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NEW VERSIONS OF THE HAHN-BANACH THEOREM

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ABSTRACT. The Hahn-Banach theorem is perhaps the most fundamental individual theorem in abstract analysis. It is in the literature in countless forms. Yet there is still demand for versions which at the same time have simple shape and admit fast and widespread application. The present note claims to present some versions of this sort. The Main Version 1.1 is due to König and has been announced in [5] without its complicated initial proof. The present simple proof via the Fundamental Lemma 1.2 is due to Fuchssteiner. We also present the extended versions of the minimax theorem which follow from the above Hahn-Banach results.

The Hahn-Banach theorem has meanwhile been lifted to a new level of abstraction in a paper of Rodé [7]. His Theorem contains our Main Version 1.1, but to obtain the latter one in this manner would be much more involved, so that our presentation seems to retain independent interest.

1. MAIN VERSION AND FUNDAMENTAL LEMMA

Let E be a real vector space, and let E^* consist of the real-linear real-valued functionals on E.

1.1 MAIN VERSION. Let $\theta : E \to \mathbb{R}$ be sublinear, and consider on the nonvoid subset $T \subset E$ the function $\tau : T \to \mathbb{R}$ with $\tau \leq \theta | T$. Assume that there is a pair of numbers $\alpha, \beta > 0$ such that

$$\operatorname*{Inf}_{w \in T} (\theta(w - \alpha u - \beta v) - \tau(w) + \alpha\tau(u) + \beta\tau(v)) \leq 0 \qquad \forall\, u,v \in T .$$

Then there exists $\varphi \in E^*$ such that $\varphi \leq \theta$ and $\tau \leq \varphi | T$.

The proof is after the usual scheme: One applies to an appropriate modified sublinear functional the primitive Hahn-Banach version that below each sublinear functional there exists a linear one. We define $Q : E \to \mathbb{R}$ to be

$$Q(x) = \operatorname*{Inf}_{\substack{u \in T \\ t>0}} (\theta(x + tu) - t\tau(u)) \qquad \forall\, x \in E ,$$

which is finite valued since the Inf is to be taken over a set of numbers $\geq -\theta(-x) > -\infty$. One verifies that

$$Q(x) \leq \theta(x) \quad \forall\, x \in E \qquad \text{and} \qquad \tau(x) \leq -Q(-x) \quad \forall\, x \in T .$$

From this one deduces for $\varphi \in E^*$ the equivalence: $\varphi \leq Q \iff \varphi \leq \theta$ and $\tau \leq \varphi|T$. Thus after the primitive Hahn-Banach theorem it remains to prove that the functional Q is sublinear. This will be done via the following Fundamental Lemma.

The Fundamental Lemma can be established on a fixed nonvoid cone $F \subset E$ (defined to be closed under addition and under multiplication with positive numbers) rather than on all of E, at the expense that we have to consider functions with values in $\mathbb{R} \cup \{-\infty\} =: \mathbb{R}^-$. In case that $0 \in F$, a sublinear $\theta : F \to \mathbb{R}^-$ is seen to fulfill $\theta(0) = 0$, except for the constant $\theta = -\infty$.

1.2 FUNDAMENTAL LEMMA. Let $P : F \to \mathbb{R}^-$ be such that for each $x \in F$ the function $t \mapsto P(tx)$ is upper semicontinuous (=: USC) on $]0,\infty[$. Assume that there is a pair of numbers $\alpha,\beta > 0$ such that

$$(*) \qquad P(\alpha x + \beta y) \leq \alpha P(x) + \beta P(y) \qquad \forall\, x,y \in F .$$

Define $Q : F \to \mathbb{R}^-$ to be

$$Q(x) = \operatorname*{Inf}_{t>0} \frac{1}{t} P(tx) \qquad \forall\, x \in F .$$

Then Q is sublinear. Furthermore, in case $\alpha + \beta \neq 1$, we have

$$Q(x) = \lim_{n \to \infty} \frac{P((\alpha + \beta)^n x)}{(\alpha + \beta)^n} \qquad \forall\, x \in F ,$$

where the limit exists since $P((\alpha + \beta)x) \leq (\alpha + \beta)\, P(x) \quad \forall\, x \in F$ in view of $(*)$.

The Fundamental Lemma will be proved in Section 2. To deduce the Main Version from the Fundamental Lemma, we assume the situation of 1.1. Define

$P : E \to \mathbb{R}$ to be

$$P(x) = \operatorname*{Inf}_{u \in T} (\theta(x + u) - \tau(u)) \qquad \forall\, x \in E ,$$

which is seen to be finite valued, as above. Then the functional $Q : E \to \mathbb{R}$ defined earlier turns out to be

$$Q(x) = \operatorname*{Inf}_{t > 0} \frac{1}{t} P(tx) \qquad \forall\, x \in E ,$$

so that P and Q are connected as in 1.2. Thus it remains to show that the functional P satisfies the assumptions of 1.2 with $F = E$.

(i) For $x,u \in E$, we have

$$|\theta(tx + u) - \theta(sx + u)| \leq \Theta((t - s)x) = |t - s|\, \Theta(x) \qquad \forall\, s,t \in \mathbb{R} ,$$

where

$$\Theta : \Theta(x) = \mathrm{Max}(\theta(x),\theta(-x)) \qquad \forall\, x \in E$$

is the associated seminorm. It follows that for $x,u \in E$ the function $t \mapsto \theta(tx + u) - \tau(u)$ is continuous on $\mathbb{R}$, and hence that for $x \in E$ the function $t \mapsto P(tx)$, as the Inf of a family of continuous functions, is USC on $\mathbb{R}$.

(ii) For $x,y \in E$ and $u,v \in T$, we have

$$\begin{aligned} P(\alpha x + \beta y) &= \operatorname*{Inf}_{w \in T} (\theta(\alpha x + \beta y + w) - \tau(w)) \\ &= \operatorname*{Inf}_{w \in T} (\theta(\alpha(x + u) + \beta(y + v) + (w - \alpha u - \beta v)) - \tau(w)) \\ &\leq \alpha\theta(x + u) + \beta\theta(y + v) + \operatorname*{Inf}_{w \in T} (\theta(w - \alpha u - \beta v) - \tau(w)) \\ &\leq \alpha(\theta(x + u) - \tau(u)) + \beta(\theta(y + v) - \tau(v)) , \end{aligned}$$

so that we obtain

$$P(\alpha x + \beta y) \leq \alpha P(x) + \beta P(y) \qquad \forall\, x,y \in E .$$

This completes the proof of 1.2 ⇒ 1.1. □

2. PROOF OF THE FUNDAMENTAL LEMMA

(i) We define $H : F \to \mathbb{R}^-$ to be

$$H(x) = \operatorname*{Inf}_{n \in \mathbb{Z}} \frac{P((\alpha + \beta)^n x)}{(\alpha + \beta)^n} = \lim_{n \to \infty} \frac{P((\alpha + \beta)^n x)}{(\alpha + \beta)^n} \qquad \forall\, x \in F .$$

It follows that

$$H(\alpha x + \beta y) \leq \alpha H(x) + \beta H(y) \qquad \forall\, x,y \in F ,$$
$$H((\alpha + \beta)x) = (\alpha + \beta)\, H(x) \qquad \forall\, x \in F ,$$

and that for each $x \in F$ the function $t \mapsto H(tx)$ is USC on $]0,\infty[$.

(ii) We claim that H is convex on F. Let M consist of the numbers $t \in [0,1]$ such that

$$H((1 - t)x + ty) \leq (1 - t)\, H(x) + tH(y) \qquad \forall\, x,y \in F$$

(with the usual convention $0(-\infty) := 0$). Then

(1) $0,1 \in M$,

(2) $s,t \in M \Rightarrow \lambda := \dfrac{\alpha s + \beta t}{\alpha + \beta} \in M$.

In fact, $\forall\, x,y \in F$ we have

$$\begin{aligned} H((1 - \lambda)x + \lambda y) &= H\left(\frac{\alpha(1 - s) + \beta(1 - t)}{\alpha + \beta}\, x + \frac{\alpha s + \beta t}{\alpha + \beta}\, y\right) \\ &= \frac{1}{\alpha + \beta}\, H(\alpha((1 - s)x + sy) + \beta((1 - t)x + ty)) \\ &\leq \frac{\alpha}{\alpha + \beta}\, H((1 - s)x + sy) + \frac{\beta}{\alpha + \beta}\, H((1 - t)x + ty) \\ &\leq \frac{\alpha}{\alpha + \beta}\, ((1 - s)H(x) + sH(y)) + \frac{\beta}{\alpha + \beta}\, ((1 - t)H(x) + tH(y)) \\ &= (1 - \lambda)\, H(x) + \lambda H(y) . \end{aligned}$$

From (1) and (2) it follows that

(3) $\overline{M} = [0,1]$,

(4) M is closed ,

and hence $M = [0,1]$, which is the assertion. In fact, $[0,1] \setminus M$ consists of the $t \in\,]0,1[$ such that

$$\exists\, x,y \in F \quad \text{with} \quad H((1 - t)x + ty)) > (1 - t)H(x) + tH(y) ,$$

or

$$\exists\ x,y \in F \quad \text{with} \quad H(x + y) > (1 - t)\ H\left(\frac{x}{1 - t}\right) + tH\left(\frac{y}{t}\right),$$

and hence is open after the USC behavior of H as described in (i).

(iii) For $x \in F$, we have

$$\operatorname*{Inf}_{t>0} \frac{1}{t} H(tx) = \operatorname*{Inf}_{t>0} \operatorname*{Inf}_{n \in \mathbb{Z}} \frac{P((\alpha + \beta)^n\ tx)}{(\alpha + \beta)^n\ t} = \operatorname*{Inf}_{t>0} \frac{1}{t} P(tx) = Q(x)\ .$$

(iv) We claim that Q is subadditive and hence sublinear on F. In fact, for $x,y \in F$ we have $\forall\ s,t > 0$, after (ii), (iii),

$$Q(x + y) \leq (s + t)\ H\left(\frac{1}{s + t}\ (x + y)\right) = (s + t)\ H\left(\frac{s}{s + t}\left(\frac{x}{s}\right) + \frac{t}{x + t}\left(\frac{y}{t}\right)\right)$$

$$\leq sH\left(\frac{x}{s}\right) + tH\left(\frac{y}{t}\right) \quad \text{and hence} \quad \leq Q(x) + Q(y)\ .$$

(v) Assume now that $\alpha + \beta \neq 1$. We claim that $H(tx) = tH(x)$ $\forall\ x \in F$ and $t > 0$, so that (iii) implies that $H = Q$, which remains to be proved. In fact, for $p \in \mathbb{Z}$ with $0 < t < (\alpha + \beta)^p$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} H([((\alpha+\beta)^p - t)(\alpha+\beta)^n + t]x) &= (\alpha+\beta)^p\ H\left(\left(1 - \frac{t}{(\alpha+\beta)^p}\right)(\alpha+\beta)^n x + \frac{t}{(\alpha+\beta)^p}\ x\right) \\ &\leq (\alpha+\beta)^p\left(\left(1 - \frac{t}{(\alpha+\beta)^p}\right)(\alpha+\beta)^n H(x) + \frac{t}{(\alpha+\beta)^p}\ H(x)\right) \\ &= ((\alpha+\beta)^p - t)(\alpha+\beta)^n\ H(x) + tH(x)\ . \end{aligned}$$

Now the function $s \mapsto H(sx)$ is convex on $]0,\infty[$ and therefore is either always $= -\infty$ or always finite valued and hence continuous. We can assume the latter case. Let $n \to \pm\infty$ such that $(\alpha + \beta)^n \to 0$. Then $H(tx) \leq tH(x)$. This holds true $\forall\ x \in F$ and $t > 0$, so that in fact we have $=$. The proof of 1.2 is complete. □

3. SPECIALIZATIONS OF THE MAIN VERSION

We start with a version of the familiar Hahn-Banach extension theorem. Here we have to take $\alpha = \beta = 1$. We want to emphasize, however, that the extension version is much less powerful and flexible than the subsequent ones.

3.1 EXTENSION VERSION. *Let* $\theta : E \to \mathbb{R}$ *be sublinear, and on an additive subgroup* $T \subset E$ *let* $\tau : T \to \mathbb{R}$ *be additive with* $\tau \leq \theta|T$. *Then there exists* $\varphi \in E^*$ *such that* $\varphi \leq \theta$ *and* $\tau = \varphi|T$.

Let us turn to more efficient specializations. First we mention the version τ = const. Here it is natural to restrict $\alpha,\beta > 0$ to $\alpha + \beta = 1$. For $\alpha = \beta = 1/2$ this has been the basic theorem in [2], [3]. Next we quote the version $\tau = 0$. It requires no restriction on $\alpha,\beta > 0$. For $\alpha = \beta = 1$, this has already been obtained in [3]. The version $\tau = 0$ will be the source for all that follows.

3.2 HOMOGENEOUS VERSION. _Let_ $\theta : E \to \mathbb{R}$ _be sublinear. Assume that the nonvoid subset_ $T \subset E$ _is such that there is a pair of numbers_ $\alpha,\beta > 0$ _with_

$$\operatorname*{Inf}_{w \in T} \theta(w - \alpha u - \beta v) \leq 0 \qquad \forall\, u,v \in T .$$

_If_ $\theta|T \geq 0$, _then there exists_ $\varphi \in E^*$ _such that_ $\varphi \leq \theta$ _and_ $\varphi|T \geq 0$.

An important special case is $E = C(X,\mathbb{R})$, with X a compact Hausdorff space $\neq \emptyset$, and

$$\theta = \text{Max}: \theta(f) = \text{Max}\, f \qquad \forall\, f \in C(X,\mathbb{R}) .$$

As in [2], [3], we extend the result to the cone $USC(X)$ of the USC functions $X \to \mathbb{R}^-$.

3.3 USC VERSION. _Let the nonvoid subset_ $T \subset USC(X)$ _and the numbers_ $\alpha,\beta > 0$ _be such that for all_ $f,g \in T$ _and_ $\varepsilon > 0$ _there exists_ $h \in T$ _with_ $h \leq \alpha f + \beta g + \varepsilon$. _If_ $\text{Max}\, f \geq 0 \;\; \forall\, f \in T$, _then there exists_ $\varphi \in \text{Prob}(X)$ _such that_ $\varphi(f) \geq 0 \;\; \forall\, f \in T$.

We turn to a close relative. On a nonvoid set X, consider $E = B(X,\mathbb{R})$, the space of bounded functions $X \to \mathbb{R}$, and

$$\theta = \text{Sup}: \theta(f) = \text{Sup}\, f \qquad \forall\, f \in B(X,\mathbb{R}) .$$

Define $AProb(X)$ to consist of the $\varphi \in B(X,\mathbb{R})^*$ with $\varphi \leq \text{Sup}$. The functionals $\varphi \in AProb(X)$ have various simple characterizations; see, for example, [1] Appendix 1. As before, we extend the result to the cone $USB(X)$ of the upper semibounded functions $X \to \mathbb{R}^-$. For $\varphi \in AProb(X)$, it is natural to define

$$\varphi(f) := \text{Inf}\{\varphi(F) : f \leq F \in B(X,\mathbb{R})\} \qquad \forall\, f \in USB(X) .$$

One verifies that in particular the extended functional $\varphi : \mathrm{USB}(X) \to \mathbb{R}^-$ remains additive, a fact which here is much more obvious than in the measure-theoretic USC situation.

3.4 USB VERSION. Let the nonvoid subset $T \subset \mathrm{USB}(X)$ and the numbers $\alpha,\beta > 0$ be such that for all $f,g \in T$ and $\varepsilon > 0$ there exists $h \in T$ with $h \leq \alpha f + \beta g + \varepsilon$. If $\mathrm{Sup}\, f \geq 0 \quad \forall\, f \in T$, then there exists $\varphi \in \mathrm{AProb}(X)$ such that $\varphi(f) \geq 0 \quad \forall\, f \in T$.

3.5 CONSEQUENCE. Let X be a compact Hausdorff space $\neq \emptyset$. For each $\varphi \in \mathrm{Prob}(X)$, there exists $\Phi \in \mathrm{AProb}(X)$ such that not only $\Phi|C(X,\mathbb{R}) = \varphi$ but also $\Phi(f) = \varphi(f) \quad \forall\, f \in \mathrm{USC}(X)$.

In fact, this results from 3.4 applied to

$$T := \{f \in B(X,\mathbb{R}) : f \geq \text{some } F \in \mathrm{USC}(X) \text{ with } \varphi(F) \geq 0\} .$$

Note that for $\varphi \in \mathrm{Prob}(X)$ and $\Phi \in \mathrm{AProb}(X)$ with $\Phi|C(X,\mathbb{R}) = \varphi$, one always has

$$\Phi(f) \leq \varphi(f) \qquad \forall\, f \in \mathrm{USC}(X) ,$$

but $<$ is possible. A simple example can be formed with the Dirac functional $\varphi = \delta_a \in \mathrm{Prob}(X)$ and $f = \chi_a \in \mathrm{USC}(X)$, where $a \in X$ is not an isolated point of X.

An important common specialization of 3.3 and 3.4 is the case that X is finite. Let us restrict our attention to finite-valued functions.

3.6 FINITE VERSION. Let the nonvoid subset $T \subset \mathbb{R}^r$ and the numbers $\alpha,\beta > 0$ be such that for all $u,v \in T$ and $\varepsilon > 0$ there exists $x \in T$ with $x \leq \alpha u + \beta v + \varepsilon$. If

$$\mathrm{Max}(x_1,\dots,x_r) \geq 0 \qquad \forall\, x \in T ,$$

then there exist $\sigma_1,\dots,\sigma_r \geq 0$ with $\sigma_1 + \dots + \sigma_r = 1$ such that

$$\sigma_1 x_1 + \dots + \sigma_r x_r \geq 0 \qquad \forall\, x \in T .$$

The above Hahn-Banach versions are powerful work horses. They often allow us to cut down lengthy proofs to a few lines and, what is more important, can lead to more adequate forms of results. Decisive for their easy application is the weak form of the assumption "There is a pair of numbers $\alpha,\beta > 0$ such that ..." instead of, for example, "For all pairs $\alpha,\beta > 0$ with $\alpha + \beta = 1$ ... ." There are numerous examples in [1], [2], [3], [4].

4. THE BARYCENTER LEMMA

It can be expected that the minimax theorem as obtained in [2],[6] admits extended versions which correspond to the above results. We start to extend the barycenter lemma [2],[6].

4.1 FINITE VERSION REFORMULATED. _Consider_ $f_1,\dots,f_r : X \to \mathbb{R}$ _on the nonvoid set_ X. _Assume that there is a pair of numbers_ $\alpha,\beta > 0$ _such that for all_ $x,y \in X$ _and_ $\varepsilon > 0$ _there exists_ $z \in X$ _with_

$$\alpha f_\ell(x) + \beta f_\ell(y) \leq f_\ell(z) + \varepsilon \qquad \forall\, \ell = 1,\dots,r .$$

_If_ $\mathrm{Min}(f_1,\dots,f_r) \leq 0$ _on_ X, _then there exists a convex combination_ $f \in \mathrm{Conv}(f_1,\dots,f_r)$ _such that_ $f \leq 0$ _on_ X.

This follows upon application of 3.3 or 3.4 or 3.6 to the set

$$T := \{\hat{x} := -(f_1(x),\dots,f_r(x)) \;:\; x \in X\} \subset \mathbb{R}^r = C(\{1,\dots,r\},\mathbb{R}) = B(\{1,\dots,r\},\mathbb{R}) .$$

We see that the functions $f_1,\dots,f_r$ could have been allowed to take values in $\mathbb{R} \cup \{\infty\}$ as well. In what follows, however, the opposite case of functions with values in $\mathbb{R} \cup \{-\infty\} = \mathbb{R}^-$ will be needed. This is a nontrivial extension, the first simple treatment of which appears to be due to Neumann [6]. For the sake of completeness, we include the explicit transfer of his idea.

4.2 EXTENDED FINITE VERSION. _Consider_ $f_1,\dots,f_r \in USB(X)$ _on the nonvoid set_ X. _Assume that there is a pair of numbers_ $\alpha,\beta > 0$ _such that for all_ $x,y \in X$ _and_ $\varepsilon > 0$ _there exists_ $z \in X$ _with_

$$\alpha f_\ell(x) + \beta f_\ell(y) \leq f_\ell(z) + \varepsilon \qquad \forall\, \ell = 1,\dots,r .$$

_If_ $\mathrm{Min}(f_1,\dots,f_r) \leq 0$ _on_ X, _then to each_ $\varepsilon > 0$ _there exists a convex combination_ $f \in \mathrm{Conv}(f_1,\dots,f_r)$ _(with the convention_ $0(-\infty) := 0$) _such that_ $f \leq \varepsilon$ _on_ X.

There are trivial examples which show that the conclusion cannot be maintained as in 4.1: Take $X = \{0,1\}$ and define $f_1, f_2 \in USB(X)$ to be $f_1(0) = 1$, $f_1(1) = -\infty$ and $f_2(0) = 0$, $f_2(1) = 1$.

Proof of 4.2. Let

$$D := \{x \in X : f_1(x), \dots, f_r(x) > -\infty\} .$$

If $D = \emptyset$, then

$$f := \frac{1}{r}(f_r + \cdots + f_r)$$

will do. If $D \neq \emptyset$, then after 4.1 applied to $f_1|D, \dots, f_r|D$ there are real $\sigma_1, \dots, \sigma_r \geq 0$ with $\sigma_1 + \cdots + \sigma_r = 1$ such that

$$\sigma_1 f_1 + \cdots \sigma_r f_r \leq 0$$

on D. Let now $M > 0$ with $f_1, \dots, f_r \leq M$ and put

$$\tau_\ell := \left(1 - \frac{\varepsilon}{M}\right)\sigma_\ell + \frac{\varepsilon}{rM} \qquad \forall\, \ell = 1, \dots, r .$$

Then it is obvious that $\tau_1 f_1 + \cdots + \tau_r f_r \leq \varepsilon$ on X. □

4.3 USB BARYCENTER LEMMA. Let the nonvoid subset $T \subset USB(X)$ on the set X and the numbers $\alpha, \beta > 0$ be such that for all $x, y \in X$ and $\varepsilon > 0$ there exists $z \in X$ with

$$\alpha f(x) + \beta f(y) \leq f(z) + \varepsilon \qquad \forall\, f \in T .$$

If for some $\varphi \in AProb(X)$ we have

$$\varphi(f) \geq 0 \qquad \forall\, f \in T ,$$

then

$$\operatorname{Sup} f \geq 0 \qquad \forall\, f \in Min(T) ,$$

where $Min(T)$ is defined to consist of the functions $f = Min(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in T$.

Proof. (i) The case $\alpha + \beta > 1$ requires separate treatment. First we show that in this case $f \leq 0$ $\forall\, f \in T$. In fact, assume that $F(a) > 0$ for some $F \in T$ and $a \in X$. Then after the assumption applied to

$$x = y = a \quad \text{and} \quad \varepsilon = \frac{1}{2}(\alpha + \beta - 1)F(a) > 0 \ ,$$

there exists $b \in X$ with

$$(\alpha + \beta)F(a) \leq F(b) + \frac{1}{2}(\alpha + \beta - 1)F(a) \ ,$$

or

$$F(b) \geq \frac{1}{2}(1 + \alpha + \beta)F(a) \ .$$

Via induction, we obtain $a_n \in X$ with

$$F(a_n) \geq \left(\frac{1}{2}(\alpha + \beta + 1)\right)^n F(a) \qquad \forall\, n \in \mathbb{N} \ .$$

It follows that $F(a_n) \to \infty$ for $n \to \infty$, which is impossible.

(ii) Let $\alpha + \beta > 1$ and fix $f = \mathrm{Min}(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in T$. We have to show that $\mathrm{Sup}\, f \geq 0$. Assume that $\mathrm{Sup}\, f < 0$ and hence $\mathrm{Sup}\, f < -\delta$ for some $\delta > 0$. Then we have a decomposition $X = X(1) \cup \dots \cup X(r)$ into pairwise disjoint $X(\ell) \subset X$ such that $f_\ell \leq -\delta$ on $X(\ell)$ $\forall\, \ell = 1, \dots, r$. In view of (i), it follows that

$$f_\ell + \delta\chi_{X(\ell)} \leq 0$$

and hence

$$\delta\varphi(\chi_{X(\ell)}) \leq \varphi(f_\ell) + \delta\varphi(\chi_{X(\ell)}) = \varphi(f_\ell + \delta\chi_{X(\ell)}) \leq 0 \ ,$$

so that $\varphi(\chi_{X(\ell)}) = 0$ $\forall\, \ell = 1, \dots, r$. Thus $\varphi(1) = 0$, which is a contradiction.

(iii) Let now $\alpha + \beta \leq 1$. Fix $f = \mathrm{Min}(f_1, \dots, f_r)$ with $f_1, \dots, f_r \in T$ and assume that $\mathrm{Sup}\, f < 0$ and hence

$$\mathrm{Sup}\, f \leq -\delta \ , \quad \text{or} \quad f + \delta = \mathrm{Min}(f_1 + \delta, \dots, f_r + \delta) \leq 0 \ ,$$

on X for some $\delta > 0$. In view of $\alpha + \beta \leq 1$, the version 4.2 can be applied to the functions $f_1 + \delta, \dots, f_r + \delta$. For $\varepsilon := \delta/2$ we thus obtain real numbers $\sigma_1, \dots, \sigma_r \geq 0$ with $\sigma_1 + \dots + \sigma_r = 1$ such that

$$\sum_{\ell=1}^{r} \sigma_\ell (f_\ell + \delta) \leq \frac{\delta}{2} \quad \text{or} \quad \sum_{\ell=1}^{r} \sigma_\ell f_\ell \leq -\frac{\delta}{2} \quad \text{on} \quad X \ .$$

It follows that

$$0 \leq \sum_{\ell=1}^{r} \sigma_\ell \varphi(f_\ell) = \sum_{\ell=1}^{r} \varphi(\sigma_\ell f_\ell) = \varphi\left(\sum_{\ell=1}^{r} \sigma_\ell f_\ell\right) \leq \mathrm{Sup}\left(\sum_{\ell=1}^{r} \sigma_\ell f_\ell\right) \leq -\frac{\delta}{2} \ .$$

We thus arrive at a contradiction, which proves the assertion. □

In order to obtain the USC barycenter lemma, we combine the above result with 3.5 and with the usual Dini theorem [2], which can be stated as follows.

4.4 DINI THEOREM. Let X be a compact Hausdorff space. For each nonvoid $T \subset USC(X)$ with $F := \mathrm{Inf}_{f \in T} f \in USC(X)$, we have $\mathrm{Inf}_{f \in \mathrm{Min}(T)} \mathrm{Max}\, f = \mathrm{Max}\, F$.

4.5 USC BARYCENTER LEMMA. Let the nonvoid subset $T \subset USC(X)$ on the compact Hausdorff space X and the numbers $\alpha, \beta > 0$ be such that for all $x, y \in X$ and $\varepsilon > 0$ there exists $z \in X$ with

$$\alpha f(x) + \beta f(y) \leq f(z) + \varepsilon \qquad \forall\, f \in T .$$

If for some $\varphi \in \mathrm{Prob}(X)$ we have

$$\varphi(f) \geq 0 \qquad \forall\, f \in T ,$$

then

$$F := \operatorname*{Inf}_{f \in T} f \in USC(X)$$

has $\mathrm{Max}\, F \geq 0$; that is, there exists $a \in X$ such that $f(a) \geq 0 \quad \forall\, f \in T$.

5. EXTENDED VERSIONS OF THE MINIMAX THEOREMS

Now as in [2] we combine 3.4 with 4.3, and 3.3 with 4.5, to obtain the following extended minimax theorems.

5.1 USB MINIMAX THEOREM. Assume that the nonvoid subset $T \subset USB(X)$ on the set X satisfies:

(i) There is a pair of numbers $\alpha, \beta > 0$ such that for all $f, g \in T$ and $\varepsilon > 0$ there exists $h \in T$ with $h \leq \alpha f + \beta g + \varepsilon$.

(ii) There is a pair of numbers $\sigma, \tau > 0$ such that for all $x, y \in X$ and $\varepsilon > 0$ there exists $z \in X$ with $\sigma f(x) + \tau f(y) \leq f(z) + \varepsilon \quad \forall\, f \in T$.

If $\mathrm{Sup}\, f \geq 0 \quad \forall\, f \in T$, then $\mathrm{Sup}\, f \geq 0 \quad \forall\, f \in \mathrm{Min}(T)$.

5.2 USC MINIMAX THEOREM. Assume that the nonvoid subset $T \subset USC(X)$ on the compact Hausdorff space X satisfies:

(i) There is a pair of numbers $\alpha, \beta > 0$ such that for all $f, g \in T$

and $\varepsilon > 0$ there exists $h \in T$ with $h \leq \alpha f + \beta g + \varepsilon$.

(ii) There is a pair of numbers $\sigma,\tau > 0$ such that for all $x,y \in X$ and $\varepsilon > 0$ there exists $z \in X$ with $\sigma f(x) + \tau f(y) \leq f z) + \varepsilon \quad \forall f \in T$.

If $\operatorname{Max} f \geq 0 \quad \forall f \in T$, then

$$F := \operatorname*{Inf}_{f \in T} f \in USC(X)$$

has $\operatorname{Max} F \geq 0$; that is, there exists $a \in X$ such that $f(a) \geq 0 \quad \forall f \in T$.

There are trivial examples which show that the conclusion in 5.1 cannot be the same as in 5.2: Let $X = [0,\infty[$ and $T \subset USB(X)$ consist of the monotone increasing functions $f : X \to \mathbb{R}$ with $f(x) \to 0$ for $x \to \infty$. The same remark applies to 4.3 and 4.5.

We conclude with the remark that both from 5.1 and from 5.2 we obtain a more familiar minimax theorem when we restrict assumptions (i) and (ii) to pairs of numbers $\alpha,\beta > 0$ with $\alpha + \beta = 1$ and $\sigma,\tau > 0$ with $\sigma + \tau = 1$: Then the assumptions and hence the conclusions carry over from T to $T - c := \{f - c : f \in T\}$ for fixed $c \in \mathbb{R}$, and thus the assertions can be formulated in the form of familiar equalities.

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