REPRESENTING ISOTONE OPERATORS ON CONES

By B. FUCHSSTEINER *and* J. D. MAITLAND WRIGHT

[Received 18 May 1976]

Introduction

LET V be a boundedly complete vector lattice, or, in other words, a Dedekind complete Riesz space [**2**], [**9**]. The object of this paper is to extend the main theorem of [**5**], which is a representation theorem for real valued states on Dini cones (see below for definitions), to a representation theorem for V-valued operators on Dini cones. In [**13**], Vincent-Smith showed by novel and elegant methods that an analogue of Choquet boundary theory can be developed for V-valued measures. As an application of our main theorem we are able to give an improved version of one of his results.

When V is weakly σ-distributive [**16**], [**3**] the situation is quite straightforward. But, for general V, the problem is, as we would expect from [**16**], [**3**], more subtle and its solution more elusive.

1. Hahn–Banach preliminaries

It is well known that when V is equipped with an order-unit then analogues of the Hahn–Banach Theorem are valid for V-valued operators, see [**8**], [**10**] and, particularly, the sophisticated results of [**12**]. As we shall need a variant of the vector-valued Hahn–Banach Theorem for which we do not know of an accessible reference, we shall give a short proof for the convenience of the reader.

Let us adjoin an element $-\infty$ to V and let $V_* = V \cup \{-\infty\}$. We extend the partial ordering of V to V_* by requiring $-\infty < x$ for all $x \in V$ and we define $x+y$ to be $-\infty$ whenever x or y is $-\infty$. Furthermore, when $\rho \in \mathbb{R}^+$ we define $\rho \cdot (-\infty)$ to be $-\infty$ if $\rho > 0$ and define $0 \cdot (-\infty)$ to be 0.

In this section F will be a convex cone with vertex the origin, that is, F is a non-empty, convex subset of a real vector space such that, whenever $x \in F$ and $\lambda \in \mathbb{R}^+$ then $\lambda x \in F$. (It is not required that F be proper.) Furthermore, F will be equipped with a partial ordering compatible with its cone structure, that is, whenever $x \leq x'$ and $y \leq y'$ then $x+y \leq x'+y'$ and, for each $\lambda \in \mathbb{R}^+$, $\lambda x \leq \lambda x'$. A function ϕ: $F \to V_*$ is *positively homogeneous* if, for each non-negative real number ρ, $\phi(\rho x) = \rho\phi(x)$ for every x in F.

A positively homogeneous function ϕ: $F \to V_*$ is said to be *sublinear* if, for all f and g in F, $\phi(f+g) \leq \phi(f) + \phi(g)$; it is said to be *superlinear* if, for all f and g in F, $\phi(f+g) \geq \phi(f)+\phi(g)$; it is said to be *isotone* (or *order-preserving*) if, whenever $f \leq g$, then $\phi(f) \leq \phi(g)$; when ϕ is both superlinear and sublinear then ϕ is said to be *linear*. This coincides with the usual notion of linear when F is a vector space for ϕ does not take the value $-\infty$ anywhere; in the general situation where F is a convex cone it means that ϕ is additive and positively homogeneous.

Since we can do so without extra effort, the result below is proved in slightly greater generality than is strictly necessary for the applications in § 2.

In the following, V_*^F, the collection of all V_*-valued functions on F, will be partially ordered pointwise, that is, $w \leq \pi$ means that $w(f) \leq \pi(f)$ for all $f \in F$.

THEOREM 1. (SANDWICH THEOREM). *Let π: $F \to V_*$ be sublinear and isotone, let w: $F \to V_*$ be superlinear and let $w \leq \pi$. Then there exists a linear function ϕ: $F \to V_*$ such that ϕ is isotone and $w \leq \phi \leq \pi$.*

Let Γ be the collection of all superlinear functions γ: $F \to V_*$ such that $w \leq \gamma \leq \pi$ and equip Γ with the partial ordering it inherits from V_*^F. Since $w \in \Gamma$, Γ is not empty. By a straightforward application of Zorn's Lemma, it follows that Γ contains a maximal element ϕ. Fix $g \in F$ and define a to be

$$\bigwedge \{\pi(g+q)-\phi(q): \ q \in F \text{ and } \phi(q) > -\infty\}.$$

For each $f \in F$ let

$$\phi_g(f) = \bigvee \{\phi(h)+\lambda a: \ (h, \lambda) \in F \times \mathbb{R}^+ \text{ and } h \times \lambda g \leq f\}.$$

Then ϕ_g is order-preserving, superlinear and $\phi \leq \phi_g \leq \pi$. It follows from the maximality of ϕ that $\phi = \phi_g$.

When $\phi(h) > -\infty$, we have

$$\pi(g+h)-\phi(h) \geq \phi(g+h)-\phi(h) \geq \phi(g) = \phi_g(g) \geq a.$$

Hence

$$\phi(g) = \bigwedge \{\pi(g+q)-\phi(q): \ \phi(q) > -\infty \text{ and } q \in F\}.$$

Since this must hold for every g in F, it now follows by an easy calculation, that ϕ is subadditive. Hence ϕ is an order-preserving, linear map with $w \leq \phi \leq \pi$.

2. The representation theorem

Let X be a non-empty set and let $F = F(X)$ be a cone of upper bounded functions on X such that the constant functions are in $F(X)$. The cone $F(X)$ is said to be a *Dini cone* [5] if, whenever (f_n) $(n = 1, 2, \ldots)$ is a monotone decreasing sequence of functions in F then

$$\sup_{x\in X}\inf_{n\in N} f_n(x)=\inf_{n\in N}\sup_{x\in X} f_n(x).$$

An example of a Dini cone would be: $F(X)=UC_\infty^+(X)+\mathbb{R}$, where $UC_\infty^+(X)$ is the cone of non-negative upper semicontinuous functions vanishing at infinity and defined on a locally compact space X. Let $\Sigma(F)$ be the smallest σ-algebra of subsets of X such that each function on $F(X)$ is measurable with respect to $\Sigma(F)$.

By a (finite) V-valued measure on $\Sigma(F)$ we mean, as in **[15]**, **[16]**, a map $m\colon \Sigma(F)\to V^+$ which is finitely additive and such that, whenever (E_n) $(n=1,2,\ldots)$ is a monotone increasing sequence of sets in $\Sigma(F)$, then $m\left(\bigcup_1^\infty E_n\right)=\bigvee_1^\infty mE_n$, where $\bigvee_1^\infty mE_n$ is the least upper bound of $\{mE_n\colon n=1,2,\ldots\}$ in V. It is convenient to adopt the convention that $\int_X f\,dm=-\infty$ if $\left\{\int_X (f\vee(-n1))\colon n=1,2,\ldots\right\}$ is not bounded below in V and f is bounded above.

Let $\phi\colon F(X)\to V_*$ be a linear map and let m be a V-valued measure on $\Sigma(F)$ **[15]**. Then m is said to be a *representing measure* for ϕ if, for each $f\in F$, $\phi(f)\le\int_X f\,dm$. If, for each $f\in F$, $\phi(f)=\int_X f\,dm$ then m is said to be a *strongly representing measure.*

Let e be any non-zero positive element of V. We define an *e-state* to be a linear $\phi\colon F(X)\to V_*$ such that $\phi(1)=e$ and, for each $f\in F$, $\phi(f)\le e\sup_X(f)$. The set of e-states inherits a partial ordering from V_*^F and an e-state is said to be *maximal* if it is maximal with respect to this partial ordering. It is an immediate consequence of Zorn's Lemma that each e-state is dominated by a maximal e-state and, by applying the Sandwich Theorem with $\pi(f)=e\sup_X(f)$ we see that each state is dominated by an isotone e-state, so that each maximal e-state is isotone. The only $\mathbb{R}_*$-valued states we shall consider will be the 1-states.

LEMMA 2. *Let $\phi\colon F(X)\to V_*$ be linear and isotone; let $F^\#(X)$ be the max-stable cone generated by $F(X)$ and let $F^\#(X)$ be equipped with the pointwise partial ordering. Then there exists a linear isotone $\phi^\#$ on $F^\#(X)$ which is an extension of ϕ. Furthermore, if for some e in V^+ the map ϕ is an e-state then we can choose $\phi^\#$ to be an e-state on $F^\#(X)$.*

Define

$$w(h)=\begin{cases}-\infty & \text{if } h\in F^\# \text{ and } h\notin F.\\ \phi(h) & \text{if } h\in F.\end{cases}$$

Define

$$\pi(h)=\bigwedge\left\{\phi(g)+\sup_{x\in X} f(x)\phi(1):\ (g,f)\in F\times F^{\#} \text{ and } g+f\geq h\right\}.$$

The lemma follows from the Sandwich Theorem.

Let S be a compact Hausdorff space such that $C(S)$ is a boundedly complete vector lattice. This is equivalent to requiring S to be extremally disconnected [12]. Let χ_S be the characteristic function of S, that is, the function on S which takes the constant value 1.

LEMMA 3. *Let ϕ: $F(X)\to C(S)_*$ be a maximal χ_S-state such that ϕ never takes the value $-\infty$. For each $s\in S$, and each $f\in F$ let $\phi_s(f)=\phi(f)(s)$; let ψ_s be a maximal (real valued) state which dominates ϕ_s. Let ψ: $F(X)\to \mathbb{R}^S$ be defined by $\psi(h)(s)=\psi_s(h)$. Then ψ is a linear $\mathbb{R}^S$-valued χ_S-state and, for each $f\in F$, the set*

$$\{s\in S:\ \phi_s(f)<\psi_s(f)\}$$

is meagre.

Let $\hat{\psi}(f)=\bigwedge\{b\in C(S):\ b\geq\psi(f)\}$ for each $f\in F(X)$. Then $\hat{\psi}\geq\phi$ and $\hat{\psi}$ is sublinear and isotone. Moreover, by a result of Stone [11], $\{s\in S:\ \hat{\psi}(f)(s)<\psi(f)(s)\}$ is meagre for each $f\in F$. So, to establish this lemma, it will suffice to show that $\phi=\hat{\psi}$.

First, we define an auxiliary function γ: $F(X)\to V$ by

$$\gamma(f)=\bigwedge\{\hat{\psi}(f+q)-\phi(q):\ q\in F(X)\}.$$

then $\phi\leq\gamma\leq\hat{\psi}$. Let us assume that for some f_0, we have $\phi(f_0)<\gamma(f_0)$. We define δ on $F(X)$ by

$$\delta(f)=\bigvee\{\phi(f_1)+r\gamma(f_0):\ (f_1,r)\in F\times\mathbb{R}^+ \text{ and } f=f_1+rf_0\}.$$

Then δ is superlinear and $\phi\leq\delta\leq\hat{\psi}$. Also $\delta(f_0)\geq\gamma(f_0)$. By the Sandwich Theorem, there exists a χ_S-state r with $\hat{\psi}\geq r\geq\delta$. Thus $r\geq\phi$ and $r(f_0)>\phi(f_0)$, which is impossible since ϕ is maximal. Hence $\phi=\gamma$.

To complete the proof we proceed as follows.

$$\begin{aligned}\phi(f)=\gamma(f)&=\bigwedge\{\hat{\psi}(f+q)-\phi(q):\ q\in F\}\\ &=\bigwedge\{b-\phi(q):\ (q,b)\in F\times C(S) \text{ and } b\geq\psi(f)+\psi(q)\}\\ &\geq\bigwedge\{b-\phi(q):\ (q,b)\in F\times C(S) \text{ and } b\geq\psi(f)+\phi(q)\}\\ &=\bigwedge\{d:\ d\in C(S) \text{ and } d\geq\psi(f)\}\\ &=\hat{\psi}(f).\end{aligned}$$

We now come to the main theorem.

THEOREM 4. *Let $F(X)$ be a convex cone of upper bounded functions on X with the constant functions contained in $F(X)$. Let e be a non-zero positive* *element of V. Then the following statements are equivalent.*

(i) *$F(X)$ is a Dini cone.*

(ii) *Whenever μ is an e-state on $F(X)$ there exists a V-valued representing measure m defined on $\Sigma(F)$ with $mX = e$.*

(iii) *Whenever ϕ is a maximal e-state on $F(X)$ there exists a V-valued strongly representing measure m defined on $\Sigma(F)$ with $mX = e$.*

The equivalence of (ii) and (iii) is immediate. That (ii)$\Rightarrow$(i) follows easily from Lemma 1 of [**5**].

Let us now suppose that $F(X)$ is a Dini cone then, as proved in [**5**], the generated max-stable cone $F^{\#}(X)$ is also a Dini cone. It follows from Lemma 2 and this observation that it is sufficient to consider the special case where $F(X) = F^{\#}(X)$.

Let $F_0(X)$ be the cone of bounded functions in $F(X)$ and ϕ_0 the restriction of ϕ to $F_0(X)$. For each $f \in F(X)$, $f \vee (-n\chi_X)$ is in F_0 for $n = 1, 2, \ldots$. Hence $\Sigma(F) = \Sigma(F_0)$. It is clear that F_0 is, itself, a Dini cone and that ϕ_0 is an e-state on F_0. Suppose that we can find a strongly representing measure m on $\Sigma(F_0) = \Sigma(F)$ for ϕ_0. Then, whenever $f \in F$,

$$\phi(f) \leq \phi(f \vee (-n\chi_X)) \leq \int_X f \vee (-n\chi_X)\, \mathrm{d}m \quad \text{for} \quad n = 1, 2, \ldots.$$

So

$$\phi(f) \leq \int_X f\, \mathrm{d}m.$$

Thus it suffices to establish (iii) in the special case where $F(X)$ is a max-stable cone of bounded functions on X and ϕ is a maximal e-state on $F(X)$. Then ϕ does not take the value $-\infty$ at any point of $F(X)$.

By the Kadison–Kakutani Theorem, see [**1**], page 76, there is a compact Hausdorff space S such that $C(S)$ is linearly order-isomorphic to $V[e] = \{b \in V\colon \exists \lambda \in \mathbb{R}^+ \text{ with } -\lambda e \leq b \leq \lambda e\}$. Clearly $C(S)$ is boundedly complete. We are now able to make use of Lemma 3.

The main theorem of [**5**] and the equivalence of (ii) and (iii) above implies that, for each $s \in S$, we can find a (real valued) probability measure ν_s on $\Sigma(F)$ such that $\psi(f)(s) = \int_X f\, \mathrm{d}\nu_s$ for all $f \in F(X)$.

Let M be the lattice of all bounded $\Sigma(F)$-measurable functions on X and let

$L = \{h \in M\colon$ There exists $\eta(h) \in C(S)$ such that $\eta(h)$ and $\psi(h)$ differ only on a meagre subset of $S\}$.

It follows from the Baire Category Theorem that whenever $h \in L$ then $\eta(h)$ is the *unique* element of $C(S)$ which differs from $\psi(h)$ only on a meagre set.

Using the linearity of ψ and applying the Baire Category Theorem we find that L is a linear subspace of M and that η is a positive linear operator on L. Let (f_n) $(n = 1, 2, \ldots)$ be an upper bounded monotone increasing sequence in L and let $f(x) = \sup f_n(x)$ for all $x \in X$. Then, by the Lebesgue Monotone Convergence Theorem,

$$\psi(f)(s) = \sup_n \psi(f_n)(s) \quad \text{for each } s.$$

But $\bigvee_1^\infty \eta(f_n)$ agrees with $\sup_n \eta(f_n)$ except on a meagre subset of S, and so $\bigvee_1^\infty \eta(f_n)$ agrees with $\psi(f)$ except on a meagre subset of S. Thus f is in L and $\eta(f) = \bigvee_1^\infty \eta(f_n)$.

By Lemma 3, L contains the vector lattice $F(X) - F(X)$ and hence $L = M$.

For each $E \in \Sigma(F)$ we define mE to be $\eta(\chi_E)$. Then m is the required strongly representing measure and the theorem is now proved.

In the following application of the above theorem let Z be a compact convex subset of a Hausdorff locally convex topological vector space; let ∂Z be the set of extreme points of Z; and let $A(Z)$ be the space of affine continuous functions on Z.

COROLLARY 5. *Let ϕ: $A(Z) \to V$ be a positive linear operator. Then there exists a V-valued quasi-regular Borel measure q on Z such that*

$$\phi(f) = \int_Z f \, dq \quad \textit{for all} \quad f \in A(Z)$$

and, whenever B is a Baire subset of Z which is disjoint from ∂Z then $qB = 0$.

By the Bauer maximum principle for compact convex sets, see [**1**], page 46, $A(Z)$ is isometrically linearly and order-isomorphic to $F(\partial Z)$, where $F(\partial Z)$ is the set of restrictions of affine continuous functions to ∂Z. Furthermore, by Bauer's maximum principle, $F(\partial Z)$ is a Dini cone.

From Theorem 4 there exists a measure m on $\Sigma(F)$ such that $\phi(f) = \int_{\partial Z} f \, dm$ for all $f \in A(Z)$. Since $\Sigma(F) = \{B \cap \partial Z\colon\ B \text{ is a Baire subset of } Z\}$, we define a Baire measure q_0 on Z by $q_0(B) = M(B \cap \partial Z)$. Then, see **[16]**, there is a unique quasi-regular Borel measure q which extends the Baire measure q_0. Clearly,

$$\phi(f) = \int_Z f \, dq \quad \text{for each} \quad f \in A(Z).$$

Remark. In **[14]** Vincent-Smith obtains a similar result to the above Corollary, but his maximal measures are shown to be boundary measures in the weaker sense that, when E is a G_δ-set disjoint from ∂Z then E has zero measure. When V is weakly σ-distributive, each V-valued Baire measure is regular **[17]** and the result above is an immediate consequence of his work. But when V is not weakly σ-distributive (and hence V-valued Baire measures need not be regular **[17]**) Corollary 5 gives a stronger result.

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer (1971).
2. D. H. Fremlin, *Topological Riesz Spaces and Measure Theory*, Cambridge University Press (1974).
3. D. H. Fremlin, 'A direct proof of the Matthes–Wright integral extension theorem', *J. London Math. Soc.* (2) 11 (1975), 276–284.
4. B. Fuchssteiner, 'Sandwich theorems and lattice semi-groups', *J. Functional Analysis* 16 (1974), 1–14.
5. B. Fuchssteiner, 'When does the Riesz Representation Theorem hold?', Arch. Math. 28 (1977), 173–181.
6. B. Fuchssteiner, 'Masse auf σ-kompakten Räumen, *Math. Z.* 142 (1975), 185–190.
7. R. V. Kadison, 'A representation theory for commutative topological algebra', *Memoirs Amer. Math. Soc.* 7 (1951).
8. J. L. Kelley, 'Banach spaces with the extension property', *Trans. Amer. Math. Soc.* 72 (1954), 323–326.
9. W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces I*, North-Holland (1971).
10. L. Nachbin, 'A theorem of Hahn-Banach type for linear transformations', *Trans. Amer. Math. Soc.* 68 (1950), 28–46.
11. M. H. Stone, 'Boundedness properties in function lattices', *Canad. J. Math.* 1 (1949), 176–186.
12. G. F. Vincent-Smith, 'The Hahn-Banach theorem for modules', *Proc. London Math. Soc.* 17 (1967), 72–90.
13. G. F. Vincent-Smith, 'A Choquet boundary theory for measures taking values in a Stone algebra', *J. London Math. Soc.* 44 (1969), 553–558.
14. J. D. M. Wright, 'Stone-algebra-valued measures and integrals', *Proc. London Math. Soc.* (3) 19 (1969), 107–122.
15. J. D. M. Wright, 'Vector lattice measures on locally compact spaces', *Math. Z.* 120 (1971), 193–203.

16. J. D. M. Wright, 'The measure extension problem for vector lattices', *Ann. Inst. Fourier* (Grenoble) 21 (1971), 65–85.

Départment de Mathématiques
École Polytechnique Fédérale de Lausanne
Lausanne

Department of Mathematics
The University
Reading