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PRODUCTION AND DISTRIBUTION

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We treat in this paper a supply-demand model with infinite commodity set. The main feature of the model is that the producer as well as the consumer do have alternatives concerning their production and consumption respectively. This generalizes a situation considered by Heinz König in his lectures on Mathematical Economy(for finite commodity sets). The principal tools needed in the investigation are disintegration - type arguments which were developed in the decomposition-theory of linear functionals.

THE PROBLEM

We consider a commodity set X and, for the moment, it is assumed that $X = \{1,\dots,n\}$ is finite. Furthermore we consider one producer and one consumer whose production and consumption are measured by the functions α and ν respectively. We look for conditions which guarantee that production can satisfy consumption. An immediate guess for such a condition is that α has to exceed ν. But in real life α and ν are rather functions on $P'(X) = \{Y \subset X \mid Y \neq \phi\}$ instead of functions on X, since the consumer allows alternatives and the producer usually has at his disposal capacities which he uses according to the market situation. So, for solving our problem, we have to establish a production plan p and a distribution plan v such that

(1) p observes the limits given by the production capacities α (p is then said to be *possible*),

(2) v is *satisfactory*, i.e. v satisfies the demand ν.

(3) we do not distribute more than we produce (p and v are then called *compatible*).

To make this more precise we assume that the producer consists of subunits $U(Y)$, $Y \in P'(X)$, where $U(Y)$ is the collection of all factories where the commodities $i \in Y$ can be produced equivalently but where production cannot switch to commodities outside of Y. Then the production capacities $\alpha : P'(X) \to \mathbb{R}_+$ are given by the numbers $\alpha(Y)$, measuring the maximal output of the subunit $U(Y)$. On the consumers side the situation is quite similar. His demand is given by a function $\nu : P'(X) \to \mathbb{R}_+$, where $\nu(Y)$ measures that fraction of his total demand which can be satisfied by assignment of an arbitrary produced by the subunit $U(Y)$. Similarily, $v(i,Y)$ measures the amount of being assigned to the demand $\nu(Y)$.

The requirements (1) - (3) immediatly lead to the following definitions:

The production plan p is said to be *possible* if for all $Y \in P'(X)$ we do have

$$(4)^* \quad \sum_{i \in Y} p(i,Y) \le \alpha(Y).$$

The distribution plan v is defined to be *satisfactory* if

$$(5)^* \quad \nu(Y) \le \sum_{i \in Y} v(i,Y) \quad \text{for all } Y \in P'(X) .$$

And finally, the plans p and v are said to be *compatible* if for every commodity $i \in X$ production exceeds distribution, i.e.

$$(6)^* \quad \sum_{\{Y | i \in Y\}} v(i,Y) \le \sum_{\{Y | i \in Y\}} p(i,Y) .$$

If we put formally $p(i,Y) = v(i,Y) = 0$ whenever $i \notin Y$ then we can rewr (4) - (6) in the following form:

$$(4) \quad \sum_{i \in X} p(i,Y) \le \alpha(Y)$$

$$(5) \quad \nu(Y) \le \sum_{i \in X} v(i,Y)$$

$$(6) \quad \sum_{Y \in P'(X)} v(i,Y) \le \sum_{Y \in P'(X)} p(i,Y) .$$

Now, our (preliminary) problem is:

Are there plans, which are possible, satisfactory and compatible ?

A simple necessary condition for the existence of these plans is easily fo For this purpose we consider an arbitrary subset $Y \subset X$. Then obviously th amount of that part of the demand which can only be satisfied by assignmen goods out of Y has to be less than or equal to the amount the producer c produce when he concentrates all his efforts on the production of goods be to Y, that is, whenever he has the alternative to produce something in in $X \setminus Y$ then he chooses the production of the good in Y. In formulas th condition reads as follows:

$$(7) \quad \sum_{\{Z \in P'(X) | Z \subset Y\}} \nu(Z) \le \sum_{\{Z \in P'(X) | Z \cap Y \neq \phi\}} \alpha(Z) \qquad \text{for all } Y \in P'(X) .$$

At this stage we have to admit that we are not interested in the problem w stated so far, but rather in its generalization to infinite commodity sets In the first moment this sounds like an intended contribution to "general nonsense", which is not so. For example, already in the simple case when o interested in the dynamical behaviour of supply-demand models one has to t into account that the assignment of a commodity today is different from it

($\mathbb{R}$ being the time scale), which is clearly infinite. For this reason, in the next chapter our problem will be reformulated (and solved) in a measure-theoretic setup.

1 Remarks : (i) Our problem originates from a problem considered by Heinz König in his beautiful lectures on "Mathematische Wirtschaftstheorie" [6]. There he gives a complete description of the case when X is a finite set and when the production plan is already known, or equivalently, when the producer has no alternatives (i.e. α is a function $X \to \mathbb{R}_+$). König's solution of this problem already incorporates the classical "Heiratssatz".

(ii) For finite X our problem can be completely solved by a (somewhat sophisticated) application of network methods. The best way to proceed is to apply Gales theorem ([5] or [1,p.38]).

(iii) In this paper we do neither deal with algorithms nor with integer-valued plans. But all this can be done in the framework of methods which is developed in the next chapter. Furthermore these methods allow the treatment of more sophisticated models. For example when further restrictions (raw-material constraints) are imposed on the production.

(iv) Finally we should mention that the case of several producers and consumers can be investigated by summing up their production-capacity functions α and their demand functions ν, and that further applications (time-tables, labor-market models, etc.) are possible.

THE PRODUCTION - DISTRIBUTION THEOREM

As commodity set we consider now a (possibly infinite) metric space X. By $\mathcal{K}$ we denote the family of its nonempty compact subsets. We endow $\mathcal{K}$ with the Hausdorff-metric [9] :

$$h(K_1,K_2) = \max(\sup_{x \in K_1} d(x,K_2), \sup_{x \in K_2} d(x,K_1)), \; K_1,K_2 \in \mathcal{K} ,$$

where $d(\cdot,\cdot)$ stands for the metric of X.

A $[0,+\infty]$- valued countably additive measure on a topological space will be called *tight* if it is a measure with respect to the σ-algebra generated by the compact sets and if it is finite on every compact set and if it is inner-regular with respect to the family of compact sets. By Σ_X and $\Sigma_{\mathcal{K}}$ we denote the σ-algebras generated by the compact subsets of X and $\mathcal{K}$ respectively.

Now, assume that there are given two tight measures α and ν on $\mathcal{K}$. We call these measures *production-capacity* and *demand-measure*. Throughout this paper we shall assume that there is no *superabundance*. That means that there is no compact set of commodities which can be produced in an infinite amount. In formulas this reads as follows:

$$(8) \quad \alpha\{Z \in \mathcal{K} \mid Z \cap K \neq \emptyset\} < \infty \quad \text{for all} \quad K \in \mathcal{K}.$$

The measures α and ν are said to fulfill the *balance condition* if:

$$(9) \quad \nu\{Z \in \mathcal{K} \mid Z \subset K\} \leq \alpha\{Z \in \mathcal{K} \mid Z \cap K \neq \emptyset\} \quad \text{for all} \quad K \in \mathcal{K} .$$

Our aim is to find a reasonable production plan and a suitable distribution plan. By this we mean tight measures p, v on $X \times \mathcal{K}$ such that:

(10) p and v are *plans*, i.e. $p(A \times B) = v(A \times B) = 0$ whenever $A \in \Sigma_X$ and $B \in \Sigma_K$ are such that

$A \cap K = \phi$ for all $K \in B$,

(11) p is *possible*, i.e. $\alpha(B) \geq p(X \times B)$ for all $B \in \Sigma_K$,

(12) v is *satisfactory*, i.e. $\nu(B) \leq v(X \times B)$ for all $B \in \Sigma_K$,

(13) p and v are *compatible*, i.e.

$$v(A \times K) \leq p(A \times K) \quad \text{for all} \quad A \in \Sigma_X .$$

These are the suitable generalizations of the properties given by (4) to (
Of course, because of the regularity condition it suffices to check (10) t
for compact A and B.

Main Theorem: *There are plans, which are possible, satisfactory and compa if and only if the balance condition does hold.*

THE PRINCIPAL TOOLS

In this chapter we gather the principal tools which will be used in the an of our supply-demand model.

I. $F = (F,\leq)$ denotes a preordered convex cone, i.e. $\leq$ is reflexive and transitive on F such that

$$f_i \leq g_i,\ 0 \leq \lambda_i \in \mathbb{R}\ (i = 1,2) \Rightarrow \lambda_1 f_1 + \lambda_2 f_2 \leq \lambda_1 g_1 + \lambda_2 g_2 .$$

Functionals are maps $\mu : F \to \mathbb{R} = \bar{\mathbb{R}} \cup \{-\infty\}$, where $0 \cdot (-\infty)$, is defined and where the other operations are extended to $\mathbb{R}$ in the obvious way. set of functionals we consider the pointwise order on F. Since no confusi arise this order relation is also denoted by $\leq$. Linear (sublinear, super means positive homogeneous (i.e. $\mu(\lambda f) = \lambda \mu(f) \quad \forall \lambda \geq 0,\ f \in F$) and additive (subadditive, superadditive). A functional μ is called *order-p ving* if $f \geq g \Rightarrow \mu(f) \geq \mu(g)$.

Sandwich Theorem ([2]): *Let π be a sublinear and order-preserving funct and let $\delta \leq \pi$ be superlinear. Then there is a linear order-preserving μ* $\delta \leq \mu \leq \pi$.

II. If T is a regular topological space then we denote by $UC_0^+(T)$ the cone of $\mathbb{R}_+$ - valued uppersemicontinuous functions with compact support. W $UC_0^+(T)$ with the T- pointwise order.

2. Lemma: If $\mu : UC_0^+(T) \to \mathbb{R}_+$ is linear and order-preserving then the a tight measure m on T such that

$$\mu(f) = \int_T f\, dm \quad \text{for all} \quad f \in UC_0^+(T) .$$

For the proof we restrict μ to the subcone $UC_0^+(K)$, where K is some c subset of T. Then we extend this restriction linearily and order-preservi to the cone $\Phi = \cup C_0^+(K) + C(K)$. This is a subcone of $\cup C_0^+(K) - \cup C_0^+(K)$, hence the extension must be unique. Then by the Riesz representation theorem we find a measure m_K on K representing our linear functional on Φ, hence on $\cup C_0^+(K)$. Taking the limit of all those m_K we obtain our measure m. □

III. We consider two regular spaces T_1, T_2 and the σ-algebras Σ_1, Σ_2 generated by the compact subsets of T_1 and T_2. Let $(A,B) \to \tau(A,B)$ be a $\mathbb{R}_+ \cup \{+\infty\}$-valued map on $\Sigma_1 \times \Sigma_2$ such that $\tau(A,B) = \sup\{\tau(K_1 \cap A, K_2 \cap B) \mid K_i \text{ compact} \subset T_i\}$ $\forall (A,B) \in \Sigma_1 \times \Sigma_2$ and having in addition the property that if one variable is put equal to a fixed subset of a compact set then the map is a tight measure in the other variable.

<u>3 Lemma</u>: There is a unique tight measure m on $T_1 \times T_2$ such that $m(A \times B) = \tau(A,B)$ $\forall (A,B) \in \Sigma_1 \times \Sigma_2$.

For the proof one considers the ring P of all finite disjoint unions of measurable rectangles [6,p 149]. Then one defines for finite disjoint unions $m(\cup A_i \times B_i) = \Sigma\, \tau(A_i, B_i)$ and gets an unambiguously defined finitely additive map on P since τ is separately additive. Then for an arbitrary rectangle $K = K_1 \times K_2$ the restriction of m to $P_K = \{S \in P \mid S \subset K\}$ extends uniquely to a Borel measure m_K on K (Riesz representation theorem). Again taking the limi[t] one finds the desired m. □

IV. The following pre-disintegration theorem (a special case can be found in [10]) is an interesting application of the sandwich theorem.

<u>Situation:</u> Let (Ω,Σ,m) be a measure space and let $\omega \to \pi_\omega$ be a map from Ω into the set of $\mathbb{R}_+$- valued order-preserving sublinear functionals on the convex cone $F = (F, \leq)$ such that $\omega \to \pi_\omega(f)$ is in $L^1(m)$ for all $f \in F$. Furthermore we consider a superlinear $\mathbb{R}_+$- valued functional δ on F with

$$\delta(f) \leq \int_\Omega \pi_\omega(f)\, dm(\omega) \quad \text{for all } f \in F.$$

<u>4 Theorem</u>: *There is a map* $\tau : \Sigma \times F \to \mathbb{R}_+$ *having the following properties:*

(14) $f \to \tau(A,f)$ *is for every* $A \in \Sigma$ *linear and order-preserving on* F,

(15) $A \to \tau(A,f)$ *is for every* $f \in F$ *a positive measure on* (Ω,Σ),

(16) $\tau(A,f) \leq \int_A \pi_\omega(f)\, dm(\omega)$ *for all* $A \in \Sigma$, $f \in F$.

(17) $\delta(f) \leq \tau(\Omega,f)$ *for all* $f \in F$.

<u>Proof</u> (compare [3, proof of the sum theorem]): We consider the cone Φ of all <u>simple</u> and measurable functions $\varphi : \Omega \to F$, where simple means that $\varphi(\Omega)$ is a finite subset of F and where measurable means that $\varphi^{-1}(\{f\}) \in \Sigma$ $\forall f \in F$. We endow Φ with the preorder

Then we define an order-preserving sublinear π and a superlinear $q \le \pi$ b

$$\pi(\varphi) = \int_\Omega \pi_\omega(\varphi(\omega))\, d\,m(\omega)$$

$$q(\varphi) = \begin{cases} \delta(f) & \text{if } \varphi \text{ is constant and equal to } f \\ -\infty & \text{otherwise.} \end{cases}$$

According to the sandwich theorem and by virtue of Zorn's lemma there is a m order-preserving and linear $\mu : \Phi \to \overline{\mathbb{R}}$ with $q \le \mu \le \pi$. Now, the desired given by

$$\tau(A,f) = \mu(1_A \cdot f) \;,$$

where $1_A(\omega) = \{1$ if $\omega \in A$ and 0 otherwise$\}$ is the characteristic fun of A. All the properties are easily checked, the only difficulty arises i proving that $A \to \tau(A,f)$ is countably additive. For this purpose it is suf to show that

$$(18) \quad \tau\Big(\bigcup_{n\in\mathbb{N}} A_n, f\Big) = \liminf_{m\to\infty} \sum_{n=1}^{m} \tau(A_n,f)$$

for every sequence $A_n \in \Sigma$ with $A_n \cap A_m = \phi$ if $n \neq m$. Given such a seq we consider the superlinear $\rho \le \pi$ defined by

$$\rho(\varphi) = \mu(1_Y \varphi) + \liminf_{m\to\infty} \sum_{n=1}^{m} \mu(1_{A_m} \varphi) \;,$$

where $Y = \Omega \setminus \bigcup_{n\in\mathbb{N}} A_n$. We claim that $\rho \ge \mu$. Then this proves (18) becaus sandwich theorem gives us an order-preserving linear $\bar{\mu}$ with $\rho \le \bar{\mu} \le \pi$. Since μ is maximal $\bar{\mu}$ must be equal to μ, hence $\rho = \mu$ and (18) is p Now, all what remains is the proof of the claim. If $Z_m = \bigcup_{n\ge m} A_n$ then we

$$\rho(\varphi) + \limsup_{m\to\infty} \mu(1_{Z_m} \cdot \varphi) \ge \mu(\varphi) \;.$$

And we obtain $\rho \ge \mu$ from

$$\limsup_{m\to\infty} \mu\,(1_{Z_m} \cdot \varphi) \le \limsup_{m\to\infty} \pi\,(1_{Z_m} \varphi) =$$

$$= \limsup_{m\to\infty} \int_{Z_m} \pi_\omega(\varphi(\omega))\, d\,m(\omega) \;,$$

where the last term ≤ 0 since $Z_{m+1} \subset Z_m$ and $\bigcap_{m\in\mathbb{N}} Z_m = \phi$.

V. Combination of the results from II to IV leads to a powerful tool.

<u>Situation</u>: Assume that we are given:
Two regular topological spaces T and Ω,

- and a map $\omega \to \pi_\omega$ from Ω into the set of $\mathbb{R}_+$- valued order-preserving sublinear functionals on $UC_0^+(T)$ such that $\omega \to \pi_\omega(f)$ is always in $L^1(m)$ and fulfills the following inequality:

$$(19) \quad \delta(f) \le \int_\Omega \pi_\omega(f)\, d\, m(\omega) \quad \text{for all} \quad f \in UC_0^+(T) .$$

<u>5 Theorem</u>: *There is a tight measure* s *on* $\Omega \times T$ *such that*

$$(20) \quad s(K_1 \times K_2) \le \int_{K_1} \pi_\omega (1_{K_2})\, d\, m(\omega)$$

$$(21) \quad \delta(1_{K_2}) \le s(\Omega \times K_2)$$

for all compact $K_1 \subset \Omega$, $K_2 \subset T$.

<u>BACK TO THE MAIN THEOREM</u>

Now, let us return to our commodity set X and the tight measures α and ν given on the metric space K consisting of the nonempty compact subsets of X.

For $K \in K$ we consider on the convex cone $UC_0^+(X)$ the sublinear functional

$$\pi_K(f) = \sup\{f(x) \mid x \in K\}$$

and the superlinear functional

$$\delta_K(f) = \inf\{f(x) \mid x \in K\} .$$

The functions $K \to \pi_K(f)$ and $K \to \delta_K(f)$ are Σ_K- measurable since for all $\varepsilon \in$
$\{K \mid \pi_K(f) < \varepsilon\} = \{K \in K \mid K \subset f^{-1}([0,\varepsilon[)\}$ is open and
$\{K \mid \delta_K(f) \ge \varepsilon\} = \{K \in K \mid K \subset f^{-1}([\varepsilon,\infty[)\}$
is either compact $(\varepsilon>0)$ or equal to K $(\varepsilon \le 0)$.
Integration with respect to α and ν yields the sublinear

$$\pi(f) = \int_{K \in K} \pi_K(f)\, d\,\alpha(K)$$

and the superlinear

$$\delta(f) = \int_{K \in K} \delta_K(f)\, d\,\nu(K) .$$

<u>6 Lemma:</u> The following are equivalent:

(i) $\delta \le \pi$

(ii) α and ν fulfill the balance condition.

<u>Proof:</u> (i) $\Rightarrow$ (ii): For $K \in K$ take f to be 1_K (the characteristic function of K). Then

$\delta(f) = \nu\{Y \in K \mid Y \subset K\}$ and $\pi(f) = \alpha\{Y \in K \mid Y \cap K \neq \phi\}$.

assume $f = \sum_{i=1}^{n} \lambda_i 1_{K_i}$, $K_i \in \mathcal{K}$, $\lambda_i \geq 0$. Without loss of generality we may a that

$$(22)\quad K_1 \subset K_2 \subset K_3 \ldots \subset K_n$$

This assumption implies

$$\pi_K(f) = \sum_{i=1}^{n} \lambda_i 1_{T_i}(K)$$

$$\delta_K(f) = \sum_{i=1}^{n} \lambda_i 1_{S_i}(K) ,$$

where 1_{T_i} and 1_{S_i} are the characteristic functions of the sets

$T_i = \{Y \in \mathcal{K} \mid Y \cap K_i \neq \phi\}$ and $S_i = \{Y \in \mathcal{K} \mid Y \subset K_i\}$. Now, we obtain from (i desired inequality:

$$\pi(f) = \sum_{i=1}^{n} \lambda_i \alpha(T_i) \geq \sum_{i=1}^{n} \lambda_i \nu(S_i) = \delta(f) . \qquad \square$$

For the final proof of our main theorem we need another lemma of a similar fl Let us define for $x \in X$ via

$$\rho_x(\varphi) = \sup\{\varphi(K) \mid x \in K\}$$

a sublinear functional on $UC_0^+(\mathcal{K})$. For fixed φ the function $x \to \rho_x(\varphi)$ is certainly Σ_X- measurable, since the set

$$(23)\quad \{x \in X \mid \rho_x(\varphi) \geq \varepsilon\} = \bigcup \{K \mid \varphi(K) \geq \varepsilon\}$$

is compact whenever $\varepsilon > 0$. Now, let m be a tight measure on X.

<u>7 Lemma</u>: The following are equivalent:

(i) $\nu\{Y \in \mathcal{K} \mid Y \subset K\} \leq m(K) \quad \forall\, K \in \mathcal{K}$

(ii) $\int_{\mathcal{K}} \varphi(K)\, d\nu(K) \leq \int_X \rho_x(\varphi) dm(x) \quad \forall\, \varphi \in UC_0^+(\mathcal{K})$.

<u>Proof</u> (compare [8,p. 26]):

(ii) ⇒ (i): For arbitrary K we consider φ to be the characteristic funct of the compact set $\mathcal{K}(K) = \{Y \in \mathcal{K} \mid Y \subset K\}$. Then according to (23) $x \to \rho_x(\varphi)$ is equal to the characteristic function of K. So, inequality (i) i special case of (ii).

(i) ⇒ (ii): Again it suffices to prove (ii) for step functions. Without loss

$$\varphi = \sum_{i=1}^{n} \lambda_i 1_{K_i} ,$$

where K_i are compact subsets of K with $K_1 \subset K_2 \subset K_3 \ldots \subset K_n$. Then the function $x \to \rho_x(\varphi)$ is equal to $\sum_{i=1}^{n} \lambda_i 1_{K_i}$, where $K_i = \bigcup\{Y | Y \in K_i\}$. From this and (i) we obtain

$$\int_X \rho_x(\varphi) d\, m(x) \geq \sum_{i=1}^{n} \lambda_i m(K_i) \geq \sum_{i=1}^{n} \lambda_i \nu\{Y \in K \mid Y \subset K_i\} \geq \sum_{i=1}^{n} \lambda_i \nu(K_i)$$

$$\geq \int_K \varphi(K)\, d\nu(K). \qquad \square$$

Proof of the Main Theorem: The fact that the existence of a production-distribution plan implies the balance condition is very easy and therefore left to the reader as an exercise.

Now, let us assume that the balance condition holds for α and ν.
Then we have $\delta \leq \pi$ (lemma 6). And theorem 5 gives us a tight measure p on $X \times K$ such that we have for compact $K_1 \subset X$ and compact $\tilde{K} \subset K$

$$(24)^* \quad p(K_1 \times \tilde{K}) \leq \int_{K \in \tilde{K}} \pi_K(1_{K_1})\, d\alpha(K)$$

$$(25)^* \quad p(K_1 \times K) \geq \int_{K \in K} \delta_K (1_{K_1})\, d\nu(K) .$$

Using the definition of δ_K and π_K we may rewrite these inequalities in the following form:

$$(24) \quad p(K_1 \times \tilde{K}) \leq \alpha\{Y \in \tilde{K} \mid Y \cap K_1 \neq \phi\}$$

$$(25) \quad p(K_1 \times K) \geq \nu\{Y \in K \mid Y \subset K_1\}.$$

The inequality (24) already implies that p is a plan in the sense of (10) and that this plan is possible (11).

Now, we define a measure m on X by

$$m(A) = p(A \times K) .$$

This is a tight measure because it is finite on compact sets which is a consequence of (24) and the fact that superabundance was not allowed. From (25) we get

$$(26) \quad \nu\{Y \in K \mid Y \subset K\} \leq m(K) \quad \text{for all } K \in K .$$

$$(27) \quad \int_K \varphi(K)\, d\nu(K) \le \int_X \rho_x(\varphi)\, dm(x) \quad \text{for all} \quad \varphi \in UC_o^+(K) .$$

Since the left-hand side is linear we may apply theorem 5 for a second time. This gives us a tight measure v on $X \times K$ with

$$(28)^* \quad v(K_1 \times \tilde{K}) \le \int_{K_1} \rho_x(1_{\tilde{K}})\, dm(x)$$

$$(29) \quad v(X \times \tilde{K}) \ge \int_K 1_{\tilde{K}}\, d\nu = \nu(\tilde{K}) ,$$

for compact $K_1 \subset X$ and compact $\tilde{K} \subset K$.

Using the definition of ρ_x the inequality $(28)^*$ may be rewritten in the following form

$$(28) \quad v(K_1 \times \tilde{K}) \le m(\bigcup\{K_1 \cap Y \mid Y \in \tilde{K}\}).$$

This shows that v is a plan in the sense of (10). Another consequence (by regularity) of (28) is

$$(30) \quad v(K_1 \times K) \le m(K_1) = p(K_1 \times K) ,$$

which implies that p and v are compatible (13). And finally, using (29) come to the conclusion that v is satisfactory. □

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