Small boundaries

By

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Let F denote some family of upper bounded functions $f\colon X \to [-\infty, \infty[$ on an arbitrary nonempty set X. As usual, a subset Y of X is called a boundary for F if every $f \in F$ attains its X-supremum at some point of Y. If X is a compact Hausdorff space and if all $f \in F$ are upper semicontinuous, an important example is furnished by the Choquet boundary for F. In the first part of this note, we construct a boundary for F which is (occasionally properly) contained in the Choquet boundary and which is, in addition, the smallest boundary as far as certain fixpoints are concerned (theorem 1).

Throughout the remaining sections, F will be a convex cone containing the constant real functions. In the second part, no topology is assumed on X. In theorem 2, boundary properties of some subset Y of X will be related to the problem of finding for every state on F a representing measure on Y. In this connexion, the main theorem of [4] is slightly improved by using a convergence lemma due to Simons [9]. Let us point out that theorem 2 even applies to subsets Y of X which do not contain all extreme points of the state space of F. In the final section, we transfer the preceding results to the case of upper semicontinuous functions on a compact Hausdorff space X. In particular, we study the role played by those boundaries which are Lindelöf spaces. Theorem 3 is an extended version of a closely related result from [8]. As an immediate consequence we obtain conditions on X and F which force the Choquet boundary to be the smallest boundary for F. We conclude with some further applications and examples.

I. The smallest fixpoint-boundary. Let X be a nonempty compact Hausdorff space and let F denote some family consisting of upper semicontinuous functions

$$f\colon X \to [-\infty, \infty[\,.$$

For each $a \in X$ let

$$[a] := \{x \in X\colon f(x) = f(a) \text{ for all } f \in F\}$$

denote the corresponding equivalence class. Given some family T of mappings $t\colon X \to X$, a subset Y of X is called *T-invariant* if $t(Y) \subset Y$ for all $t \in T$. And T is termed *F-exposed* if for all $f \in F$ and all T-invariant closed $Y \subset X$ the set

$$Y(f) := \{y \in Y\colon f(y) = \sup_{u \in Y} f(u) =: \sup_Y f\}$$

is also T-invariant. Finally, let

$$\operatorname{Fix}(T, F) := \{x \in X : t(x) \in [x] \text{ for all } t \in T\}$$

denote the set of common fixpoints for T with respect to F. Clearly, $\operatorname{Fix}(T, F)$ coincides with the set of common fixpoints for T if F separates the points of X (which is the most convenient case). The following example indicates that F-exposed families may be useful in the investigation of small boundaries.

Example. Let X be a compact convex set in some Hausdorff locally convex vector space. Then the vector space $F = A(X)$ of all continuous affine functions $f: X \to \mathbb{R}$ separates the points of X. We shall refer to this setting as the *geometric situation.* Now, for each $z \in X$ let $t_z: X \to X$ be given by $t_z(x) := z$ if $2x - z \in X$ and $t_z(x) := x$ otherwise. Then it is easy to see that the family $T := \{t_z : z \in X\}$ is $A(X)$-exposed and has exactly the extreme points of X as common fixpoints.

In the subsequent theorem, we shall introduce and characterize the smallest boundary for F within a certain class of boundaries determined by fixpoints.

Theorem 1. *For every F-exposed family T of mappings $t: X \to X$, $\operatorname{Fix}(T, F)$ is a boundary for F. There exists a smallest boundary of this type. This smallest fixpoint-boundary coincides with $\operatorname{Max}(X, F)$, where $\operatorname{Max}(X, F)$ consists of those $a \in X$ such that for every nonempty compact $K \subset X$ with $K \not\subset [a]$ there is some $f \in F$ satisfying $f(a) \geqq \sup_K f$ and $f(a) > \inf_K f$.*

Proof. 1) Consider some F-exposed family T and let $\mathfrak{K}$ denote the system of all T-invariant compact and nonempty subsets of X. Note that $\mathfrak{K}$ is inductively ordered by inclusion. Now consider an arbitrary $g \in F$. Then $X(g) \in \mathfrak{K}$ since T is F-exposed. By Zorn's lemma there exists a minimal $Y \in \mathfrak{K}$ such that $Y \subset X(g)$. Again, since T is F-exposed, for all $f \in F$ we have $Y(f) \in \mathfrak{K}$ and hence $Y(f) = Y$ in view of minimality. We conclude that each $f \in F$ is constant on Y which implies $Y \subset \operatorname{Fix}(T, F)$. On the other hand $\emptyset \neq Y \subset X(g)$. Consequently, g attains its X-maximum at some point of $\operatorname{Fix}(T, F)$ which proves the first assertion.

2) Let T^* denote the union of all F-exposed families. T^* is not empty since the identity mapping of X belongs to T^*. Obviously, T^* is F-exposed and fulfills $\operatorname{Fix}(T^*, F) \subset \operatorname{Fix}(T, F)$ whenever T is F-exposed. The second assertion follows.

3) Finally, we have to prove $\operatorname{Fix}(T^*, F) = \operatorname{Max}(X, F)$. First, let $a \in \operatorname{Max}(X, F)$ be arbitrary and let $\mathfrak{K}^*$ denote the system of all T^*-invariant compact $K \subset X$ such that $[a] \subset K$. Define $K \in \mathfrak{K}^*$ to be the intersection of all $L \in \mathfrak{K}^*$. If $K \not\subset [a]$ then in contradiction to the minimality of K there would be some $f \in F$ with $f(a) \geqq \sup_K f$ and $f(a) > \inf_K f$ so that $K(f) \in \mathfrak{K}^*$ with $K(f) \neq K$. Therefore $K = [a]$. In particular, $[a]$ turns out to be T^*-invariant which implies $a \in \operatorname{Fix}(T^*, F)$. Conversely, consider some $a \in X \setminus \operatorname{Max}(X, F)$. Then there exists some compact $K \subset X$ with $K \not\subset [a]$ such that every $f \in F$ with $f(a) \geqq \sup_K f$ is constant $= f(a)$ on K. For each $z \in K$ let $t_z: X \to X$ be given by $t_z(x) := z$ if $x \in [a]$ and $t_z(x) := x$ otherwise. It is not hard to verify that $T := \{t_z : z \in K\}$ is F-exposed. Obviously $t_z(a) \notin [a]$ whenever $z \in K \setminus [a]$ so that $a \notin \operatorname{Fix}(T, F)$, in particular $a \notin \operatorname{Fix}(T^*, F)$. This finishes the proof.

$\mathrm{Max}(X, F)$ has been investigated in [3]. In particular, it is well-known and easily seen that $\mathrm{Max}(X, F)$ is always contained in the Choquet boundary

$$\mathrm{Ch}(X, F) := \{a \in X\colon \operatorname{supp} \lambda \subset [a] \text{ for all Radon probability measures } \lambda \text{ on } X \text{ such that } f(a) \leqq \int_X f\, d\lambda \text{ for all } f \in F\}.$$

Note that theorem 1 provides a new proof of the fact that $\mathrm{Max}(X, F)$ is a boundary for F and hence of Bauer's maximum principle. Even in the geometric situation $\mathrm{Max}(X, F)$ can be a proper subset of $\mathrm{Ch}(X, F)$. More precisely we have:

Remark. Let X be a compact convex and balanced set of infinite dimension in some locally convex Hausdorff space. Then there exists a compact convex subset $\tilde{X}$ in X such that $\mathrm{Max}(\tilde{X}, A(\tilde{X})) \subsetneqq \mathrm{Ch}(\tilde{X}, A(\tilde{X}))$.

This can be proved via a slight modification of a construction given in [3]. By contrast, $\mathrm{Max}(X, A(X)) = \mathrm{Ch}(X, A(X))$ for every finite dimensional compact convex set X. The same relation holds for every Choquet simplex X as can be immediately derived from [1, Cor. II.5.20]. Some relations between the set of maximum points and potential theory have been studied in [2].

II. Dini cones and representing measures. In this section we shall consider the following *non-topological situation*: Let $\emptyset \neq Y \subset X$ be arbitrary sets and let F denote some convex cone consisting of upper bounded functions $f\colon X \to [-\infty, \infty[$. We assume that F contains all constant real functions.

First, let us carefully enlarge the given cone. By VF we denote the max-stable convex cone generated by F. Thus VF consists of all functions of the form

$$f_1 \vee \cdots \vee f_n\colon f_1 \vee \cdots \vee f_n(x) := \max(f_1(x), \ldots, f_n(x))$$

where $f_1, \ldots, f_n \in F$ and $n \in \mathbb{N}$. Further, let $V_B F$ be an abbreviation for the convex cone of all bounded functions in VF, and let $\mathbb{F}$ denote the convex cone consisting of all uniformly converging sums $\sum_{n=1}^{\infty} f_n$ of functions $f_n \in F \vee \mathbb{R} := \{f \vee r\colon f \in F, r \in \mathbb{R}\}$. Note that $\mathbb{F}$ is contained in the sup-norm-closure $\overline{V_B F}$ of $V_B F$.

Next, let us slightly extend the notion of Dini cone from [4]. F is said to be a *Dini cone on* Y if

$$\inf_{n \in \mathbb{N}} \sup_{x \in X} f_n(x) \leqq \sup_{y \in Y} \inf_{n \in \mathbb{N}} f_n(y)$$

whenever $(f_n)_{n \in \mathbb{N}}$ is a sequence in F that decreases pointwise on X. Since for every $f \in F$ the sequence of functions $f_n := n(f - \sup_X f) \in F$ is pointwise decreasing on X, the Dini property with respect to Y certainly implies that Y is a boundary for F. The converse implication does not hold in general. However, some positive result in this direction will be obtained in the next theorem.

Finally, we shall be interested in representing states on F by measures on Y. As usual, a positive-homogeneous and additive map $\mu\colon F \to [-\infty, \infty[$ is called a *state on* F if $\mu(f) \leqq \sup_X f$ for all $f \in F$. And a probability measure m on Y (always with respect to the coarsest σ-algebra on Y such that all the restrictions $f|Y$ for

$f \in F$ are measurable) is called a *representing measure for the state* μ *on* Y if $\mu(f) \leqq \int_Y f\,dm$ for all $f \in F$. Of course, all measures have to be σ-additive.

The emphasis of the following theorem lies on the equivalence of the assertions (ii), (iii) and (vii); for completeness we list some more equivalences. The equivalence (ii) $\Leftrightarrow$ (iii) is closely related to [8, Satz 1.2]. But it should be remarked that here we need neither topological assumptions nor a generalization of Simons' convergence lemma [8, 9]. This advantage is made even by the fact that we had to assume some weak kind of lattice structure in the definition of $\mathbb{F}$. The most important special case of theorem 2 arises when $Y = X$; as indicated by condition (iv), the case $Y = X$ is in a certain sense equivalent to the general situation. Let $F \mid Y$ denote the convex cone of the restrictions of all $f \in F$ to Y.

Theorem 2. *The following assertions are equivalent:*

(i) *Y is a boundary for $\overline{V_B F}$.*

(ii) *Y is a boundary for $\mathbb{F}$.*

(iii) *F is a Dini cone on Y.*

(iv) *$F \mid Y$ is a Dini cone on Y, and $\sup_X f = \sup_Y f$ for all $f \in F$.*

(v) *VF is a Dini cone on Y.*

(vi) *$\overline{V_B F}$ is a Dini cone on Y.*

(vii) *Every state on F has a representing measure on Y.*

(viii) *Whenever $0 \geqq f_n \in F$ for $n = 1, 2, \ldots$, then for every state μ on F we have*

$$\limsup_{n\to\infty} \mu(f_n) \leqq \sup_{y\in Y} \limsup_{n\to\infty} f_n(y)\,.$$

(ix) *Whenever $0 \geqq f_n \in F$ for $n = 1, 2, \ldots$,*

$$\sum_{n=1}^{\infty} \mu(f_n) > -\infty \quad \textit{for some state } \mu \textit{ on } F \textit{ implies} \quad \sup_{y\in Y} \sum_{n=1}^{\infty} f_n(y) > -\infty\,.$$

(x) *Whenever $0 \geqq f_n \in F$ for $n = 1, 2, \ldots$,*

$$\inf_{m\in\mathbb{N}} \sup_{x\in X} \sum_{n=1}^{m} f_n(x) > -\infty \quad \textit{implies} \quad \sup_{y\in Y} \sum_{n=1}^{\infty} f_n(y) > -\infty\,.$$

Proof. (i) $\Rightarrow$ (ii): is trivial.

(ii) $\Rightarrow$ (iii): Let $(h_n)_n$ be a decreasing sequence chosen arbitrarily in F such that $\alpha := \inf_{n\in\mathbb{N}} \sup_{x\in X} h_n(x) > -\infty$. Consider some real $\beta < \alpha$ and let $f_n := h_n \vee \beta$ for all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $F \vee \mathbb{R}$ which is uniformly bounded on X. In particular, for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real $\lambda_n \geqq 0$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, the series $\sum_{n=1}^{\infty} \lambda_n f_n$ uniformly converges on X and hence attains its X-supremum at some point of Y because of (ii). Therefore, from Simons' convergence lemma [9, p. 704] we conclude that

$$\inf \left\{ \sup_{x\in X} \sum_{n=1}^{m} \lambda_n f_n(x) :\ \lambda_1, \ldots, \lambda_m \geqq 0,\ \sum_{n=1}^{m} \lambda_n = 1 \right\} \leqq \sup_{y\in Y} \limsup_{n\to\infty} f_n(y)\,.$$

Since $(f_n)_{n\in\mathbb{N}}$ decreases it follows that

$$\inf_{n\in\mathbb{N}}\sup_{x\in X} f_n(x) \leqq \sup_{y\in Y}\inf_{n\in\mathbb{N}} f_n(y)$$

and hence

$$\inf_{n\in\mathbb{N}}\sup_{x\in X} h_n(x) \leqq \sup_{y\in Y}\inf_{n\in\mathbb{N}} h_n(y)$$

because of $\beta < \alpha$. This forces F to be a Dini cone on Y.

(iii) $\Rightarrow$ (x): is trivial.

(x) $\Leftrightarrow$ (ix): By a well-known consequence of Dini's lemma [4, Lemma 1] or by a suitable application of the sandwich theorem [5, Lemma 1], for every decreasing sequence $(h_n)_{n\in\mathbb{N}}$ in F there is an order-preserving state μ on X such that

$$\inf_{n\in\mathbb{N}}\sup_{x\in X} h_n(x) = \inf_{n\in\mathbb{N}} \mu(h_n)\,.$$

Thus the equivalence (x) $\Leftrightarrow$ (ix) is obvious.

(x) $\Rightarrow$ (iv): First apply (x) to the constant sequence of functions $f_n := f - \sup_X f$ to derive $\sup_X f = \sup_Y f$ for any $f \in F$. Now consider an arbitrary sequence of subsets $Y_m \subset Y$ such that $Y_m \uparrow Y$. From (x) we conclude that

$$\inf_{n\in\mathbb{N}} \sup_Y h_n > -\infty \quad \text{implies} \quad \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}}\sup_{Y_m} h_n > -\infty$$

whenever $(h_n)_n$ is a sequence in $F\,|\,Y$ satisfying $h_{n+1} = h_n + g_n$ with $0 \geqq g_n \in F\,|\,Y$ for all n. Because of the construction in [5, p. 157-8], this property enables us to apply the decomposition theorem [5, p. 158] to $F\,|\,Y$ so that

$$(*) \qquad \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}}\sup_{Y_m} h_n = \inf_{m\in\mathbb{N}} \sup_Y h_n$$

for every decreasing sequence $(h_n)_n$ in $F\,|\,Y$. From (*) it is easy to see that $F\,|\,Y$ is a Dini cone on Y. Indeed: Consider some decreasing sequence $(h_n)_n$ in $F\,|\,Y$, let $\beta := \sup_{y\in Y}\inf_{n\in\mathbb{N}} h_n(y)$ and fix an arbitrary real $\alpha > \beta$. Then $Y_m := \{y \in Y : h_m(y) \leqq \alpha\} \uparrow Y$ as $m \to \infty$ so that (*) implies $\alpha \geqq \inf_{n\in\mathbb{N}} \sup_Y h_n$. This proves the desired Dini property since $\alpha > \beta$ was arbitrarily chosen.

(iv) $\Rightarrow$ (v): In view of [4, Lemma 5] or [5, Cor. 1], $VF\,|\,Y$ is a Dini cone on Y. Because of $\sup_Y f = \sup_X f$ for all $f \in VF$, this forces VF to be a Dini cone on Y.

(v) $\Rightarrow$ (vi): Consider some decreasing sequence $(g_n)_n$ in $\overline{V_B F}$ and choose $h_n \in V_B F$ such that $|g_n - h_n| \leqq 1/2^{n+2}$ on X for $n = 1, 2, \ldots$. Note that the sequence of functions $f_n := h_n + 1/2^n \in V_B F$ decreases on X. And the Dini property with respect to Y immediately carries over from $(f_n)_n$ to $(g_n)_n$. Hence (v) implies (vi).

(vi) $\Rightarrow$ (i): As remarked earlier, Y is a boundary for any convex cone containing the constants which is a Dini cone on Y.

(iv) $\Rightarrow$ (vii): Let μ be an arbitrary state on F. Because of $\sup_Y f = \sup_X f$ for all $f \in F$, an easy consequence of the sandwich theorem [4, Cor. 1] supplies us with a state $\tilde{\mu}$ on $F\,|\,Y$ such that $\mu(f) \leqq \tilde{\mu}(f\,|\,Y)$ for all $f \in F$. Now, by the main theorem of [4] there is a representing measure on Y for $\tilde{\mu}$ and hence for μ.

(vii) $\Rightarrow$ (viii): follows immediately from Fatou's lemma.

(viii) $\Rightarrow$ (ix): is obvious.

This completes the proof of theorem 2.

III. Lindelöf boundaries and applications. Throughout this last section, let X be a nonempty compact Hausdorff space and let F be a convex cone consisting of upper semicontinuous functions $f\colon X \to [-\infty, \infty[$ such that F contains the constant real functions. For some arbitrary nonempty subset Y of X, let us consider the following six assertions:

(i) $Y \cap [a] \neq \emptyset$ for all $a \in \mathrm{Ch}(X, F)$.

(ii) Y is a boundary for $\mathbb{F}$.

(iii) Every state on F has a representing measure on Y.

(iv) $\sup\limits_{x \in X} \inf\limits_{n \in \mathbb{N}} f_n(x) = \sup\limits_{y \in Y} \inf\limits_{n \in \mathbb{N}} f_n(y)$ for every decreasing sequence $(f_n)_{n \in \mathbb{N}}$ in F.

(v) For every F_σ-subset Z of X with $Y \subset Z$ we have $Z \cap [a] \neq \emptyset$ $\forall a \in \mathrm{Ch}(X, F)$.

(vi) For every Lindelöf set Z in X with $Y \subset Z$ we have $Z \cap [a] \neq \emptyset$ $\forall a \in \mathrm{Ch}(X, F)$.

Of course, a subset Z of X is termed Lindelöf if each open covering contains a countable subcovering, i.e. if Z is a Lindelöf space in the topology coming from X. The preceding properties are always related as follows:

Theorem 3. *The following implications hold in general:*

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow \text{(v)} \Leftrightarrow \text{(vi)}.$$

Proof. Obviously $\mathrm{Ch}(X, F) = \mathrm{Ch}(X, \mathbb{F})$. Hence (i) implies (ii) by Bauer's maximum principle. In view of Dini's lemma, assertion (iv) exactly means that F is a Dini cone on Y. Therefore, the equivalence of (ii), (iii) and (iv) follows from theorem 2. And (iv) $\Rightarrow$ (v) is immediate from [8, Satz 2.6]; a different approach to this implication can be given by suitable direct constructions based on well-known maximum properties of the Choquet boundary [3, p. 417]. Since F_σ-sets are clearly Lindelöf, all that remains to prove is (v) $\Rightarrow$ (vi). Let Z be a Lindelöf subset of X with $Y \subset Z$ and suppose $Z \cap [a] = \emptyset$ for some $a \in \mathrm{Ch}(X, F)$. It is well-known and easily seen that $[a]$ is closed [8, p. 187]. Consequently, every $z \in Z$ has an open neighbourhood $U(z)$ such that $\overline{U(z)} \cap [a] = \emptyset$. Because of the Lindelöf property we obtain a sequence $(z_n)_{n \in \mathbb{N}}$ in Z such that $Z \subset \bigcup\limits_{n=1}^{\infty} U(z_n)$. It follows that $Y \subset \bigcup\limits_{n=1}^{\infty} \overline{U(z_n)} =: \tilde{Z}$ and hence $\tilde{Z} \cap [a] \neq \emptyset$ in view of (v). This contradiction to the choice of $U(z)$ completes the proof.

A counterexample due to Simons [9, p. 706] shows that even in the geometric situation the implication (ii) $\Rightarrow$ (i) fails to be true in general. However, when imposing additional assumptions on X, Y or F, we shall be able to derive a number of positive results from theorem 3. First of all, let us state an immediate consequence.

Corollary 1. *For every Lindelöf subset Y of X, all the assertions* (i)—(vi) *are equivalent.*

Corollary 2. *Suppose that each $f \in F$ is Baire measurable. Then for arbitrary $Y \subset X$ the assertions* (ii)—(vi) *are equivalent.*

Proof. In view of theorem 3, it suffices to prove (v) $\Rightarrow$ (ii). Consider an arbitrary $f \in \mathbb{F}$. Then f is upper semicontinuous and Baire measurable because of our additional assumption. Thus $\{x \in X: f(x) \geqq \max_X f\}$ is a compact Baire set and hence a G_δ. We conclude that $Z := \{x \in X: f(x) < \max_X f\}$ is an F_σ-set. Now assume that f does not attain its X-maximum at some point of Y. Then $Y \subset Z$ so that $Z \cap [a] \neq \emptyset$ for every $a \in \mathrm{Ch}(X, F) = \mathrm{Ch}(X, \mathbb{F})$. By Bauer's maximum principle Z turns out to be a boundary for $\mathbb{F}$ which is clearly impossible. This contradiction proves assertion (ii).

In particular, corollary 2 applies whenever $F \subset C(X)$, i.e. whenever F consists of continuous functions on X. Another important special case arises when the σ-algebra of Borel sets in X coincides with the Baire σ-algebra. But in this situation the result can be extended to include property (i).

Corollary 3. *Suppose that each Borel subset of X is a Baire set. Then for arbitrary $Y \subset X$, all the assertions* (i)—(vi) *are equivalent.*

Proof. We shall prove (v) $\Rightarrow$ (i). Assume that there is some $a \in \mathrm{Ch}(X, F)$ such that $Y \subset X \backslash [a] =: Z$. Since $[a]$ is compact and hence a compact Baire set, it follows that Z is an F_σ. From (v) we conclude $Z \cap [a] \neq \emptyset$ which is impossible. The result follows.

Corollary 4. *Suppose that X is first countable and that F separates the points of X. Then for arbitrary $Y \subset X$, all the assertions* (i)—(vi) *are equivalent. In particular,* $\mathrm{Ch}(X, F)$ *is the smallest boundary for $\mathbb{F}$ and coincides with* $\mathrm{Max}(X, \mathbb{F})$.

Proof. Again, let us prove (v) $\Rightarrow$ (i). Consider an arbitrary $a \in \mathrm{Ch}(X, F)$. From our assumptions it follows that $[a] = \{a\}$ is a G_δ. Hence $Z := X \backslash [a]$ is an F_σ-set. As in the preceding proof, we conclude from (v) that $a \in Y$. The last statement is immediate from part I.

In this corollary, the assumption on X is essential. This is clear from [9, p. 706] as well as from the subsequent example.

Example. Let $X = \beta Y$ be the Stone-Czech compactification of some completely regular Hausdorff space Y and let F consist of all continuous real-valued functions on X. Then $F = \mathbb{F}$, $\mathrm{Ch}(X, F) = X$, and Y is a boundary for F iff it is pseudocompact. Now, the equivalence (ii) $\Leftrightarrow$ (vi) from corollary 2 gives immediately: *A completely regular Hausdorff space Y is pseudocompact if and only if every Lindelöf subset of its Stone-Czech compactification that contains Y is compact* (cf. [6, p. 72 and 115]). Further, let us note that in this case assertion (iii) is a well-known result of Glicksberg (cf. [7] or [10]). Finally suppose that Y is pseudocompact, but not compact. Then Y is a boundary for $\mathbb{F}$ which is strictly contained in $\mathrm{Ch}(X, F)$. Thus corollary 4 ceases to be true if X is not assumed to be first countable.

In the geometric situation, corollary 4 can be used to study how exposed extreme points are:

Example. Let X be a compact convex set in some locally convex Hausdorff space. Suppose that X is first countable. *Then $x \in X$ is an extreme point of X if and only if for every compact $K \subset X$ with $\emptyset \neq K \neq \{x\}$ there is a continuous convex function $f \colon X \to \mathbb{R}$ such that $f(x) \geqq \sup_K f$ and $f(x) > \inf_K f$.* Indeed, let F consist of all continuous convex functions $f \colon X \to \mathbb{R}$. Then the assertion is immediate from corollary 4 because $\operatorname{Ch}(X, F)$ is the set of extreme points of X and $F = \mathbb{F}$. Of course, this characterization remains valid if one replaces the cone of continuous convex functions on X by $\mathbb{A}(X)$. But unfortunately we have no satisfying notion of what $\mathbb{A}(X)$ looks like. And we do not know if the above extreme point characterization holds if X is not assumed to be first countable. It would be nice to have a counterexample.

Finally, let us present an application concerning fixpoint-boundaries in the non-compact situation. We shall use the same notion of F-exposed families as in the compact case.

Corollary 5. *Let F be a convex cone consisting of continuous real-valued functions on some Lindelöf topological space Y. Suppose that F separates the points of Y and contains all constant real functions. Assume further that F is a Dini cone on Y. Then for every F-exposed family T of mappings $t \colon Y \to Y$, the set of common fixpoints for T is a boundary for F.*

Proof. Let X denote the set of all lattice-preserving states on the vector lattice $E := VF - VF$. Endow X with the weak-*-topology $\sigma(E^*, E)$. Then X is a compact Hausdorff space, and the mapping $i \colon Y \to X$ given by $i(y)(g) := g(y)$ is a continuous injection so that $i(Y)$ is a Lindelöf subset of X. As usual, for $f \in F$ define $\hat{f}(\mu) := \mu(f)$ for all $\mu \in X$. Then $\hat{f}$ extends f in the sense that $\hat{f}(i(y)) = f(y)$ for all $y \in Y$. Since F is a Dini cone on Y, the equivalence (i) $\Leftrightarrow$ (iv) from corollary 1 gives $\operatorname{Ch}(X, \hat{F}) \subset i(Y)$ where $\hat{F} := \{\hat{f} \colon f \in F\}$. Therefore $\operatorname{Max}(X, \hat{F}) \subset i(Y)$. Now, it is easy to see that $\hat{T} := \{\hat{t} \colon t \in T\}$ is $\hat{F}$-exposed on X where $\hat{t} \colon X \to X$ is defined by $\hat{t}(\mu) := i(t(y))$ if $\mu = i(y)$ for some $y \in Y$ and $\hat{t}(\mu) := \mu$ otherwise. By theorem 1 we know that $\operatorname{Max}(X, \hat{F})$ is a boundary for $\hat{F}$ with the property $\operatorname{Max}(X, \hat{F}) \subset \operatorname{Fix}(\hat{T}, \hat{F})$. We conclude that $\operatorname{Max}(X, \hat{F}) \subset \operatorname{Fix}(\hat{T}, \hat{F}) \cap i(Y) = i(\operatorname{Fix}(T, F))$. The result follows.

References

[1] E. M. ALFSEN, Compact convex sets and boundary integrals. Berlin-Heidelberg-New York 1971.

[2] J. BLIEDTNER and K. JANSSEN, A generalization of H. Bauer's minimum principle. Arch. Math. **25**, 505–510 (1974).

[3] B. FUCHSSTEINER und W. HACKENBROCH, Maximumspunkte. Arch. Math. **23**, 415–421 (1972).

[4] B. FUCHSSTEINER, When does the Riesz representation theorem hold? Arch. Math. **28**, 173–181 (1977).

[5] B. FUCHSSTEINER, Decomposition theorems. Manuscripta Math. **22**, 151–164 (1977).

[6] L. GILLMAN and M. JERISON, Rings of continuous functions. New York-Toronto-London-Melbourne 1960.

[7] I. GLICKSBERG, The representation of functionals by integrals. Duke Math. J. **19**, 253–261 (1952).

[8] M. Neumann, Varianten zum Konvergenzsatz von Simons und Anwendungen in der Choquettheorie. Arch. Math. 28, 182–192 (1977).
[9] S. Simons, A convergence theorem with boundary. Pacific J. Math. **40**, 703–708 (1972).
[10] V. S. Varadarajan, Measures on topological spaces. Amer. Math. Soc. Translations **48**, 161–228 (1965).

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