique solution $u(t) = u(t,u_0)$ which is differentiable in u_0.

An important role is played by the linear perturbation equation associated with (1):

(2) $v_t = K'(u)[v]$

where $K'(u)$ denotes the derivative with respect to u, i.e.

$$K'(u)[v] = \frac{\partial}{\partial \varepsilon} K(u+\varepsilon v)_{|\varepsilon = 0} ,$$

and where u is a given solution of (1).

A symmetry [10] $\sigma(\cdot)$ is a map $\sigma : S \to S$ such that $\sigma(u(t))$ is a solution of (2) whenever $u(t)$ is a solution of (1). Examples are $K(\cdot)$ and, if (1) is translation invariant, $u \to u_x$.

A conserved covariant $\gamma(\cdot)$ is a map $\gamma : S \to S^*$ (algebraic dual) such that $\langle\gamma(u(t)), v(t)\rangle$ is time independent whenever $u(t)$ and $v(t)$ are solutions of (1) and (2) respectively. Of course, $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form on (S^*,S). Examples for conserved covariants are the gradients of conservation laws, i.e. maps $p: S \to \mathbb{R}$ such that $p(u(t))$ is time independent for every solution $u(t)$ of (1).

Finally, a strong symmetry (called recursion operator in [9]) $\Phi(\cdot)$ is a map Φ: $S \to L(S,S)$(linear operators in S) such that $\Phi(u(t))\, v(t)$ is a solution of (2) whenever $u(t)$ and $v(t)$ are solutions of (1) and (2) respectively. Differentiable $\Phi(\cdot)$ are strong symmetries if and only if one of the following two equivalent [4, theorem 1.3] conditions holds

(3) $[\Phi(u(t)), \frac{\partial}{\partial t} - K'(u(t))] = 0$ if $u(t)$ is a solution of (1),

(4) $\Phi'(w)K(w) - [K'(w), \Phi(w)] = 0$ for all $w \in S$.

Here, [,] stands for the commutator and $\Phi'(w)K(w)$ denotes the operator $\frac{\partial}{\partial \varepsilon} \Phi(w + \varepsilon K(w))|_{\varepsilon = 0}$.

Now, let $\sigma(\cdot)$, $\gamma(\cdot)$ and $\Phi(\cdot)$ be as above and let Φ^T denote the transposed of Φ. Then elementary considerations lead to:

<u>1. Properties of strong symmetries</u> [4]:

(i) $\Phi(\cdot)\sigma(\cdot)$ <u>is a symmetry</u>

(ii) $\Phi^T(\cdot)\gamma(\cdot)$ <u>is a conserved covariant</u>

(iii) If $u(t)$ <u>is a solution of</u> (1) <u>then the eigenvalues of</u> $\Phi(u(t))$ <u>are time independent</u>. <u>Furthermore</u>, <u>if</u>

$$\sigma(u(t_0)) = \sum_{n=1}^{N} \omega_n(t_0) \quad ,$$

<u>then for all times</u> t <u>we have</u>

$$(5) \quad \sigma(u(t)) = \sum_{n=1}^{N} \omega_n(t) \quad ,$$

<u>where the</u> $\omega_n(t)$ <u>are eigenvectors of</u> $\Phi(u(t))$ <u>with eigenvalues</u> λ_n .

The only nontrivial fact is the <u>structural stability</u> observed in (iii). The reason for this stability (which, by asymptotic considerations) turns out to be the <u>soliton decomposition</u>) is that for a fixed eigenvalue one can choose the corresponding eigenvector to be a solution of the perturbation equation.

So, roughly spoken, a nontrivial strong symmetry describes completely all the features which set soliton equations apart from ordinary nonlinear evolution equations.

Now, under additional restrictions, our method for finding strong symmetries is as follows:

<u>Consider</u> an eigenvalue $\lambda(u)$ of $L(u)$ and determine its gradient grad(λ) given by

$$\langle \mathrm{grad}(\lambda), v\rangle = \frac{\partial\lambda(u+\varepsilon v)}{\partial\varepsilon}\Big|_{\varepsilon=0} \qquad \text{for all} \quad v \in S .$$

Find then for grad (λ) a differential-integral equation (for a rather important class of operators this can be done by elementary linear algebra). If this equation can be rewritten in eigenvalue-form:

$$(6) \quad \Psi(u)\ \mathrm{grad}(\lambda) = f(\lambda)\ \mathrm{grad}(\lambda),$$

where $f(\lambda) \in \mathbb{R}$ and $\Psi(u)$ is λ-independent, then the transposed $\Psi^T(u)$ of $\Psi(u)$ is a strong symmetry. In addition it is a strong symmetry for the following class of evolution equations.

$$(7) \quad u_t = (\Psi^T(u))^n\, u_x .$$

And it turns out, that the knowledge of (1) is not at all necessary. Since our method deals only with the spectral gradient of $L(u)$ it enables us to determine those evolution equations having with respect to $L(u)$ the required properties.

Before developing the theory associated with this <u>recipe</u> we like to illustrate this procedure at the example given by the Korteweg -de Vries equation (KdV).

<u>2. Example KdV</u>: It is well known ([5] to [8]) that the eigenvalues of the

one-dimensional Schrödinger operator $L(u) = D^2 + u$ (where D stands for taking the x-derivative) are time-independent if $u(t)$ evolves according to the KdV.

(8) $u_t = u_{xxx} + 6uu_x$.

Let $w = w(u)$ the eigenvector for a fixed eigenvalue $\lambda(u)$, then the spectral gradient is easily calculated to be

(9) $\frac{\partial \lambda(u+\varepsilon v)}{\partial \varepsilon}\Big|_{\varepsilon = 0} = \frac{1}{\langle w,w\rangle} \langle w^2,v\rangle$,

where $\langle\ ,\ \rangle$ is the usual scalar product in L^2. Elementary manipulation with $L(u)$ yields the following differential equation for w^2:

(10) $4\lambda^2 Dw^2 = (D^3 + 2Du + 2uD)w^2$.

Hence, the desired $\Psi(u)$ must be

$$\Psi(u) = D^{-1}(D^3 + 2Du + 2uD) ,$$

where D^{-1} stands for the operator of integration from $-\infty$ to x.

The transposed

(11) $\Psi(u)^T = D^2 + 2u + 2DuD^{-1}$ (Lenard -operator)

is then a strong symmetry for the KdV and for

(12) $u_t = (\Psi^T(u))^n (u_{xxx} + 6uu_x)$ (generalized KdV's).

Since for these evolution equations $u \to u_x$ and $u \to u$ are a symmetry and a conserved covariant respectively, sequences of symmetries and conserved covariants are given by

(13) $\sigma_n(u) = (\Psi^T(u))^n u_x$

(14) $\gamma_n(u) = (\Psi(u))^n u$.

Furthermore, a multisoliton solution is given if and only if for some t_0 (and therefore all t) we have

$$(15) \quad u_x = \sum_{n=1}^{N} \omega_n \quad ,$$

where ω_n are eigenvectors of $\Psi^T(u)$. Because of $D\Psi(u) = \Psi(u)^T D$ this is equivalent to

$$(16) \quad u = \sum_{n=1}^{N} \tilde{\omega}_n \quad ,$$

where the $\tilde{\omega}_n$ are eigenvectors of $\Psi(u)$ (or squares of eigenvectors of $L(u)$). This soliton description is well known ([3] or [5]).

II. The local-global principle

In this section we briefly describe a principle which enables us to determine the validity of certain operator identities for operators depending on a parmeter $v \in S$. These identities will be checked by testing them against the eigenvectors of another operator-valued function on S. Due to the special structure of the operators under consideration, this test will be sufficient even if the linear span of the eigenvectors is far from being dense.

First a word on notation. By $S^-(= S^-_{\mathbb{R}}$ or $S^-_{\mathbb{C}})$ we denote the (real or complex) ∞-often differentiable functions which vanish rapidly at $-\infty$ and are of at most polynomial growth at $+\infty$. We embed S^- in S^* by defining for $f \in S^-$, $v \in S$

$$\langle f,v\rangle \doteq \int_{-\infty}^{+\infty} (f_1 v_1 + f_2 v_2)d\xi \quad ,$$

where the $f_{1,2}$ and $v_{1,2}$ are real such that $f = f_1 + if_2$, $v = v_1 + iv_2$.

Now, for two functions f, g on $\mathbb{R}$ we write

(17) $f(x) \equiv g(x)$

if f equals g in a neighborhood of x. A map $K : S \to S^-$ (or $K : S^- \to S^-$) is called <u>local</u> if $v(x) \equiv \tilde{v}(x)$ always implies

$$(K(v))(x) \equiv (K(\tilde{v}))(x).$$

Similarly, a function $\Psi\colon\ S \times S^- \to S^-$ is said to be local if it is separately local in each variable. And $\Phi\colon\ S \to L(S^-,S^-)$ (linear operators on S^-) is called a local operator-valued function if $(u,v) \to \Phi(u)v$ is local and if for all $u \in S$ and almost all $x \in \mathbb{R}$ the operator $\Phi(u)$ is locally either zero (i.e. $(\Phi(u)v)(x) \equiv 0\ \forall v \in S^-$) or has a finite dimensional local kernel. Here, the local kernel at x is, of course, the quotient of $E = \{v \in S^- | (\Phi(u)v)(x) \equiv 0\}$ with respect to the equivalence relation given by (17).

<u>3. Example</u>: Let $\mathcal{A}(D)$ be the algebra of operator-valued functions generated by the constant function being everywhere D, and by the multiplication-operator-valued functions $v \to g(v)$, where g is supposed to be an entire function. If $S = S_{\mathbb{R}}$ then we require in addition that g is real on $\mathbb{R}$, and if $S = S_{\mathbb{C}}$ then we require in addition that $\mathcal{A}(D)$ is closed against taking the complex conjugate of multiplication operators. The elements of $\mathcal{A}(D)$ are local.

Since (17) depends solely on the topology of $\mathbb{R}$ we may also consider other topologies, for example when $\{(-\infty,\alpha) | \alpha \in \mathbb{R}\}$ are the open sets. To set this situation apart from the usual one we then use <u>semilocal</u> instead of local.

4. Example: Let $\mathcal{A}(D,D^{-1})$ be the algebra of operator-valued functions generated by $\mathcal{A}(D)$ and the constant function being everywhere D^{-1}. Then the elements of $\mathcal{A}(D,D^{-1})$ are semilocal.

Now, let T be a multivalued function on S assigning to each $v \in S$ elements $w_1(v), \ldots, w_n(v) \in S^-$, where $n = n(v)$. Let $R(T(v))$ be the linear span of these vectors and let $P(T(v))(x)$ the linear span of

$$\{R(T(\tilde{v})) \mid \tilde{v} \in S, v(x) \equiv \tilde{v}(x)\}.$$

If $P(T(v))(x)$ has, for every $x \in \mathbb{R}$ and $v \in S$, infinite dimension then T is called global (or semiglobal in case of the unusual topology on $\mathbb{R}$).

5. Example: Denote by $w_1(v), \ldots, w_n(v)$ the eigenvectors in S of the Schrödinger operator $D^2 + v$, then $v \to (w_1(v), \ldots, w_n(v))$ is global, and moreover semiglobal. This is so, because for fixed x, one can generate arbitrary many eigenvectors by changing (deeping) the potential on the right of x.

6. Proposition: Let $T(\cdot)$ be global (semiglobal) and let $\Phi(\cdot)$ be a local (semilocal) operator-valued function. If $\Phi(v)\, w(v) = 0$ for all v and $w(v) \in R(T(v))$ then $\Phi(\cdot) = 0$.

The obvious proof is an immediate consequence of the above definitions and the observation that $(\Phi(v)\, w(v)(x) \equiv 0$ implies $\Phi(v)\, w(\tilde{v})(x) \equiv 0$ whenever $v(x) \equiv \tilde{v}(x)$.

III. The spectral gradient

Assume that in a vector space E we have a bilinear functional $\langle \cdot,\cdot \rangle$ and a $\langle \cdot,\cdot \rangle$-symmetric, differentiable operator family $L(u)$, $u \in S$. Let $w_1(u), \ldots, w_n(u)$ be the set of eigenvectors of $L(u)$ with $\langle w_i(u), w_i(u) \rangle = 1$

and denote by $\lambda_1(u), \ldots, \lambda_n(u)$, the associated eigenvalues.
For simplicity only, we assume that the multiplicity of these eigenvectors is 1. By $G_i(u)$ we denote the gradients of the $\lambda_i(u)$, i.e.

$$\frac{\partial \lambda_i(u+\varepsilon v)}{\partial \varepsilon}\Big|_{\varepsilon = 0} = \langle G_{\lambda_j}(u),v\rangle \quad \text{for all} \quad v \in S.$$

By differentiation of the eigenvalue equation we get

$$(18)\quad \langle G_{\lambda_j}(u),v\rangle = \langle w_j(u), L'(u)[v]w_j(u)\rangle .$$

We assume in the following that $L(\cdot)$ has a <u>global spectral gradient</u>, i.e. $u \to (G_{\lambda_1}(u), \ldots, G_{\lambda_n}(u))$ is global. We say, that $L(\cdot)$ <u>admits a spectral gradient operator</u> if there is some nonzero $\Psi(\cdot) \in \mathcal{A}(D,D^{-1})$ such that for all u we have

$$(19)\quad \Psi(u)\, G_{\lambda_i}(u) = \beta_i(u)\, G_{\lambda_i}(u)\ ,\ \beta_i(u) \text{ scalar, } i = 1,2, \ldots$$

and such that $\Psi^T(u)$ maps $S \to S$.

Now, consider some $K\colon\ S \to S$ of the form $K(u) = \Gamma(u)u$ with $\Gamma(\cdot) \in \mathcal{A}(D,D^{-1})$. Then $K(\ )$ is differentiable with $K'(\ ) \in \mathcal{A}(D,D^{-1})$.

<u>7. Lemma</u>: <u>Let $u(t)$ evolve according to $u_t = K(u)$. Then the eigenvalues of $L(u(t))$ are always time-independent if and only if $K(\cdot)$ is orthogonal to the spectral gradient of $L(\cdot)$, i.e. for all $v \in S$ we have $\langle G_{\lambda_j}(v), K(v)\rangle = 0$, $j = 1, 2, \ldots$.</u>

The proof is an immediate consequence of

$$(20)\quad \frac{d\lambda_j}{dt} = \frac{\partial \lambda_j(u+\varepsilon u_t)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \langle G_{\lambda_j}(u), u_t\rangle = \langle G_{\lambda_j}(u), K(u)\rangle\ ,$$

and the assumption that we can prescribe every $v \in S$ as initial condition for $u_t = K(u)$. ∎

8. <u>Corollary</u>: <u>Let $K(\cdot)$ be orthogonal to the spectral gradient of $L(\cdot)$. Then the eigenvalues of $L(u(t))$ are time-independent whenever $u(t)$ evolves according to one of the following evolution equations</u>:

$$(21) \quad u_t = K_m(u) \doteq (\Psi^T(u))^m u.$$

<u>Proof</u>: Because of

$$\langle G_{\lambda_i}(u), K_m(u)\rangle = \langle \Psi(u)^m G_{\lambda_i}(u), K(u)\rangle = \beta_i(u)^m \langle G_{\lambda_i}(u), K(u)\rangle$$

$K_m(\cdot)$ must be orthogonal to the spectral gradient. ∎

9. <u>Theorem</u>: <u>Let $K(\cdot)$ be orthogonal to the spectral gradient of $L(\cdot)$. Then for all u we have</u>

$$\mathcal{V}(u) = \frac{\Psi(u+\varepsilon K(u))}{\varepsilon}\Big|_{\varepsilon=0} + [(K'(u))^T, \Psi(u)] = 0.$$

<u>Proof</u>: Because of the assumption we have

$$\langle G_\lambda(u+\varepsilon v), K(u+\varepsilon v)\rangle = 0.$$

Taking the ε-derivative we obtain for arbitrary v:

$$(22) \quad \langle G_\lambda'(u)[v], K(u)\rangle + \langle G_\lambda(u), K'(u)[v]\rangle = 0.$$

And using the fact that the gradient $G_\lambda(u)$ has a potential (namely λ) we obtain (by interchanging the second derivatives of the potential):

$$(23) \quad \langle G_\lambda'(u)[v], w\rangle = \langle G_\lambda'(u)[w], v\rangle \quad \text{for all } v, w.$$

Insertion in (22) gives:

$$(24) \quad G_\lambda'(u)[K(u)] + (K'(u))^T G_\lambda(u) = 0.$$

Now, considering the ε-derivative of

$$\Psi(u+\varepsilon K(u))\ G_\lambda(u+\varepsilon K(u)) = \beta(u+\varepsilon K(u))\ G_\lambda(u+\varepsilon K(u))$$

then by use of (24) we obtain:

$$\Psi'(u)\ G_\lambda(u) = 0.$$

And from the local-global principle follows $\Psi'(u) = 0$. ∎

Taking the transposed of $\Psi'(u)$ we get with the help of (4) and Corollary 8:

<u>10. Theorem</u>: <u>Let $K(\cdot)$ be orthogonal to the spectral gradient of $L(\cdot)$, and let $\Psi(\cdot)$ be a spectral gradient operator for $L(\cdot)$, then $\Psi(u)^T$ is a strong symmetry for any of the following evolution equations</u>

$$u_t = (\Psi(u)^T)^m\ K(u), \quad m = 0,1,\ \ldots\ .$$

In the light of this theorem our procedure in Example 2 is completely justified.

<u>11. Remark</u>: In applications (cf. the sine-Gordon equation) it is sometimes tedious to check the orthogonality condition for $K(u)$ in the above theorem. Of course, if the eigenvalues of $L(u)$ are nonzero (or if the spectral gradient associated with the nonzero eigenvalues is still global) then it suffices to check the orthogonality condition for $\Psi^T(u)\ K(u)$.

IV. Applications

Let T be an invertible two-by-two matrix and consider in $E = S \oplus S = \{\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_{1,2} \in S\}$ the bilinear form

$$\left(\vec{u},\vec{v} \right) = \sum_{j,k=1}^{2} T_{jk} \, \langle u_k, v_j \rangle \ ,$$

where $\langle \cdot,\cdot \rangle$ is as defined in section II. Linear operators on E are again represented by matrices

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with operators B_{jk} on S. For simplicity operators $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ are denoted by A. The transposed B^* of B with respect to $\left(\cdot,\cdot \right)$ is given by

$$(25) \quad B^* = T^{-1} \begin{pmatrix} B_{11}^T & B_{21}^T \\ B_{12}^T & B_{22}^T \end{pmatrix} T.$$

where B^T denotes transposition in (S,S^*).

We want to calculate the spectral gradient operator for

$$(26) \quad L(u) = DA + B + u\Gamma + \bar{u}\Gamma^* \doteq DA + P(u),$$

where A, B, Γ are fixed multiplication operators (i.e. the matrix elements are operators of multiplication by functions). We assume $A = -A^{*}$ and $B = B^{*}$, then $L(u) = L(u)^{*}$. Further we assume that A is invertible. Operators of this form were considered in the celebrated AKNS-paper ([2] or [1]).

Now, we fix an eigenvector w of $L(u)$ with $\{w,w\} = 1$ and with associated eigenvalue λ. For multiplication operators $\Lambda(v)$ which depend real-linear on v we define a function $G_\lambda(u,\Lambda)$ by

$$(27)\quad \langle G_\lambda(u,\Lambda),v\rangle = \{w,\Lambda(v)w\} .$$

The spectral gradient given by (18) is of this form, namely

$$(28)\quad G_\lambda(u) = G_\lambda(u,L'(u)) \;, \quad \text{where } L'(u)[v] = \frac{\partial}{\partial\varepsilon} L(u+\varepsilon v)_{|\varepsilon=0}.$$

The importance of the next lemma lies in the fact that the x-derivative of $G_\lambda(u,\Lambda)$ can be expressed by a sum of functions of the same type

<u>12. Lemma</u>: <u>The x-derivative of</u> $G_\lambda(u,\Lambda)$ <u>is given by</u>:

$$G_\lambda(u,\Lambda)_x = G_\lambda(u,\tau(\Lambda)) + G_\lambda(u,\partial\Lambda),$$

<u>where</u>

$$\tau(\Lambda)(v) \doteq \lambda[\Lambda(v), A^{-1}] + P(u)\, A^{-1}\, \Lambda(v) \quad - \Lambda(v)\, A^{-1}P(u)$$

$$(\partial\Lambda)(v) \doteq [D,\Lambda(v)] - \Lambda(v_x).$$

<u>Proof</u>: Because of $D^{*} = -D$ we have

$$(29)\quad \langle G_\lambda(u,\Lambda)_x,v\rangle = -\langle G_\lambda(u,\Lambda),v_x\rangle = -\{w,\Lambda(v_x)w\}$$

$$= \{w,(\partial\Lambda)(v)w\} + \{Dw,\Lambda(v)w\} + \{w,\Lambda(v)Dw\} \quad .$$

From $L(u)w = \lambda w$ we obtain:

$$Dw = \lambda\, A^{-1}\, w - A^{-1}\, P(u)w \;,$$

Introducing this in (29) leads to the desired result. ∎

13. Theorem: For the function $G_\lambda(u)$ there is a linear ordinary differential equation of order ≤ 4.

Proof: Let $\sigma_0, \ldots, \sigma_3$ be a basis in the space of 2×2 - matrices.

Define G_j by:

$$\langle G_j, v\rangle = \frac{1}{2}\left(w, (v\sigma_j + \bar{v}_j\, \sigma_j^{*})w\right).$$

Then $G_\lambda(u)$ may be written as:

$$(30) \quad G_\lambda(u) = \sum_{j=0}^{3} \phi_j\, G_j \quad ,$$

where ϕ_j are suitable functions in x. From lemma 12 we obtain

$$\frac{d^m G_\lambda(u)}{d_x{}^m} = \sum_{j,k=0}^{3} \phi_{jk}\, G_k \quad , \qquad m=0,1,2,3,$$

with suitable functions ϕ_{jk}. From this system of linear equations for the unknown functions G_j we obtain the G_j in terms of derivatives of the spectral gradient with functions as coefficients. This solution inserted in (30) gives the desired differential equation. ∎

Now, if one can rewrite this equation in eigenvalue-form then our spectral gradient operator is found.

Since our problem of finding strong symmetries is now reduced to an exercise in elementary linear algebra every reader can furnish for himself a lot of examples and applications. Nevertheless we cannot resist the temptation to present some explicit calculations.

14. Example: In the 2×2-matrices we choose the basis given by the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These fulfill

$\sigma_k\sigma_k = I$, $k = 0, \ldots, 3$ and $\sigma_\ell\sigma_k = -\sigma_k\sigma_\ell = i\sigma_r$ if (ℓ,k,r) is a cyclic permutation of $(1,2,3)$. Furthermore we put $T = \sigma_1$ and we restrict $u, v \in S_{\mathbb{R}}$. Then $\sigma_{0,1}^* = \sigma_{0,1}$, $\sigma_{2,3}^* = -\sigma_{2,3}$ and

(30) $L(u) = D\sigma_3 + u\sigma_1$

is a symmetric operator. And in this case we have

$$G_\lambda(u) = G_\lambda(u,\sigma_1)$$

The transformation τ of lemma 12 is easily seen to be:

$$\tau(\Lambda) = \lambda[\Lambda,\sigma_3] - iu\{\sigma_2,\Lambda\} ,$$

where $\{\alpha,\beta\} \doteqdot \alpha\beta + \beta\alpha$. Lemma 12 yields then

$$\begin{aligned} DG_0 &= -2iu\, G_2 \\ DG_1 &= -2i\lambda\, G_2 \\ DG_2 &= 2i\lambda G_1 - 2iu\, G_0 , \end{aligned}$$

where $G_j = G_\lambda(u,\sigma_j)$, i.e. $\langle G_j,v\rangle = \langle w,v\sigma_j w\rangle$.

This gives for $G = G_\lambda(u)$ the differential equation:

$$4\lambda^2 D(\tfrac{G}{u}) = D(\tfrac{1}{u} D^2G) + 4uDG.$$

Hence, the spectral gradient operator is

(31) $\Psi(u) = D^2 + 4uD^{-1}\, uD$

and

(32) $\Psi(u)^T = D^2 + 4u_x D^{-1}\ u + 4u^2$

must be a strong symmetry for any of the following evolution equations

(33) $u_t = (\Psi(u)^T)^k u_x \quad , \qquad k \in \mathbb{Z}$,

since $u_t = u_x$ leaves the eigenvalues of $L(u(t))$ time independent (translation invariance!).

For $k = 1$ this is the modified KdV

$$u_t = u_{xxx} + u_x u^2,$$

and for $k = -1$ this is the sine-Gordon equation:

$$u_t(x,t) = \frac{1}{2} \sin(2\int_{-\infty}^{x} u(\xi,t)d\xi).$$

<u>14.1 Remark</u>: If, instead of (30), one considers

(34) $L(u) = D\sigma_3 + \frac{i}{2}(1-u)\sigma_2 - \frac{1}{2}(1+u)\sigma_1$

then the spectral gradient operator becomes the transposed of the Lenard operator.

<u>15. Example</u>: We keep the notation of the last example, but we choose T to be equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and we put $S = S_{\mathbb{C}}$. We are interested in

(34) $L(u) = iD\sigma_3 + \frac{u}{2}(\sigma_1 - i\sigma_2) + \frac{\bar{u}}{2}(\sigma_1 + i\sigma_2)$.

Then $L(u) = L(u)^*$. If σ is a matrix, then we define G_σ by

$$\langle G_\sigma, v\rangle = \frac{1}{2}\langle w,(\sigma v + \sigma^*\bar{v})w\rangle \quad \text{for all} \quad v \in S,$$

where w is the eigenvector of $L(u)$ with eigenvalue λ and with $\langle w,w\rangle = 1$.

Then we always have $G_{i\sigma} = -iG_{\sigma}$ and the spectral gradient turns out to be

$$G \doteq G_{\lambda}(u) = G_{\sigma_1} + iG_{\sigma_2} = G_{(\sigma_1 - i\sigma_2)} .$$

Lemma 12 yields for the derivative

$$D\, G_{\sigma} = i\lambda\, G_{[\sigma,\sigma_3]} - \mathrm{Re}(u)\, G_{\{\sigma_2,\sigma\}} + \mathfrak{Im}(u)\, G_{\{\sigma_1,\sigma\}},$$

where $\mathrm{Re}(\ )$ and $\mathfrak{Im}(\ )$ denote the real and the imaginary part (i.e. $u = \mathrm{Re}(u) + i\mathfrak{Im}(u)$). From this we easily obtain

$$DG = 2i\lambda\, G - 2iuG_{\sigma_0}$$

$$DG_{\sigma_0} = 2\mathrm{Re}(i\bar{u}G).$$

So, the differential equation we are searching for is

(35) $2\lambda D(\frac{1}{u}\, G) = -iD(\frac{1}{u}\, DG) + 4\mathrm{Re}(i\bar{u}G).$

Hence the spectral gradient operator and its transposed are

(36) $\Psi(u) = -iD + 4uD^{-1}\, \mathrm{Re}(i\bar{u}\ \bullet)$

(37) $\Psi(u)^T = \Psi(iu) = -iD + 4iuD^{-1}\, \mathrm{Re}(\bar{u}\ \bullet)$.

And $\Psi(iu)$ must be a strong symmetry for any of the following evolution equations

(38) $u_t = \Psi(iu)^m(u_x),$

since $L(u)$ is translation invariant. The first one of these equations is the well known Sakharov-Shabat equation ([12],[13])

$$u_t = -iu_{xx} + 2iu(u\bar{u}).$$

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