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Nonlinear Reformulation of Heisenberg's Dynamics

Martin Ziegler
ziegler@upb.de

Benno Fuchssteiner
benno@mupad.de

University of Paderborn

Abstract

A structural similarity between Classical Mechanics (CM) and Quantum Mechanics (QM) was revealed by P.A.M. Dirac in terms of Lie Algebras: while in CM the dynamics is determined by the Lie algebra of Poisson brackets on the manifold of scalar fields for classical position/momentum observables Q/P, QM evolves (in Heisenberg's picture) according to the formally similar Lie algebra of commutator brackets of the corresponding operators:

$$\tfrac{d}{dt}Q = \{Q,H\} \quad \tfrac{d}{dt}P = \{P,H\} \qquad \text{vs.} \qquad \tfrac{d}{dt}\mathbb{Q} = \tfrac{i}{\hbar}\,[\mathbb{Q},\mathbb{H}] \quad \tfrac{d}{dt}\mathbb{P} = \tfrac{i}{\hbar}\,[\mathbb{P},\mathbb{H}]$$

where $\mathbb{QP} - \mathbb{PQ} = i\hbar$. A further common framework for comparing CM and QM is the category of symplectic manifolds. Other than previous authors, this paper considers phase space of Heisenberg's picture, i.e., the manifold of pairs of operator observables $(\mathbb{Q},\mathbb{P})$ satisfying commutation relation. On a sufficiently high algebraic level of abstraction — which we believe to be of interest on its own — it turns out that this approach leads to a truly *non*linear yet Hamiltonian reformulation of QM evolution.

1 Introduction

QM nowadays is generally accepted as appropriate for describing very small particles and their physical interactions and was put into axiomatic form by von Neumann [17]. Since that time, much research has been spent on structural similarities and differences between QM and Classical Mechanics (CM). The usual approach is to propose a common mathematical category where both theories fit into and to then compare the additional axioms satisfied by either theory. A simple example are the commutation relations among Cartesian position/momentum observables:

$$QP - PQ = 0 \quad \text{in CM}, \qquad [\mathbb{Q},\mathbb{P}] := \mathbb{QP} - \mathbb{PQ} \stackrel{!}{=} i\hbar \quad \text{in QM}. \tag{1}$$

In the category of algebras, classical observables thus form a *commutative* one whereas the quantum case becomes commutative only as $\hbar$ tends to zero: one consequence of Bohr's famous Correspondence Principle.

Another category for comparing CM with QM arises from the respective dynamics $\frac{d}{dt}A = \{A,H\}$ and $\frac{d}{dt}\mathbb{A} = \frac{i}{\hbar}\,[\mathbb{A},\mathbb{H}]$, where we adopted the *Heisenberg picture* that observables $A(t)$ and $\mathbb{A}(t)$ — rather than states — evolve with time. Here $\{\,\cdot\,,\,\cdot\,\}$ denotes the *Poisson bracket* and $[\,\cdot\,,\,\cdot\,]$ the Commutator bracket. Since both brackets satisfy Jacobi's identity, one arrives at a second common category for QM and CM: *Lie Algebras*, cf. [6, 14].

The present work adds to these perspectives a further category by proposing to consider *Hamiltonian systems* in the classical sense, a notion well-known in CM [1], however on abstract operator manifolds. Hamiltonian systems have successfully been generalized from finite to infinite dimensional manifolds [3, 4] and proven to be the key to integrability of many difficult nonlinear partial differential equations. [21, 7]. We recall that if the generator K of an evolutionary equation

$$\tfrac{d}{dt}u(t) \quad = \quad K\big(u(t)\big) \tag{2}$$

is *Hamiltonian* as a vector field[1], then conserved quantities relate to symmetries in the sense of Noether. Thus, in our approach, this relation holds also in case of QM, thereby adding rich structural properties to QM. Our work proceeds in three steps:

- We turn the phase space of Heisenberg's picture into an (infinite dimensional) Banach manifold.
- We consider on this manifold the generator of the Heisenberg dynamics and regard it as purely algebraic an object.
- We show that this abstract object is a *nonlinear* Hamiltonian vector field in a sense similar to symplectic mechanics.

Related Work considered, in order to fit QM into the framework of Hamiltonian systems, the Schrödinger picture, i.e., an evolution on either the set of vectors [3, 13] or on the set of rays (pure states) in a Hilbert Space [12, 5]. This however, leads necessarily to *linear* dynamics. Heisenberg's dynamics, restricted to *spin* space, was already employed in [8] to embed the evolution of spin chains into a Hamiltonian framework. The dynamic focused on in the present work is again Heisenberg's but restricted to *phase* space, i.e., the equation

$$\frac{d}{dt}\begin{pmatrix} \mathbb{Q} \\ \mathbb{P} \end{pmatrix} \quad = \quad \begin{pmatrix} \frac{i}{\hbar}\,[\mathbb{Q},\mathbb{H}] \\ \frac{i}{\hbar}\,[\mathbb{P},\mathbb{H}] \end{pmatrix} \quad =: \quad \mathbb{K}(\mathbb{Q},\mathbb{P}) \tag{3}$$

which is assumed to take place on the set $\mathcal{M}$ of all tuples $(\mathbb{Q},\mathbb{P})$ of selfadjoint Hilbert space operators satisfying, in the sense of Weyl, the canonical commutation relation (1). The question then is: In what sense can this in general nonlinear dynamics be considered as a classical Hamiltonian flow? In fact notice that for instance *an*harmonic potential $\mathbb{H} = \mathbb{P}^2 + \mathbb{Q}^4$ leads to a *non*linear generator $\mathbb{K}(\mathbb{Q},\mathbb{P}) = (2\mathbb{P}, -4\mathbb{Q}^3)$. This does not come to surprise as classical Hamilton equations

$$\frac{d}{dt}\begin{pmatrix} Q \\ P \end{pmatrix} \quad = \quad \begin{pmatrix} \{Q,H\} \\ \{P,H\} \end{pmatrix} \quad =: \quad K(Q,P) \tag{4}$$

become nonlinear, too, for $H(Q,P) = P^2 + Q^4$. Of course [1], the classical Equation (4) is always Hamiltonian, and our distant goal is to show that also (3) is a classical Hamiltonian flow, however on a new manifold of high dimension. Once that aim is reached, then indeed we have revealed a genuine nonlinear aspect of Quantum mechanics.

Overview: As a first step, Section 3 turns the phase space $\mathcal{M}$ into an infinite-dimensional Banach manifold $\mathcal{M}$. However as $\mathbb{Q},\mathbb{P}$ are not both bounded, the usual subtleties arise [3]: weak vs. strong symplectic forms, issues of domains, and so on. We circumvent these difficulties by considering in Section 4, for Hamiltonian operators that depend polynomially on phase space variables (such as for example the above $\mathbb{H} = \mathbb{P}^2 + \mathbb{Q}^4$), the induced generator of Heisenberg's dynamics (3) as purely *algebraic* an object. Theorem 4.3, relying on on a result in combinatorial algebra [15], formally justifies this identification. In fact, [7] revealed the basic properties of usual Hamiltonian systems that relate symmetries to conserved quantities and asserted the complete integrability of so many important nonlinear flows [21, 11, 10] to be expressible in *algebraic* terms only, too. Our major result (Section 5) proves each such abstract generator of Heisenberg's dynamics to be Hamiltonian in this generalized algebraic sense. But let us start with a brief review of some basic notions from dynamical systems and differential geometry.

[1] not to be confused with the Hamiltonian *operator*, i.e., the QM observable for energy…

2 Manifolds, Flows, and Integrability

The present section contains a short introduction to infinite dimensional manifolds, differential equations thereon, and the impact of Hamiltonian generators to the integrability of such equations. This presentation closely follows [4, 7].

Definition 2.1 *Let E, F denote Banach spaces and U some open subset of E. A function $f : U \to F$ is called* differentiable *at $x \in U$ if there exists a continuous linear map $T = T[\cdot] : E \to F$ such*[2] *that*

$$\|f(x+v) - f(x) - T[v]\|/\|v\| \;\to\; 0 \qquad \textit{as} \qquad E \ni v \to 0 \;.$$

In that case, T is unique and denoted $T = f'(x)$.

More generally, it usually suffices for E, F to be *locally convex Hausdorff* rather than Banach vector spaces and f to be *Hadamard*-differentiable, cf. [7]. This allows for a variety of manifolds, e.g., modeled over the space of rapidly decreasing functions $\mathcal{S}$ or equipped with some inductive limit topology. In fact for the following considerations, the exact notion of differentiability is of minor importance as long as it satisfies properties expressible in purely algebraic terms:

- chain rule for differentiation of composite functions;
- product rule for differentiation;
- symmetry of second derivatives;
- and (occasionally) the implicit function theorem.

By means of charts, differentiability is then carried over to functions $f : \mathcal{M} \to \mathcal{N}$ on manifolds $\mathcal{M}$ and $\mathcal{N}$ modeled over E and F, respectively; see [4]. In particular, for a differentiable scalar field $H : \mathcal{M} \to \mathbb{R}$ and $u \in \mathcal{M}$, $I'(u)$ is a continuous linear mapping from tangential space $\mathcal{T}_u\mathcal{M}$ to $\mathbb{R}$, i.e., a covector $\boldsymbol{d}I(u) = I'(u) \in \mathcal{T}_u^*\mathcal{M}$, and $\boldsymbol{d}I : \mathcal{M} \to \mathcal{T}^*\mathcal{M}$ a covector field.

For a vector field $K : \mathcal{M} \to \mathcal{T}\mathcal{M}$, $\mathcal{M} \ni u \mapsto K(u) \in \mathcal{T}_u\mathcal{M}$, consider the general-type evolutionary equation (2). Its solution $t \mapsto u(t)$ for given initial value $u(0)$ is called a *flow* or *integral curve*. In the sequel, in order not to obscure the main ideas by technical details, we shall, for simplicity, assume that K is such that this solution always exists, is unique, and well-behaved (e.g., C^k). Similarly, fields are assumed to be smooth enough such that all occurring derivatives make sense.

Indeed, the basic properties that lead to infinitely many conserved quantities and symmetries for (2) are usually stated in purely algebraic terms [21, 7][3] and a sound analytical foundation is given only later by supplying $\mathcal{M}$ with a suitable topology. Observe that equations of the form (2) cover evolving physical systems ranging from simple pendulum

$$\tfrac{d^2}{dt^2}\varphi + \sin\varphi = 0, \qquad \text{i.e.} \quad \mathcal{M} := \mathbb{R}^2, \quad u := (\varphi, \dot\varphi), \quad K(u) := (\dot\varphi, -\sin\varphi)$$

up to complicated partial differential equations like the one due to Korteweg and de Vries describing one-dimensional long water waves $u = u(x,t)$

$$\partial_t u \quad = \quad 6u \cdot \partial_x u + \partial_x^3 u \quad =: \quad K\big(u\big) \tag{5}$$

on some suitable manifold of functions in one real variable. Solving the latter used to be inherently difficult, even numerically, due to its nonlinearity. The celebrated breakthrough in [21] was to show that (5) possesses an infinite number of conserved quantities related to symmetries by virtue of (a variant of) Noether's theorem.

[2] We adopt the convention that arguments entering linearly are written in square brackets.
[3] See also Section 4 of the present work.

Definition 2.2 *A* conserved quantity *for (2) is a scalar field $I : \mathcal{M} \to \mathbb{R}$ such that*

$$\boldsymbol{d}I[K] : \mathcal{M} \to \mathbb{R}, \qquad u \mapsto I'(u)[K(u)]$$

is identically zero. A symmetry *is a vector field $G : \mathcal{M} \to \mathcal{TM}$ such that the following function vanishes identically:*

$$[\![K, G]\!] : \mathcal{M} \to \mathcal{TM}, \qquad [\![K, G]\!](u) := G'(u)[K(u)] - K'(u)[G(u)] \tag{6}$$

Notice that I is a conserved quantity iff, for each flow $t \mapsto u(t)$ of (2), $t \mapsto I\big(u(t)\big)$ remains constant; cf. PROPOSITION 3.4.2 in [1]. Similarly, G is a *symmetry* iff the one-parameter groups of flows induced by K and G, respectively, commute; cf. e.g. OBSERVATION 2.2 in [7] or THEOREM, P.150 in [4]. We remark that $[\![\cdot, \cdot]\!]$ turns the set of vector fields into a Lie Algebra. Indeed, $G'[K] - K'[G]$ is chart independent (which, e.g., $G'[K]$ only is not), antisymmetric, and satisfies Jacobi's identity (due to chain rule of differentiation and symmetry of second derivatives).

Conserved quantities permit to reduce the dimension of the manifold under consideration. cf. EXERCISE 5.2H in [1] or P.125 in [3]. This explains the notion integrable for systems (2) that exhibit a *complete* collection of conserved quantities/symmetries, see DEFINITION 5.2.20 in [1]. It was therefore quite celebrated when researchers discovered the famous Korteweg-de Vries Equation (5) to be integrable, that is, soluble in a rather explicit and practical sense [21, 16, 10]. As we now know, integrability also applies to a vast number of other important nonlinear partial differential equations such as, e.g., Gardner's, Burger's, nonlinear Schrödinger, and sine Gordon. Furthermore, abstract integrability turned out to be closely related to Hamiltonian structure [11, 7] in a purely algebraic sense. One important aspect of this relation is expressed by the following well-known (variant of a) result due to Emmy Noether:

Meta-Theorem 2.1 *Let K denote a Hamiltonian vector field on $\mathcal{M}$. Then, to every conserved quantity of (2), there corresponds a symmetry.*

Here, a vector field $K : \mathcal{M} \to \mathcal{TM}$ is called *Hamiltonian* if some *symplectic* 2-form $\omega : \mathcal{TM} \times \mathcal{TM} \to \mathbb{R}$, identified with $\omega : \mathcal{TM} \to \mathcal{T}^*\mathcal{M}$, maps it ($K$) to the gradient of a scalar field[4] $H : \mathcal{M} \to \mathbb{R}$, that is, an exact covector field:

$$\omega \circ K : \mathcal{M} \to \mathcal{T}^*\mathcal{M} \qquad \stackrel{!}{=} \qquad \boldsymbol{d}H : \mathcal{M} \to \mathcal{T}^*\mathcal{M} \ ; \tag{7}$$

compare., e.g., P.12 in [3], DEFINITION 5.5.2 in [1], or SECTION VII.A.2 in [4]. Also notice that one of the requirements for ω to be symplectic is that at each $u \in \mathcal{M}$, the linear map $\omega(u) : \mathcal{T}_u\mathcal{M} \to \mathcal{T}_u^*\mathcal{M}$ has a continuous inverse. Therefore, K is uniquely determined by H and ω; more precisely, $K = \omega^{-1}[\boldsymbol{d}H]$ for linear antisymmetric $\omega^{-1}(u) : \mathcal{T}_u^*\mathcal{M} \to \mathcal{T}_u\mathcal{M}$.

As was later observed, the proof of Noether's theorem in fact exploits only algebraic properties (e.g., symmetry of second derivatives or Jacobi's identity). Indeed for $K : \mathcal{M} \to \mathcal{TM}$, the mappings in Definition 2.2 — $\boldsymbol{L}_K : I \mapsto \boldsymbol{d}I[K]$ on the set $\mathcal{F}$ of scalar fields and $\boldsymbol{L}_K : G \mapsto [\![K, G]\!]$ on the set Γ of vector fields — as well as their canonical extensions to the set Γ^* of covector fields and to tensor fields of higher rank according to SECTION III.C.2 in [4], are *derivations* in the algebraic sense; cf. PROPOSITION, P.148 in [4]. In fact, SECTION 4 of [7] and SECTION 2 of [9] gradually stripped down the prerequisites of Theorem 2.1 and found it to holds on a far more abstract level:

[4] usually the *energy* functional associated with the system...

Definition 2.3 *Let* $(\Gamma, \llbracket\cdot,\cdot\rrbracket)$ *denote a Lie algebra and* $\mathcal{F}$ *a vector space — called (abstract)* vector *and* scalar fields*, respectively. Suppose that, for each* $K \in \Gamma$, $\boldsymbol{L}_K : \mathcal{F} \to \mathcal{F}$ *is such that* $K \mapsto \boldsymbol{L}_K$ *is injective and a Lie algebra homomorphism, i.e., satisfies*

$$\boldsymbol{L}_K \boldsymbol{L}_G - \boldsymbol{L}_G \boldsymbol{L}_K \quad = \quad \boldsymbol{L}_{\llbracket K,G \rrbracket} \ . \tag{8}$$

For $H \in \mathcal{F}$ *call the linear map*

$$\boldsymbol{d}H : \Gamma \to \mathcal{F}, \qquad K \mapsto \boldsymbol{L}_K(H) \quad =: \quad \boldsymbol{d}H[K]$$

a (abstract) gradient *or* covector field*; the set of all of them being denoted by* Γ^*.

Now extend, similarly to SECTION III.C.2 *in [4],* Lie derivative $\boldsymbol{L}_K$ *from scalar to vector and covector fields* $G \in \Gamma$ *and* $\boldsymbol{d}H \in \Gamma^*$*, respectively*

$$\boldsymbol{L}_K G := \llbracket K, G \rrbracket, \qquad (\boldsymbol{L}_K \boldsymbol{d}H)[G] := \boldsymbol{L}_K(\boldsymbol{d}H[G]) - \boldsymbol{d}H[\boldsymbol{L}_K G] \tag{9}$$

and finally to tensors of higher rank; cf. EQUATION (2.4) *in [9]. A linear antisymmetric mapping* $\Theta : \Gamma^* \to \Gamma$*, is called* Noetherian *if* $\boldsymbol{L}_{\Theta[\boldsymbol{d}H]}(\Theta) = 0$ *for all* $\boldsymbol{d}H \in \Gamma^*$*. Call* $K \in \Gamma$ Hamiltonian *if* $K = \Theta[\boldsymbol{d}H]$ *for some* $H \in \mathcal{F}$*.*

In the usual setting, $\mathcal{F}$ is the commutative algebra of (sufficiently well-behaved, at least C^1) scalar fields on manifold $\mathcal{M}$, Γ the set of vector fields on $\mathcal{M}$, and Γ^* the set of all (conventional) gradients, i.e., a proper subset of all continuous linear local functionals on Γ. But Definition 2.3 allows for $\boldsymbol{L}_K$ to operate also *non*locally on $\mathcal{F}$; in fact, no underlying manifold is required at all as long as $\mathcal{F}$, Γ, and $\boldsymbol{L}_K$ satisfy algebraic conditions similar to conventional scalar/vector fields and Lie derivatives on some $\mathcal{M}$. Concerning the notion of a Noetherian operator: THEOREM 4.5 in [7] contains six equivalent characterizations for antisymmetric linear $\theta : \Gamma^* \to \Gamma$ to satisfy this requirement. They reveal that "*loosely speaking,* θ *has the algebraic behavior of the inverse of a* symplectic *operator*" [7, P.223]. In other words: rather than imposing (regularity and other) conditions on a $(2,0)$-tensor ω such that $K = \omega^{-1}[\boldsymbol{d}H]$, the last part of Definition 2.3 considers $K = \theta[\boldsymbol{d}H]$ and imposes conditions on the $(0,2)$-tensor θ *directly*. It thus generalizes Equation (7) while avoiding explicit nondegeneracy requirements which, in infinite dimension, become subtly ambiguous (injective/surjective/bijective) anyway. Relevance of these dramatic generalizations is illustrated by, among others[5], the following

Theorem 2.4 *Let* $K = \Theta[\boldsymbol{d}H] \in \Gamma$ *be Hamiltonian and* $I \in \mathcal{F}$ *s.t.* $\boldsymbol{d}I[K] = 0$ *(i.e., an* abstract conserved quantity*). Then,* $G := \Theta[\boldsymbol{d}I] \in \Gamma$ *satisfies* $\llbracket K, G \rrbracket = 0$ *(i.e., is an* abstract symmetry*).*

Proof: See THEOREM 3.3 in [7] or Appendix A in `quant-ph/0210198`. □

This gives late justification why the purely formal manipulations in [21] actually did yield infinitely many conserved quantities in the conventional sense. More precisely, Definition 2.3 and Theorem 2.4 permit to separate algebraic conditions from analytic ones; the latter are, for the nonlinear partial differential equations already mentioned, usually taken care of later by choosing as manifold $\mathcal{M}$ some appropriate function space with a suitable topology.

3 Phase Space Manifold of Heisenberg's Picture

Consider a QM system with one spacial degree of freedom and let $\mathcal{H}$ denote some infinite-dimensional separable Hilbert space. In Heisenberg's picture, phase space

[5] e.g., OBSERVATIONS 5.2 and 5.3 in [7]...

$\mathcal{M}$ consists of all pairs $(\mathbb{Q},\mathbb{P})$ of selfadjoint operators on $\mathcal{H}$ satisfying, in the sense of Weyl, the canonical commutation relation (1).

The below considerations are easily generalized to QM systems with $f > 1$ spacial degrees of freedom, where phase space consists of $2f$-tuples $(\mathbb{Q}_1,\mathbb{P}_1,\ \mathbb{Q}_2,\mathbb{P}_2,\ \ldots,\ \mathbb{Q}_f,\mathbb{P}_f)$ of selfadjoint operators satisfying

$$\mathbb{Q}_k\mathbb{Q}_l = \mathbb{Q}_l\mathbb{Q}_k, \qquad \mathbb{P}_k\mathbb{P}_l = \mathbb{P}_l\mathbb{P}_k, \qquad \mathbb{Q}_k\mathbb{P}_l - \mathbb{P}_l\mathbb{Q}_k = i\hbar\delta_{kl}\ . \tag{10}$$

It is merely for notational convenience that in this section we will focus on the case $f = 1$ and show how to turn the set $\mathcal{M}$ into a Banach manifold in the sense of [4, Section VII.A.1].

To this end recall von Neumann's celebrated result that each such pair $(\mathbb{Q},\mathbb{P})$ is unitarily equivalent to a fixed pair $(\mathbb{Q}_0,\mathbb{P}_0)$; cf. e.g. Theorem 4.3.1 in [18] or Theorem VIII.14 in [19]: $\mathbb{Q} = \mathbb{U}\mathbb{Q}_0\mathbb{U}^* \ \wedge\ \mathbb{P} = \mathbb{U}\mathbb{P}_0\mathbb{U}^*$ for some unitary $\mathbb{U}$. We explicitly disallow multiplicities/direct sums because systems with *one* degree of freedom correspond to *irreducible* representations of Schrödinger couples. Since conversely, every pair $(\mathbb{U}\mathbb{Q}_0\mathbb{U}^*, \mathbb{U}\mathbb{P}_0\mathbb{U}^*)$ does satisfy (1), it suffices to consider the set $\mathcal{U}(\mathcal{H})$ of all unitary operators on $\mathcal{H}$. We assert that this set actually indeed *is* a manifold.

Theorem 3.1 [6] *The set $\mathcal{U}(\mathcal{H})$ of unitary operators on a separable Hilbert space $\mathcal{H}$ is a real C^∞ Banach manifold.*

4 Polynomials in Operator-Variables

In CM, the phase space[7] manifold $\mathcal{M}$ consists of canonical position/momentum variables (Q,P), and the Hamilton function depends smoothly (say, rationally or polynomially) on these variables.

In Heisenberg's picture of QM, the phase space[7] manifold $\mathcal{M}$ consists of canonical position/momentum observables, that is, of tuples $(\mathbb{Q},\mathbb{P})$ of selfadjoint Hilbert space operators satisfying, in the sense of Weyl, commutation relations (1). This time, the Hamilton function is an operator-valued function of such tuples namely a Hamiltonian operator like $\mathbb{H}(\mathbb{Q},\mathbb{P}) = \mathbb{Q}^4 + \mathbb{P}^2$.

We will in the sequel focus on Hamiltonians depending *polynomially* on $\mathbb{Q}$ and $\mathbb{P}$, and the aim of this section is to make this notion mathematically sound. In algebra, polynomials $p \in \mathbb{C}[Q,P]$ in two variables are of course well-defined objects. The following four definitions are known to be equivalent:

Definition 4.1 *The set $\mathbb{C}[X_1,\ldots,X_m]$ of* polynomials *over $\mathbb{C}$ in m (commutating) variables is*

a) the free commutative $\mathbb{C}$-algebra generated by $\{X_1,\ldots,X_m\}$;

b) the set of finite unbounded sequences (namely the coefficients preceding monomials) with convolution as product;

c) the smallest family of mappings $\hat{p} : \mathbb{C}^m \to \mathbb{C}$ containing constants and projections $(x_1,\ldots,x_m) \mapsto x_j$ and being closed under addition and multiplication;

d) the collection of all differentiable mappings $\hat{p} : \mathbb{C}^m \to \mathbb{C}$ for which differentiation is nilpotent, i.e. for some k, the k-th derivative vanishes[8].

[6] Due to a page limit, proofs had to be moved into an appendix available at `quant-ph/0210198`.
[7] of systems with one spacial degree of freedom
[8] Recall that, according to complex analysis, any differentiable function on $\mathbb{C}$ is necessarily C^∞.

While appropriate for classical (i.e., commuting) observables, this type of polynomials however does not reflect the non-commutativity in general exhibited by quantum observables. Instead consider a definition of polynomials in *non*commuting variables similar to a):

Definition 4.2 *The set* $\mathbb{C}\langle X_1,\ldots,X_m\rangle$ *of* polynomials *over* $\mathbb{C}$ *in non-commuting variables is the free non-commutative (but associative and distributive)* $\mathbb{C}$*-algebra generated by* $\{X_1,\ldots,X_m\}$*. A* monomial *in* $\mathbb{C}\langle X_1,\ldots,X_m\rangle$ *of* degree d *is of the form* $\prod_{n=1}^{d} X_{k_n}$ *with* $k\in\{1,\ldots,m\}^d$*.*

As each such polynomial is obviously a linear combination of finitely many monomials and vice versa, one easily obtains an equivalent characterization in terms of coefficient sequences similar to Definition 4.1b); the convolution just doesn't look as nice any more. But how about analogues to c) and d), that is, a way to identify polynomials with certain differentiable mappings on *quantum* observables?

Of course for some $\mathbb{C}$-algebra $\mathcal{A}$ — such as the set of selfadjoint linear operators on some Hilbert space — every $p\in\mathbb{C}\langle X_1,\ldots,X_m\rangle$ gives rise to a mapping $\hat{p}:\mathcal{A}^m\to\mathcal{A}$ where $\hat{p}(A_1,\ldots,A_m)$ is defined by substituting X_j with B_j. But is $\hat{p}$ differentiable? Moreover, $p\mapsto\hat{p}$ is a homomorphism; but in order to *identify* p with $\hat{p}$, this homomorphism should be injective! As one can easily imagine, this heavily depends on $\mathcal{A}$ to be sufficiently rich; for example the two polynomials in two non-commuting variables $p,q\in\mathbb{C}\langle X,Y\rangle$ with $p:=X\cdot Y$ and $q:=Y\cdot X$ differ whereas $\hat{p}$ and $\hat{q}$ agree on $\mathcal{A}:=\mathbb{C}$. Similarly, $p:=(X\cdot Y-Y\cdot X)^2\cdot Z$ and $q:=Z\cdot(X\cdot Y-Y\cdot X)^2$ from $\mathcal{A}:=\mathbb{C}\langle X,Y,Z\rangle$ satisfy, according to *Hall's Identity*, $\hat{p}=\hat{q}$ on the algebra $\mathcal{A}$ of 2×2 matrices.

This section's main result asserts that already the set of compact symmetric linear operators on infinite-dimensional Hilbert space $\mathcal{H}$ is sufficiently rich to identify polynomials with polynomial mappings. Furthermore on bounded linear operators, these polynomial mappings are differentiable.

Theorem 4.3 *Let* $\mathcal{H}$ *denote some separable infinite-dimensional Hilbert space,* $p,q\in\mathbb{C}\langle X_1,\ldots,X_m\rangle$*.*

a) *Let* $\mathcal{A}$ *be a set of linear operators on* $\mathcal{H}$ *containing at least the compact symmetric ones. Then* $\hat{p}\big|_{\mathcal{A}^m}=\hat{q}\big|_{\mathcal{A}^m}$ *implies* $p=q$*.*

b) *Let* $\mathcal{B}$ *be the Banach algebra of bounded linear operators on* $\mathcal{H}$*. Then* $\hat{p}:\mathcal{B}^m\to\mathcal{B}$ *is differentiable.*

c) *Its derivative* $\hat{p}'(\mathbb{A}_1,\ldots,\mathbb{A}_m)[\mathbb{V}_1,\ldots,\mathbb{V}_m]$ *is the* $\widehat{\ }$*-transform of some unique polynomial in* $2m$ *non-commuting variables and thus differentiable as well.*

We may thus — and will from now on — use *polynomial* (in non-commuting variables) and *polynomial mapping* synonymously. By virtue of part c), every polynomial is C^∞, and one may write $p'(X_1,\ldots,X_m)[V_1,\ldots,V_m]\in\mathbb{C}\langle X_1,\ldots,X_m;V_1,\ldots,V_m\rangle$ for the derivative of $p\in\mathbb{C}\langle X_1,\ldots,X_m\rangle$.

As next step one has to take into account the commutation relation (1) satisfies by quantum phase space observables. Indeed, the polynomials $p:=QP-PQ$ and $q:=i\hbar$ are different in $\mathbb{C}\langle P,Q\rangle$ whereas for position/momentum observables $\mathbb{Q}/\mathbb{P}$ it holds $\mathbb{QP}-\mathbb{PQ}=i\hbar$. One therefore wants to identify polynomials $\sum_{k=1}^{K}p_k\cdot(QP-PQ-i\hbar)\cdot q_k$ in $\mathbb{C}\langle Q,P\rangle$ with 0 while maintaining the structure of an algebra such as being closed under addition and multiplication.

Fortunately, exactly this is offered by the *quotient algebra*

$$\begin{aligned}\mathbb{C}\langle Q,P\rangle/\mathcal{J} \;&:=\; \Big\{p/\mathcal{J}:p\in\mathbb{C}\langle Q,P\rangle\Big\}, \qquad p/\mathcal{J}:=\{p+q:q\in\mathcal{J}\},\\ (p_1/\mathcal{J})+(p_2/\mathcal{J})\cdot(p_3/\mathcal{J}) \;&:=\; (p_1+p_2\cdot p_3)/\mathcal{J}\end{aligned} \tag{11}$$

where $\mathcal{J}$ denotes some appropriate ideal. Recall that a (two-sided) ideal is a subset of an algebra which is closed under addition and closed under multiplication (both left and right) with arbitrary elements not only from $\mathcal{J}$ but from the whole algebra. In our case, take the ideal spanned by $QP-PQ-i\hbar \in \mathbb{C}\langle Q,P\rangle$, that is, the smallest[9] ideal containing $QP-PQ-i\hbar$; explicitly:

$$\mathcal{J} \quad = \quad \Big\{ \sum_{k=1}^{K} p_k \cdot (QP-PQ-i\hbar) \cdot q_k \ : \ K \in \mathbb{N}_0,\ p_k, q_k \in \mathbb{C}\langle Q,P\rangle \Big\} \ .$$

Considering $\mathbb{C}\langle Q,P\rangle/\mathcal{J}$ rather than $\mathbb{C}\langle Q,P\rangle$, it now holds $\mathbb{QP}-\mathbb{PQ}=i\hbar$ for elements $\mathbb{Q} := Q/\mathcal{J}$ and $\mathbb{P} := P/\mathcal{J}$. Let us abbreviate $\mathbb{C}\langle \mathbb{Q},\mathbb{P}\rangle := \mathbb{C}\langle Q,P\rangle/\mathcal{J}$ and remark that $\mathbb{Q},\mathbb{P}$ like $Q,P \in \mathbb{C}\langle Q,P\rangle$, are in some sense not 'variables' but very specific and purely algebraic objects. On the other hand recall that by virtue of the above considerations, each element $\mathbb{H}$ from $\mathbb{C}\langle Q,P\rangle/\mathcal{J}$ does give rise to and can be identified with a mapping $\hat{\mathbb{H}}$ on the phase space manifold of all pairs of quantum position/momentum observables. We are thus led to the following

Definition 4.4 *The algebra $\mathcal{F} := \mathbb{C}\langle \mathbb{Q},\mathbb{P}\rangle$ is called the set of* abstract scalar fields *on quantum phase space.*

5 Hamiltonian Heisenberg's Dynamics

We will now use and extend the purely algebraic approach from Section 4 to prove that the generator of Heisenberg's dynamics is Hamiltonian in the sense of Definition 2.3.

A first step, the set $\mathcal{F}$ of abstract scalar fields has already been introduced in Definition 4.4. This formalized the class Hamiltonian operators of interest: polynomials $\mathbb{H}=\mathbb{H}(\mathbb{Q},\mathbb{P})$ in phase space variables $(\mathbb{Q},\mathbb{P})=u$. According to Definition 2.3, next we need is a Lie algebra Γ to serve as (abstract) vector fields, i.e., containing generators $\mathbb{K}=\mathbb{K}(u)$ of a dynamics (2) on phase space manifold $\mathcal{M}=\big\{(\mathbb{Q},\mathbb{P}) : \mathbb{QP}-\mathbb{PQ}=i\hbar\big\}$. In order for corresponding flows $t\mapsto u(t)=\big(\mathbb{Q}(t),\mathbb{P}(t)\big)$ to stay on $\mathcal{M}$, $\mathbb{K}=(\mathbb{K}_Q,\mathbb{K}_P)$ must not alter commutation relation (1), i.e.,

$$\begin{aligned} 0 \quad &\overset{!}{=} \quad \tfrac{d}{dt}0 \quad = \quad \tfrac{d}{dt}\big(\mathbb{Q}(t)\mathbb{P}(t)-\mathbb{P}(t)\mathbb{Q}(t)-i\hbar\big) \\ &\overset{(*)}{=} \quad \big(\tfrac{d}{dt}\mathbb{Q}(t)\big)\mathbb{P}(t)+\mathbb{Q}(t)\big(\tfrac{d}{dt}\mathbb{P}(t)\big)-\big(\tfrac{d}{dt}\mathbb{P}(t)\big)\mathbb{Q}(t)-\mathbb{P}(t)\big(\tfrac{d}{dt}\mathbb{Q}(t)\big) \\ &\overset{(2)}{=} \quad \mathbb{K}_Q\mathbb{P}+\mathbb{Q}\mathbb{K}_P-\mathbb{K}_P\mathbb{Q}-\mathbb{P}\mathbb{K}_Q \quad = \quad [\mathbb{K}_Q,\mathbb{P}\underline{]}\ -\ [\mathbb{K}_P,\mathbb{Q}\underline{]} \end{aligned} \qquad (12)$$

where at $(*)$ we used the product rule of differentiation and exploited that $\hbar$ does not vary over time. The algebraic excerpt of these considerations is subsumed in

Definition 5.1 *The set of* abstract vector fields *on quantum phase space is*

$$\Gamma \quad = \quad \big\{\mathbb{K}=(\mathbb{K}_Q,\mathbb{K}_P) \ : \ \mathbb{K}_Q,\mathbb{K}_P \in \mathbb{C}\langle\mathbb{Q},\mathbb{P}\rangle\ ,\ [\mathbb{K}_Q,\mathbb{P}\underline{]}\ =\ [\mathbb{K}_P,\mathbb{Q}\underline{]}\ \big\}$$

For $\mathbb{K}=(K_Q/\mathcal{J},K_P/\mathcal{J})\in\Gamma$ *and* $\mathbb{H}=H/\mathcal{J}\in\mathcal{F}$ *with* $K_Q,K_P,H\in\mathbb{C}\langle Q,P\rangle$,

$$\boldsymbol{L}_{\mathbb{K}}\mathbb{H} \quad := \quad \big(H'[(K_Q,K_P)]\big)/\mathcal{J} \quad \in \quad \mathcal{F}$$

denotes the Lie derivative *of $\mathbb{H}$ with respect to $\mathbb{K}$. Finally equip Γ with the bracket*

$$[\![\mathbb{K},\mathbb{G}]\!] \quad := \quad \big(\boldsymbol{L}_{\mathbb{K}}\mathbb{G}_Q-\boldsymbol{L}_{\mathbb{G}}\mathbb{K}_Q,\boldsymbol{L}_{\mathbb{K}}\mathbb{G}_P-\boldsymbol{L}_{\mathbb{G}}\mathbb{K}_P\big) \ .$$

These structures indeed comply with the requirements in Definition 2.3:

[9] reflecting that *no other* identifications than $QP-PQ=i\hbar$ are to be made...

Theorem 5.2 *$\boldsymbol{L}_{\mathbb{K}}\mathbb{H}$ in Definition 5.1 is well-defined. $[\![\cdot,\cdot]\!]$ turns the vector fields Γ into a Lie algebra. $\mathbb{K} \mapsto \boldsymbol{L}_{\mathbb{K}}$ is an injective Lie algebra homomorphism.*

Remember our goal to find a Hamiltonian formulation for the generator $\mathbb{K} := \big(\frac{i}{\hbar}\,[\mathbb{Q},\mathbb{H}]\,,\frac{i}{\hbar}\,[\mathbb{P},\mathbb{H}]\,\big)$ of Heisenberg's dynamics on phase space (3). To this end, one needs some Noetherian $\Theta : \Gamma^* \to \Gamma$ such that $\mathbb{K}$ as $\Theta[\boldsymbol{d}\mathbb{H}]$. Compare this to the situation in CM (4) where the generator is well-known [1] to have a Hamiltonian formulation $K = \theta[\boldsymbol{d}H]$ by means of $\theta := \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$. Indeed when identifying gradients (i.e., covectors) with vectors, θ is obviously linear, antisymmetric, and even constant hence $\boldsymbol{L}_K\theta = 0$ for any K. Moreover the well-known *Darboux theorem* states that conversely *every* symplectic tensor on a $2f$-dimensional manifold is of the form $\begin{pmatrix} \mathbb{O}_f & +\mathbb{I}_f \\ -\mathbb{I}_f & \mathbb{O}_f \end{pmatrix}$ at least locally, where $\mathbb{O}_f$, $\mathbb{I}_f$ denote the zero and identity $(f \times f)$-matrix, respectively.

Of course on *in*finite-dimensional manifolds, covectors from Γ^* cannot in general be identified with vectors from Γ. But still the following consideration conveys the idea that turns out to carry over to our algebraic setting where $\Theta : \Gamma^* \to \Gamma$ need not be bijective. To this end observe that a two-dimensional covector w^* on $\mathbb{R}^2$, i.e., a linear function $w^* : \mathbb{R}^2 \to \mathbb{R}$, is identified with a vector $w \in \mathbb{R}^2$ via

$$w \quad = \quad \big(\, w^*[(1,0)]\,,\; w^*[(0,1)]\,\big) \quad \in \quad \mathbb{R}^2 \;\text{, i.e.,}$$

by evaluating w^* at arguments $(1,0)$ and $(0,1)$ forming the canonical basis for $\mathbb{R}^2$.

Definition 5.3 *For abstract covector field $\mathbb{W}^* \in \Gamma^*$, let*

$$\Theta[\mathbb{W}^*] \quad := \quad \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{W}^*[(1/\mathcal{J}, 0/\mathcal{J})] \\ \mathbb{W}^*[(0/\mathcal{J}, 1/\mathcal{J})] \end{pmatrix} \quad \in \quad \Gamma$$

Theorem 5.4 *$\Theta : \Gamma^* \to \Gamma$ is well-defined and Noetherian. Furthermore for each abstract scalar field $\mathbb{H} \in \mathcal{F}$, the (thus Hamiltonian) vector field $\Theta[\boldsymbol{d}\mathbb{H}]$ coincides with $\mathbb{K}$ according to (3).*

Although Θ resembles the classical θ, the proof in Appendix E proceeds entirely different. In fact already the antisymmetry of Θ is far from obvious and heavily relies on $\mathcal{F}$ being the quotient algebra with respect to $\mathcal{J}$.

6 Conclusion

We showed that, for Hamiltonian operators that depend polynomially on observables $\mathbb{Q}$ and $\mathbb{P}$, Heisenberg's dynamics on phase space is Hamiltonian at least in an *abstract algebraic* sense. This constitutes an important step and in fact can serve as a guide towards a Hamiltonian formulation of QM dynamics as *analytical* flow on a *concrete* manifold like the one considered in Section 3. In contrast to previous works, the *non*linearity of our approach gives, in connection with Noether's theorem, rise to interesting nontrivial symmetries which deserve further investigation.

For ease of notation, the presentation focused on systems with $f = 1$ spacial degree of freedom. In fact our considerations also apply to the general case $f \in \mathbb{N}$. Here, phase space $\mathcal{M}$ consists of all $2f$-tuples of Cartesian position/momentum observables $(\mathbb{Q}_1, \mathbb{P}_1, \ldots, \mathbb{Q}_f, \mathbb{P}_f)$ satisfying commutation relations (10). Correspondingly for the set of abstract scalar fields (polynomial mappings on $\mathcal{M}$), we now choose $\mathcal{F} = \mathbb{C}\langle\mathbb{Q}_1, \mathbb{P}_1, \ldots, \mathbb{Q}_f, \mathbb{P}_f\rangle$, i.e., the quotient algebra $\mathbb{C}\langle Q_1, \ldots, P_f\rangle/\mathcal{J}$ with respect to the ideal $\mathcal{J}$ spanned by

$$\big\{ Q_kQ_l - Q_lQ_k\;,\; P_kP_l - P_lP_k\;,\; Q_kP_l - P_lQ_k - i\hbar\delta_{kl} \;:\; 1 \le k,l \le f \big\}\;.$$

Abstract covector fields $\mathbb{K} \in \Gamma$ thus become $2f$-tuples $\mathbb{K} = (\mathbb{K}_{q1}, \ldots, \mathbb{K}_{pf})$ s.t.

$$\left.\begin{array}{rcl} \mathbb{K}_{qk}\mathbb{Q}_l + \mathbb{Q}_k\mathbb{K}_{ql} & = & \mathbb{Q}_l\mathbb{K}_{qk} + \mathbb{K}_{ql}\mathbb{Q}_k \\ \mathbb{K}_{pk}\mathbb{P}_l + \mathbb{P}_k\mathbb{K}_{pl} & = & \mathbb{P}_l\mathbb{K}_{pk} - \mathbb{K}_{pl}\mathbb{P}_k \\ \mathbb{K}_{qk}\mathbb{P}_l + \mathbb{Q}_k\mathbb{K}_{pl} & = & \mathbb{P}_l\mathbb{K}_{qk} - \mathbb{K}_{pl}\mathbb{Q}_k \end{array}\right\} \qquad \forall 1 \leq k, l \leq f$$

where again time-independence of Planck's constant entered. Finally for $\mathbb{W}^* \in \Gamma^*$,

$$\Theta[\mathbb{W}^*] \quad := \quad \begin{pmatrix} \mathbb{O}_f & +\mathbb{I}_f \\ -\mathbb{I}_f & \mathbb{O}_f \end{pmatrix} \cdot \begin{pmatrix} \mathbb{W}^*[(1/\mathcal{J}, 0/\mathcal{J}, \ldots, 0/\mathcal{J})] \\ \mathbb{W}^*[(0/\mathcal{J}, 1/\mathcal{J}, \ldots, 0/\mathcal{J})] \\ \vdots \\ \mathbb{W}^*[(0/\mathcal{J}, 0/\mathcal{J}, \ldots, 1/\mathcal{J})] \end{pmatrix} \quad \in \quad \Gamma$$

is the abstract Noetherian tensor.

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A Postponed Proof of Theorem 2.4

First notice that the gradient of a conserved quantity is an invariant covector field:

$$\boldsymbol{d}I[K]=0 \qquad \Longrightarrow \quad \boldsymbol{L}_K(\boldsymbol{d}I)=0 \ .$$

Indeed for any $G\in\Gamma$,

$$\boldsymbol{L}_K(\boldsymbol{d}I)[G] \overset{(9)}{=} \boldsymbol{L}_K(\underbrace{\boldsymbol{d}I[G]}_{=\boldsymbol{L}_G I}) - \boldsymbol{d}I[\underbrace{\boldsymbol{L}_K G}_{[\![K,G]\!]}] \ = \ \boldsymbol{L}_K\boldsymbol{L}_G I - \boldsymbol{L}_{[\![K,G]\!]} I \overset{(8)}{=} \boldsymbol{L}_G \underbrace{\boldsymbol{L}_K I}_{\boldsymbol{d}=0}$$

Thus, according to Observation 2.1 in [9],

$$[\![K,G]\!] \ = \ \boldsymbol{L}_K(\theta\boldsymbol{d}I) \ \overset{!}{=} \ \theta\boldsymbol{L}_K(\boldsymbol{d}I) \ = \ 0 \ .$$

B Postponed Proof of Theorem 3.1

Let the reader be reminded that an operator $\mathbb{U}$ on $\mathcal{H}$ is called *unitary* iff it is
a) linear and bounded, b) invertible, and c) satisfies $\mathbb{U}\mathbb{U}^*=\mathbb{I}$.

Now the set $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is, equipped with operator norm, of course a Banach algebra and thus in particular a (flat, C^∞) manifold. Similarly, the set $\mathcal{S}(\mathcal{H})$ of all *symmetric* bounded linear operators is a (real!) Banach space and hence a manifold as well. Let $\mathcal{B}(\mathcal{H})^+$ denote the set of *invertible* bounded linear operators. This subset is known to be open in $\mathcal{B}(\mathcal{H})$ and therefore also constitutes a manifold, cf. e.g. [20, Theorem 10.11]; similarly, $\mathcal{S}(\mathcal{H})^+$ is open in $\mathcal{S}(\mathcal{H})$ and therefore a manifold, too.

So it holds $\mathcal{U}(\mathcal{H})=f^{-1}(\mathbb{I})$ for $f:\mathcal{B}(\mathcal{H})^+\to\mathcal{S}(\mathcal{H})^+$, $\mathbb{A}\mapsto\mathbb{A}\mathbb{A}^*$. We are going to show that f is in fact a *submersion* on $\mathcal{U}(\mathcal{H})$.

Definition B.1 *Let X, Y denote Banach manifolds. Consider a C^1 mapping $f: X\to Y$ and $S=f^{-1}(c)\subseteq X$ for some $c\in Y$. Call f a* submersion *on S if for all $x\in S$, $f'(x):T_xX\to T_{f(x)}Y$ is surjective and has a complementable kernel.*

Lemma B.2 *In that case, S is a submanifold of X.*

Proof: See page 550 in [4]. □

Recall that a closed subspace M of a topological vector space E is called *complementable* if there exists a closed subspace N of E such that $E=M+N$ and $M\cap N=\{0\}$; cf. e.g. [20, Definition 4.20].

Lemma B.3 *$f:\mathcal{B}(\mathcal{H})^+\to\mathcal{S}(\mathcal{H})^+$, $\mathbb{A}\mapsto\mathbb{A}\mathbb{A}^*$ is continuously differentiable with derivative $f'(\mathbb{A})[\mathbb{V}]=\mathbb{V}\mathbb{A}^*+\mathbb{A}\mathbb{V}^*$.*

For $\mathbb{U}\in\mathcal{U}(\mathcal{H})$, the linear map $f'(\mathbb{U}):\mathcal{B}(\mathcal{H})\to\mathcal{S}(\mathcal{H})$ is surjective; its kernel $\mathcal{N}=\big\{\mathbb{B}\in\mathcal{B}(\mathcal{H}):\mathbb{U}\mathbb{B}^*+\mathbb{B}\mathbb{U}^*=0\big\}$ is complemented by $\mathcal{M}:=\big\{\mathbb{B}\in\mathcal{B}(\mathcal{H}): \mathbb{U}\mathbb{B}^*-\mathbb{B}\mathbb{U}^*=0\big\}$.

Proof: Straight-forward calculation yields differentiability of f:

$$f(\mathbb{A}+\mathbb{V})-f(\mathbb{A})-(\mathbb{V}\mathbb{A}^*+\mathbb{A}\mathbb{V}^*) \quad = \quad \mathbb{V}\mathbb{V}^*$$

tends to zero as $\mathbb{V}\to 0$, even when divided by $\|\mathbb{V}\|$.

For $\mathbb{S}\in\mathcal{S}(\mathcal{H})$, let $\mathbb{V}:=\frac{1}{2}\mathbb{S}\mathbb{U}\in\mathcal{B}(\mathcal{H})$. Then, using $\mathbb{V}^*=\frac{1}{2}\mathbb{U}^*\mathbb{S}^*$, $\mathbb{S}^*=\mathbb{S}$, and $\mathbb{U}\mathbb{U}^*=\mathbb{I}$, it follows that $f'(\mathbb{U})[\mathbb{V}]=\mathbb{S}$; hence $f'(\mathbb{U})$ is surjective.

$\mathcal{N}\cap\mathcal{M}=\{0\}$ is trivial. To show $\mathcal{N}+\mathcal{M}=\mathcal{B}(\mathcal{H})$, consider $\mathbb{B}\in\mathcal{B}(\mathcal{H})$; now verify that $(\mathbb{B}-\mathbb{U}\mathbb{B}^*\mathbb{U})/2\in N$ and $(\mathbb{B}+\mathbb{U}\mathbb{B}^*\mathbb{U})/2\in M$. Since the sum of both yields $\mathbb{B}$, this concludes the proof. □

C Postponed Proof of Theorem 4.3

For a) one may presume w.l.o.g. that $q = 0$. Let d denote the degree of $p \neq 0$. According to, e.g., [15] there exist symmetric $(\lfloor \frac{d}{2}+1 \rfloor \times \lfloor \frac{d}{2}+1 \rfloor)$-matrices[10] matrices $A_1, \ldots, A_m$ such that $\hat{p}(A_1, \ldots, A_m) \neq 0$. By extending the linear mappings A_j from $\mathbb{C}^{\lfloor d/2+1 \rfloor}$ to $\mathcal{H}$, the obtained symmetric compact operators still satisfy $\hat{p}(\mathbb{A}_1, \ldots, \mathbb{A}_m) \neq 0$.

For b) and c), we are going to algebraically *define* a mapping $\mathbb{C}\langle\vec{X}\rangle \ni p \mapsto p' \in \mathbb{C}\langle\vec{X};\vec{V}\rangle$ and verify that its image under $\widehat{\ }$ coincides with the derivative of $\hat{p}$. As the latter is unique on Hausdorff spaces, this proves the claim.

Definition C.1 *Write $\vec{X} = (X_1, \ldots, X_m)$ and $\vec{V} = (V_1, \ldots, V_m)$. For monomial $p = \prod_{n=1}^{d} X_{k_n} \in \mathbb{C}\langle\vec{X}\rangle$, its* partial derivative *with respect to X_l is given by*

$$\frac{\partial p}{\partial X_l}[V_l] \quad := \quad \sum_{n:k_n=l} \Big(\prod_{s<n} X_{k_s}\Big) \cdot V_l \cdot \Big(\prod_{s>n} X_{k_s}\Big) \quad \in \quad \mathbb{C}\langle\vec{X};V_l\rangle$$

The partial derivative of a linear combination of monomials is the linear combination of their respective partial derivatives. The derivative *and* second derivative *of a polynomial $p = p(\vec{X}) \in \mathbb{C}\langle\vec{X}\rangle$ are given by*

$$p' \;=\; p'(\vec{X})[\vec{V}] \;=\; p'[\vec{V}] \quad := \quad \sum_{l=1}^{m} \frac{\partial p}{\partial X_l}[V_l] \quad \in \quad \mathbb{C}\langle\vec{X};\vec{V}\rangle$$

$$p''(\vec{X})[\vec{V},\vec{W}] \;:=\; \sum_{l=1}^{m} \frac{\partial}{\partial X_l}\big(p'(\vec{X})[\vec{V}]\big)[W_l] \quad \in \quad \mathbb{C}\langle\vec{X};\vec{V},\vec{W}\rangle$$

respectively.

It's easy to verify the following properties:

Lemma C.2 *Let $p, q \in \mathbb{C}\langle\vec{X}\rangle$, $\alpha \in \mathbb{C}$.*

a) Linearity

$$(\alpha p + q)'(\vec{X})[\vec{V}] \;=\; \alpha p'(\vec{X})[\vec{V}] + q'(\vec{X})[\vec{V}]$$

b) Non-commutative product rule

$$(p \cdot q)'(\vec{X})[\vec{V}] \;=\; p(\vec{X})[\vec{V}] \cdot q'(\vec{X}) + q'(\vec{X})[\vec{V}] \cdot q(\vec{X})$$

c) Symmetry of second derivatives $\quad p''[\vec{V},\vec{W}] = p''[\vec{W},\vec{V}]$

d) Chain rule: Let $\vec{q} = (q_1, \ldots, q_m) \in \mathbb{C}\langle\vec{X}\rangle^m$. Then

$$p\big(\vec{q}(\vec{X})\big)'[\vec{V}] \;=\; p'\big(\vec{q}(\vec{X})\big)\big[\vec{q}'(\vec{X})[\vec{V}]\big]$$

with $\vec{q}' = (q_1', \ldots, q_m')$. In particular,

$$\big(p'[\vec{q}]\big)'[\vec{V}] \;=\; p''[\vec{q},\vec{V}] + p'\big[\vec{q}'[\vec{V}]\big] \;. \tag{13}$$

The first two items say that $p \mapsto p'$ is sort of a derivation. Now finally coming to Claims b) and c), it suffices to consider monomials; the rest follows from linearity of differentiation.

[10] The famous Amitsur-Levitzki-Theorem states that this matrix dimension is in fact optimal.

Let us first remark that the $\widehat{\ }$-transform of each polynomial p is a *continuous* map $\hat{p}:\mathcal{B}^m\to\mathcal{B}$. Indeed, p is a finite linear combination of products of projections $(\mathbb{A}_1,\ldots,\mathbb{A}_m)\mapsto\mathbb{A}_j$; the latter are continuous, and so are products of continuous functions because the operator norm $\|\cdot\|$ on $\mathcal{B}$ is submultiplicative.

The proof that $\hat{p}:\mathcal{B}^m\to\mathcal{B}$ is differentiable for every monomial $p\in\mathbb{C}\langle\vec{X}\rangle$ proceeds by easy induction on the degree d of p, being obvious for $d\leq 1$. For induction step $d\mapsto d+1$ let $p\cdot q$ be the product of two monomials p,q of degree at most d each. By induction hypothesis, both $\hat{p}$ and $\hat{q}$ are differentiable with respective directional derivatives $\widehat{p'}(\vec{\mathbb{A}})[\vec{\mathbb{V}}]$ and $\widehat{q'}(\vec{\mathbb{A}})[\vec{\mathbb{V}}]$, $\vec{\mathbb{A}},\vec{\mathbb{V}}\in\mathcal{B}^m$. Recall that this means that $\hat{p}(\mathbb{A}+\mathbb{V})-\hat{p}(\mathbb{A})-\widehat{p'}(\mathbb{A})[\mathbb{V}]$ tends to 0 even when divided by $\|\mathbb{V}\|\to 0$ and similarly for $\hat{q}$. We want to show that $\widehat{p\cdot q'+p'\cdot q}$ is the derivative of $\widehat{p\cdot q}$. Because of $\widehat{p\cdot q}=\hat{p}\cdot\hat{q}$, it indeed follows

$$\widehat{p\cdot q}(\mathbb{A}+\mathbb{V})\;-\;\widehat{p\cdot q}(\mathbb{A})\;-\;\widehat{p'}(\mathbb{A})[\mathbb{V}]\cdot\hat{q}(\mathbb{A}+\mathbb{V})\;-\;\hat{p}(\mathbb{A})\cdot\widehat{q'}(\mathbb{A})[\mathbb{V}]\quad=$$
$$\underbrace{\big(\hat{p}(\mathbb{A}+\mathbb{V})-\hat{p}(\mathbb{A})-\widehat{p'}(\mathbb{A})[\mathbb{V}]\big)}_{\bullet/\|\mathbb{V}\|\to 0\text{ as }\mathbb{V}\to 0}\cdot\underbrace{\hat{q}(\mathbb{A}+\mathbb{V})}_{\to\hat{q}(\mathbb{A})\text{ as }\mathbb{V}\to 0}+\hat{p}(\mathbb{A})\cdot\underbrace{\big(\hat{q}(\mathbb{A}+\mathbb{V})-\hat{q}(\mathbb{A})-\widehat{q'}(\mathbb{A})[\mathbb{V}]\big)}_{\bullet/\|\mathbb{V}\|\to 0\text{ as }\mathbb{V}\to 0}$$

since $\hat{q}$ is continuous. As $\|\cdot\|$ satisfies subadditivity and submultiplicativity, not only the indicated factors but the whole expression tends to 0 even when divided by $\|\mathbb{V}\|\to 0$. This shows that $p\cdot q$ is differentiable and its derivative is the $\widehat{\ }$-transform of $p(\vec{X})\cdot q'(\vec{X})[\vec{V}]+p'(\vec{X})[\vec{V}]\cdot q(\vec{X})\in\mathbb{C}\langle\vec{X};\vec{V}\rangle$ which completes the induction step and eventually proves Claims b) and c). □

D Postponed Proof of Theorem 5.2

Remember that, for a (two-sided) ideal $\mathcal{J}$ in some (non-commutative) algebra $\mathcal{A}$, the relation $A\equiv_{\mathcal{J}} B \;:\Leftrightarrow\; A-B\in\mathcal{J}$ satisfies

$$A\equiv_{\mathcal{J}} B\quad\Rightarrow\quad A+C\equiv_{\mathcal{J}} B+C\quad\wedge\quad A\cdot C\equiv_{\mathcal{J}} B\cdot C\quad\wedge\quad C\cdot A\equiv_{\mathcal{J}} C\cdot B$$

for $A,B,C\in\mathcal{A}$. Equivalently: (11) is well-defined. A representative for $\mathbb{A}\in\mathcal{A}/\mathcal{J}$ is some $A\in\mathcal{A}$ such that $\mathbb{A}=A/\mathcal{J}$

Well-definition of $\boldsymbol{L}_{\mathbb{K}}\mathbb{H}$ means independence of the representatives $H\in\mathbb{C}\langle Q,P\rangle$ for $\mathbb{H}\in\mathbb{C}\langle\mathbb{Q},\mathbb{P}\rangle$ and similarly $(K_Q,K_P)=\vec{K}$ for $(\mathbb{K}_Q,\mathbb{K}_P)=\mathbb{K}\in\Gamma$. So suppose $\mathbb{H}=0$ and we have to show that $H'[(K_Q,K_P)]\equiv_{\mathcal{J}} 0$ for each $H\equiv_{\mathcal{J}} 0$. Indeed linearity allows to presume w.l.o.g. $H=p\cdot(QP-PQ-i\hbar)\cdot q$ for some $p,q\in\mathbb{C}\langle Q,P\rangle$. Then Lemma C.2b) yields $H'[\vec{K}]\;=$

$$\underbrace{p'[\vec{K}]\cdot(QP-PQ-i\hbar)\cdot q}_{\equiv_{\mathcal{J}} 0}+p\cdot\underbrace{(QP-PQ-i\hbar)'[\vec{K}]}_{=K_QP+QK_P-K_PQ-PK_Q}\cdot q+\underbrace{p\cdot(QP-PQ-i\hbar)\cdot q'[\vec{K}]}_{\equiv_{\mathcal{J}} 0}$$

and the middle term is $\equiv_{\mathcal{J}} 0$ as well because (K_Q,K_P), being a representative for $(\mathbb{K}_Q,\mathbb{K}_P)\in\Gamma$, satisfies $[K_Q,P]+[Q,K_P]\equiv_{\mathcal{J}} 0$ according to Definition 5.1.

Derivatives in direction of vector fields thus basically 'commute' with taking factors w.r.t. $\mathcal{J}$:

$$(H/\mathcal{J})'[\vec{K}/\mathcal{J}]\quad=\quad\big(H'[\vec{K}]\big)/\mathcal{J}\qquad\text{for }(\vec{K})/\mathcal{J}\in\Gamma$$

For *partial* derivatives, this does in general not hold: Take $H_1:=QP-PQ$, $H_2=i\hbar$, and $V:=Q$; then $H_1/\mathcal{J}=H_2/\mathcal{J}$ but

$$\Big(\frac{\partial H_1}{\partial Q}[V]\Big)/\mathcal{J}\;=\;(QP-PQ)/\mathcal{J}\;=\;(i\hbar)/\mathcal{J}\quad\neq\quad 0/\mathcal{J}\;=\;\Big(\frac{\partial H_2}{\partial Q}[V]\Big)/\mathcal{J}\;.$$

If however $VP \equiv PV$, then $(V,0)/\mathcal{J}$ belongs to Γ and $\left(\frac{\partial H}{\partial Q}[V]\right)/\mathcal{J} = \left(H'[(V,0)]\right)/\mathcal{J}$ is independent of H as some representative for $H/\mathcal{J}$; same for $\left(\frac{\partial H}{\partial P}[V]\right)/\mathcal{J} = \left(H'[(0,V)]\right)/\mathcal{J}$ whenever $VQ \equiv QV$. In particular for $V = 1$, we therefore have

Lemma D.1 *Let $\mathbb{H} \in \mathcal{F}$ with $\mathbb{H} = H/\mathcal{J}$. Then*

$$\frac{\partial \mathbb{H}}{\partial \mathbb{Q}} \quad := \quad \left(\frac{\partial H}{\partial Q}[1]\right)/\mathcal{J}, \qquad \frac{\partial \mathbb{H}}{\partial \mathbb{P}} \quad := \quad \left(\frac{\partial H}{\partial P}[1]\right)/\mathcal{J}$$

is well-defined. Furthermore it holds

$$\frac{i}{\hbar}\,[\mathbb{H},\mathbb{Q}] \quad = \quad +\frac{\partial \mathbb{H}}{\partial \mathbb{P}}, \qquad \frac{i}{\hbar}\,[\mathbb{H},\mathbb{P}] \quad = \quad -\frac{\partial \mathbb{H}}{\partial \mathbb{Q}} \tag{14}$$

Condition (12) for $(\mathbb{K}_Q, \mathbb{K}_P) \in \Gamma$ may thus be rewritten as

$$\frac{\partial \mathbb{K}_Q}{\partial \mathbb{Q}} \quad = \quad -\frac{\partial \mathbb{K}_P}{\partial \mathbb{P}}$$

which resembles the Cauchy-Riemann equation for the complex function $f(q+ip) = k_q(q,p) - ik_p(q,p)$ to be differentiable.

Proof of Lemma D.1: Let $\mathbb{H} = H/\mathcal{J} \in \mathcal{F}$. For linearity reasons, it suffices to prove $\frac{i}{\hbar}\,[\mathbb{H},\mathbb{Q}] = \frac{\partial \mathbb{H}}{\partial \mathbb{P}}$ for monomials H. We proceed by induction on the degree of H, cases $H = 1$, $H = Q$, and $H = P$ being obvious. So let $H = H_1 \cdot H_2$ with monomials H_1, H_2 of lower degree, $\mathbb{H}_1 = H_1/\mathcal{J}$, $\mathbb{H}_2 = H_2/\mathcal{J}$. Then

$$\begin{aligned}
\frac{\partial \mathbb{H}}{\partial \mathbb{P}} \quad &= \quad \left(\frac{\partial (H_1 \cdot H_2)}{\partial P}[1]\right)/\mathcal{J} \\
&\overset{\text{C.2}b}{=} \quad \left(\frac{\partial H_1}{\partial P}[1] \cdot H_2 + H_1 \cdot \frac{\partial H_2}{\partial P}[1]\right)/\mathcal{J} \quad \overset{(11)}{=} \quad \frac{\partial \mathbb{H}_1}{\mathbb{P}} \cdot \mathbb{H}_2 + \mathbb{H}_1 \cdot \frac{\partial \mathbb{H}_2}{\mathbb{P}} \\
&\overset{(*)}{=} \quad \tfrac{i}{\hbar}\,[\mathbb{H}_1,\mathbb{Q}] \cdot \mathbb{H}_2 + \mathbb{H}_1 \cdot \tfrac{i}{\hbar}\,[\mathbb{H}_2,\mathbb{Q}] \quad = \quad \tfrac{i}{\hbar}\,[\mathbb{H}_1 \cdot \mathbb{H}_2,\mathbb{Q}]
\end{aligned}$$

where at $(*)$ the inductive presumption entered. □

Next claim is that $[\![\mathbb{K},\mathbb{G}]\!]$ belongs to Γ for $\mathbb{K},\mathbb{G} \in \Gamma$. To this end, take corresponding representatives (K_Q, K_P) and (G_Q, G_P) — which ones doesn't matter as we have just shown — and verify that the representative $\big(G'_Q[K] - K'_Q[G], G'_P[K] -$

$K'_P[G]\big)$ for $[\![\mathbb{K},\mathbb{G}]\!]$ satisfies

$$
\begin{aligned}
&\big[G'_Q[K]-K'_Q[G],P\underline{\big]} \;+\; [Q,G'_P[K]-K'_P[G]\underline{]} \\
\overset{\text{C.1}}{=}\quad & \left[\frac{\partial G_Q}{\partial Q}[K_Q],P\underline{\right]} + \left[\frac{\partial G_Q}{\partial P}[K_P],P\underline{\right]} - \left[\frac{\partial K_Q}{\partial Q}[G_Q],P\underline{\right]} - \left[\frac{\partial K_Q}{\partial P}[G_P],P\underline{\right]} \\
+\quad & \left[Q,\frac{\partial G_P}{\partial Q}[K_Q]\underline{\right]} + \left[Q,\frac{\partial G_P}{\partial P}[K_P]\underline{\right]} - \left[Q,\frac{\partial K_P}{\partial Q}[G_Q]\underline{\right]} - \left[Q,\frac{\partial K_P}{\partial P}[G_P]\underline{\right]} \\
\overset{\text{C.2}b}{=}\quad & \frac{\partial}{\partial Q}\big(\,[G_Q,P\underline{]}\,\big)[K_Q] \;+\; \frac{\partial}{\partial P}\big(\,[G_Q,P\underline{]}\,\big)[K_P] - [G_Q,K_P\underline{]} \\
-\quad & \frac{\partial}{\partial Q}\big(\,[K_Q,P\underline{]}\,\big)[G_Q] \;-\; \frac{\partial}{\partial P}\big(\,[K_Q,P\underline{]}\,\big)[G_P] + [K_Q,G_P\underline{]} \\
+\quad & \frac{\partial}{\partial Q}\big(\,[Q,G_P\underline{]}\,\big)[K_Q] - [K_Q,G_P\underline{]} \;+\; \frac{\partial}{\partial P}\big(\,[Q,G_P\underline{]}\,\big)[K_P] \\
-\quad & \frac{\partial}{\partial Q}\big(\,[Q,K_P\underline{]}\,\big)[G_Q] + [G_Q,K_P\underline{]} \;-\; \frac{\partial}{\partial P}\big(\,[Q,K_P\underline{]}\,\big)[G_P] \\
\overset{\text{C.2}a}{=}\quad & \frac{\partial}{\partial Q}\big(\,[G_Q,P\underline{]} + [Q,G_P\underline{]}\,\big)[K_Q] + \frac{\partial}{\partial P}\big(\,[G_Q,P\underline{]} + [Q,G_P\underline{]}\,\big)[K_P] \\
-\quad & \frac{\partial}{\partial Q}\big(\,[K_Q,P\underline{]} + [Q,K_P\underline{]}\,\big)[G_Q] - \frac{\partial}{\partial P}\big(\,[K_Q,P\underline{]} + [Q,K_P\underline{]}\,\big)[G_P] \\
\overset{\text{C.1}}{=}\quad & \big(\underbrace{[G_Q,P\underline{]} + [Q,G_P\underline{]}}_{\equiv_{\mathcal{J}}0}\big)'[K] - \big(\underbrace{[K_Q,P\underline{]} + [Q,K_P\underline{]}}_{\equiv_{\mathcal{J}}0}\big)'[G] \quad \equiv_{\mathcal{J}} \quad 0 \;.
\end{aligned}
$$

Indeed, Lemma C.2b) implies

$$
\frac{\partial}{\partial Q}\,[A,B\underline{]}\,[V] \quad = \quad \left[\frac{\partial A}{\partial Q}[V],B\underline{\right]} + \left[A,\frac{\partial B}{\partial Q}[V]\underline{\right]}
$$

and for $B=Q$, the last term is equal to $[A,V\underline{]}$ whereas it vanishes for $B=P$.

The mapping $\Gamma\ni\mathbb{K}\mapsto\boldsymbol{L}_{\mathbb{K}}$ is a Lie algebra homomorphism because, for $\mathbb{K}=K/\mathcal{J},\mathbb{G}=G/\mathcal{J}\in\Gamma$, and $H=H/\mathcal{J}\in\mathcal{F}$,

$$
\begin{aligned}
\boldsymbol{L}_{\mathbb{K}}\boldsymbol{L}_{\mathbb{G}}-\boldsymbol{L}_{\mathbb{G}}\boldsymbol{L}_{\mathbb{K}}:\mathbb{H} \quad &\mapsto \quad \Big(\big(H'[G]\big)'[K]-\big(H'[K]\big)'[G]\Big)/\mathcal{J} \\
&\underset{(13)}{\overset{\text{C.2}d)}{=}} \quad \Big(H''\big[G,K\big]+H'\big[G'[K]\big]-H''\big[K,G\big]-H'\big[K'[G]\big]\Big)/\mathcal{J} \\
&\underset{a+c}{\overset{\text{C.2}}{=}} \quad \Big(H'\big[G'[K]-K'[G]\big]\Big)/\mathcal{J} \quad = \quad \boldsymbol{L}_{[\![\mathbb{K},\mathbb{G}]\!]}\mathbb{H} \;.
\end{aligned}
$$

This is furthermore injective as can be seen by evaluating $\boldsymbol{L}_{\mathbb{K}}\mathbb{H}=_{\mathbb{G}}\mathbb{H}$ on $\mathbb{H}:=\mathbb{Q}$ and on $\mathbb{H}:=\mathbb{P}$. In particular, $[\![\,\cdot\,,\,\cdot\,]\!]$ satisfies antisymmetry and Jacobi's identity.

E Postponed Proof of Theorem 5.4

Let us first emphasize the importance of ideal $\mathcal{J}$ by omitting it, that is, by considering

$$
\tilde{\Theta}[W^*] \quad := \quad \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} W^*[(1,0)] \\ W^*[(0,1)] \end{pmatrix} \qquad \text{on} \qquad \tilde{\mathcal{F}}=\mathbb{C}\langle Q,P\rangle \;.
$$

This linear mapping from $\tilde{\Gamma}^* \to \tilde{\Gamma}$ is, in spite of its suggestive writing, not even antisymmetric: For $H := Q^2 \in \tilde{\mathcal{F}}$ and $F := PQP \in \tilde{\mathcal{F}}$,

$$
\begin{aligned}
&\boldsymbol{d}F\big[\tilde{\Theta}[\boldsymbol{d}H]\big] + \boldsymbol{d}H\big[\tilde{\Theta}[\boldsymbol{d}F]\big] \\
&\quad = \quad \boldsymbol{d}F\big[(0,-2Q)\big] + \boldsymbol{d}H\big[(QP+PQ,-P^2)\big] \\
&\quad = \quad \big(-2Q^2P - 2PQ^2\big) + \big(Q(QP+PQ) + (QP+PQ)Q\big) \\
&\quad = \quad 2QPQ - Q^2P - PQ^2 \quad \neq \quad 0\ .
\end{aligned}
$$

Now returning to the proof of Theorem 5.4, $\mathbb{W}^*\big[(1/\mathcal{J}, 0/\mathcal{J})\big]$ is well-defined because $\mathbb{K} = (1/\mathcal{J}, 0/\mathcal{J})$ satisfies $[\mathbb{K}_Q, \mathbb{P}]_- = [\mathbb{K}_P, \mathbb{Q}]_-$, thus belongs to Γ on which $\mathbb{W}^* : \Gamma \to \mathcal{F}$ operates.

Next we exploit that according to Definition 2.3, Γ^* consists of abstract gradients (i.e., *closed* covector fields) only. Namely to show $\Theta[\boldsymbol{d}\mathbb{H}] \in \Gamma$, take $\mathbb{H} = H/\mathcal{J}$ and compute for $K = (K_Q, K_P) := \tilde{\Theta}[\boldsymbol{d}H]$

$$
\begin{aligned}
[K_Q, P]_- + [Q, K_P]_- \quad &= \quad \left[\frac{\partial H}{\partial P}[1], P\right]_- - \left[Q, \frac{\partial H}{\partial Q}[1]\right]_- \\
&\overset{\text{D.1}}{\mathrel{\hbar\!\!\!\!=}} \quad \left[\tfrac{i}{\hbar}\,[H,Q]_-\,, P\right]_- + \left[Q, \tfrac{i}{\hbar}\,[H,P]_-\,\right]_- \\
&\overset{(*)}{=} \quad -\left[\tfrac{i}{\hbar}\,[Q,P]_-\,, H\right]_- \quad \mathrel{\hbar\!\!\!\!=} \quad [1,H]_- \quad = \quad 0
\end{aligned}
$$

where at $(*)$, Jacobi's identity was used:

$$
[\,[A,B]_-\,, C]_- \;+\; [\,[B,C]_-\,, A]_- \;+\; [\,[B,A]_-\,, C]_- \quad = \quad 0\ .
$$

Θ obviously satisfies linearity; for proving that it is furthermore antisymmetric and Noetherian, the following tool turns out to be quite useful:

Proposition E.1 *For $m,n,M,N \in \mathbb{N}_0$, $\alpha,\beta \in \mathbb{C}$, $A := \alpha P^m Q^n$ and $B := \beta P^M Q^N$ satisfy*

$$
\frac{\partial A}{\partial Q}\Big[\frac{\partial B}{\partial P}[1]\Big] \quad \mathrel{\hbar\!\!\!\!=} \quad \frac{\partial B}{\partial P}\Big[\frac{\partial A}{\partial Q}[1]\Big]\ .
$$

Notice that both terms in general coincide only with respect to the relation $\mathrel{\hbar\!\!\!\!=}$ induced by identifying $QP - PQ$ with $i\hbar$; consider, e.g., $A = Q^2$ and $B = P^3$. Furthermore, the particular form of A and B (with all P's to the left and Qs to the right) is important; consider, e.g., $A = Q^2$ and $B = PQP$. The latter results from the fact that, say, $\frac{\partial B}{\partial P}[1]$ is usually not the (representative of the) first component of a vector field and thus partial derivative $\frac{\partial A}{\partial Q}$ in this direction not necessarily independent of changing representatives for A.

Nevertheless, Proposition E.1 helps calculating, for $\mathbb{F}, \mathbb{H} \in \mathcal{F}$, $\boldsymbol{d}\mathbb{F}\big[\Theta[\boldsymbol{d}\mathbb{H}]\big] + \boldsymbol{d}\mathbb{H}\big[\Theta[\boldsymbol{d}\mathbb{F}]\big] = 0$. Indeed, one may presume w.l.o.g. that $\mathbb{F} = F/\mathcal{J}$ and $\mathbb{H} = H/\mathcal{J}$ for $F = P^m Q^n$ and $H = P^M Q^N$ because of (bi-)linearity and since every monomial in $\mathcal{H}$ can be brought to this form; cf. Lemma F.1. Then,

$$
\begin{aligned}
&\boldsymbol{d}\mathbb{F}\big[\Theta[\boldsymbol{d}\mathbb{H}]\big] \;+\; \boldsymbol{d}\mathbb{H}\big[\Theta[\boldsymbol{d}\mathbb{F}]\big] \\
&\overset{5.3}{=} \quad \frac{\partial F}{\partial Q}\Big[\frac{\partial H}{\partial P}[1]\Big]/\mathcal{J} \;-\; \frac{\partial F}{\partial P}\Big[\frac{\partial H}{\partial Q}[1]\Big]/\mathcal{J} \;+\; \frac{\partial H}{\partial Q}\Big[\frac{\partial F}{\partial Q}[1]\Big]/\mathcal{J} \;-\; \frac{\partial H}{\partial P}\Big[\frac{\partial F}{\partial P}[1]\Big]/\mathcal{J} \\
&\overset{\text{E.1}}{=} \quad 0 \;+\; 0\ .
\end{aligned}
$$

We now prove that the generalization of usual Poisson brackets

$$
(\mathbb{F}, \mathbb{H}) \;\mapsto\; \boldsymbol{d}\mathbb{H}[\Theta\boldsymbol{d}\mathbb{F}]
$$

turns the abstract scalar fields into a Lie algebra. According to [7, THEOREM 4.5], this is equivalent (among others) to Θ being Noetherian and furthermore to

$$\Theta\boldsymbol{d}\big[\boldsymbol{d}\mathbb{H}[\Theta\boldsymbol{d}\mathbb{F}]\big] \quad = \quad [\![\Theta\boldsymbol{d}\mathbb{F},\Theta\boldsymbol{d}\mathbb{H}]\!]$$

for all closed covectors $\boldsymbol{d}\mathbb{F},\boldsymbol{d}\mathbb{H}\in\Gamma^*$; cf. EQUATION (2.10) in [9]. So let, again without loss of generality, $\mathbb{F}=F/\mathcal{J}$ and $\mathbb{H}=H/\mathcal{J}$ with $F=P^mQ^n$ and $H=P^MQ^N$. Consider $A:=\frac{\partial H}{\partial Q}\Big[\frac{\partial F}{\partial P}[1]\Big]-\frac{\partial H}{\partial P}\Big[\frac{\partial F}{\partial Q}[1]\Big]$, that is, $A/\mathcal{J}=\boldsymbol{d}\mathbb{H}[\Theta\boldsymbol{d}\mathbb{F}]$; then the first component of $(\mathbb{K}_Q,\mathbb{K}_P)=\Theta\boldsymbol{d}\big[\boldsymbol{d}\mathbb{H}[\Theta\boldsymbol{d}\mathbb{F}]\big]$ equals

$$\begin{aligned}
\mathbb{K}_Q \quad &\overset{5.3}{=} \quad \frac{\partial}{\partial P}A[1]/\mathcal{J}\\
&\underset{(13)}{\overset{\text{C.2}d}{=}} \quad \frac{\partial^2 H}{\partial P\partial Q}\Big[1,\frac{\partial F}{\partial P}[1]\Big]/\mathcal{J} \;+\; \frac{\partial H}{\partial P}\Big[\frac{\partial}{\partial Q}\Big(\frac{\partial F}{\partial P}[1]\Big)[1]\Big]/\mathcal{J}\\
&\quad - \quad \frac{\partial^2 H}{\partial P\partial P}\Big[1,\frac{\partial F}{\partial Q}[1]\Big]/\mathcal{J} \;-\; \frac{\partial H}{\partial P}\Big[\frac{\partial}{\partial P}\Big(\frac{\partial F}{\partial Q}[1]\Big)[1]\Big]/\mathcal{J}\\
&\overset{(*)}{=} \quad \frac{\partial}{\partial Q}\Big(\frac{\partial H}{\partial P}[1]\Big)\Big[\frac{\partial F}{\partial P}[1]\Big]/\mathcal{J} \;+\; \frac{\partial H}{\partial Q}\Big[\frac{\partial}{\partial P}\underbrace{\Big(\frac{\partial F}{\partial P}[1]\Big)}_{=mP^{m-1}Q^n=:B}[1]\Big]/\mathcal{J}\\
&\quad - \quad \frac{\partial}{\partial P}\Big(\frac{\partial H}{\partial P}[1]\Big)\Big[\frac{\partial F}{\partial Q}[1]\Big]/\mathcal{J} \;-\; \frac{\partial H}{\partial P}\Big[\frac{\partial}{\partial Q}\underbrace{\Big(\frac{\partial F}{\partial P}[1]\Big)}_{=B}[1]\Big]/\mathcal{J}\\
&\overset{\text{E.1}}{=} \quad \frac{\partial}{\partial Q}\Big(\frac{\partial H}{\partial P}[1]\Big)\Big[\frac{\partial F}{\partial P}[1]\Big]/\mathcal{J} \;+\; \frac{\partial}{\partial P}\Big(\frac{\partial F}{\partial P}[1]\Big)\Big[\frac{\partial H}{\partial Q}[1]\Big]/\mathcal{J}\\
&\quad - \quad \frac{\partial}{\partial P}\Big(\frac{\partial H}{\partial P}[1]\Big)\Big[\frac{\partial F}{\partial Q}[1]\Big]/\mathcal{J} \;-\; \frac{\partial}{\partial Q}\Big(\frac{\partial F}{\partial P}[1]\Big)\Big[\frac{\partial H}{\partial P}[1]\Big]/\mathcal{J}\\
&\overset{5.3}{=} \quad \Big(\frac{\partial\mathbb{H}}{\partial P}[1]\Big)'[\Theta\boldsymbol{d}\mathbb{F}] \;-\; \Big(\frac{\partial\mathbb{F}}{\partial P}[1]\Big)'[\Theta\boldsymbol{d}\mathbb{H}]
\end{aligned}$$

which is the first component of $[\![\Theta[\boldsymbol{d}\mathbb{F}],\Theta[\boldsymbol{d}\mathbb{H}]]\!]$; that second components agree as well can be verified quite similarly. Let us emphasize that at $(*)$, we used

$$\frac{\partial}{\partial X_k}\Big(\frac{\partial H}{\partial X_l}[1]\Big)[V] \underset{(13)}{\overset{\text{C.2}d}{=}} \frac{\partial^2 H}{\partial X_k\partial X_l}[V,1] \;+\; \Big(\frac{\partial H}{\partial X_l}[1]\Big)'\Big[\underbrace{\frac{\partial 1}{\partial X_k}[V]}_{=0}\Big]$$

for $H,V\in\mathbb{C}\langle X_1,\dots,X_m\rangle$.

That $\Theta[\boldsymbol{d}\mathbb{H}]$ agrees with $\mathbb{K}$ according to (3) follows from Lemma D.1.

F Postponed Proof of Proposition E.1

First thing to notice is that because of linearity, one may presume $\alpha=\beta=1$.

Next, the claim follows from $A=Q^n$ and $B=P^M$ via induction. Indeed, once it holds for A and B, we have for $\tilde{B}=B\cdot Q$:

$$\frac{\partial(B\cdot Q)}{\partial P}\Big[\frac{\partial A}{\partial Q}[1]\Big] \overset{\text{C.2}b}{=} \frac{\partial B}{\partial P}\Big[\frac{\partial A}{\partial Q}[1]\Big]\cdot Q \overset{\text{I.H.}}{=} \frac{\partial A}{\partial Q}\Big[\frac{\partial B}{\partial P}[1]\Big]\cdot Q \overset{(*)}{=} \frac{\partial A}{\partial Q}\Big[\frac{\partial B}{\partial P}[1]\cdot Q\Big]$$

where at $(*)$ we used

$$\frac{\partial A}{\partial Q}[H\cdot Q] \;\overset{\text{C.1}}{=}\; \sum_{k=1}^{n} P^mQ^{k-1}(H\cdot\underbrace{Q)Q^{n-k}}_{=Q^{n-k}\cdot Q} \;=\; \sum_{k=1}^{n} P^mQ^{k-1}HQ^{n-k}\cdot Q \;=\; \frac{\partial A}{\partial Q}[H]\cdot Q\;;$$

the induction step proceeds similarly for $\tilde{A} = P \cdot A$.

It thus remains to prove

$$\frac{\partial(Q^n)}{\partial Q}\big[m \cdot P^{m-1}\big] \quad \overset{\hbar}{=} \quad \frac{\partial(P^m)}{\partial P}\big[n \cdot Q^{n-1}\big] \tag{15}$$

for any $n, m \in \mathbb{N}$. To this end, we need the commutation properties of $\mathbb{Q}^n$ and $\mathbb{P}^m$. For $n = 1 = m$, they are revealed by (1); and based on that, induction yields:

Lemma F.1 *Consider $\mathbb{Q}^n, \mathbb{P}^m \in \mathcal{F}$. Then*

$$\mathbb{Q}^n\mathbb{P} - \mathbb{P}\mathbb{Q}^n \; = \; n\hbar i \mathbb{Q}^{n-1}, \qquad \mathbb{Q}\mathbb{P}^m - \mathbb{P}^m\mathbb{Q} \; = \; m\hbar i \mathbb{P}^{m-1},$$

$$\mathbb{Q}^n\mathbb{P}^m - \mathbb{P}^m\mathbb{Q}^n \; = \; \sum_{r=1}^{\min(n,m)} \binom{m}{r}\binom{n}{r} r!(i\hbar)^r \mathbb{P}^{m-r}\mathbb{Q}^{n-r}$$

With the agreement that $\binom{k}{r} = 0$ for $k < r$, we may omit the minimum and let the sum range up to n or to m whatever seems preferable. Now turning to the proof of (15):

$$\begin{aligned}
&\frac{\partial(Q^n)}{\partial Q}\big[m \cdot P^{m-1}\big] \; - \; \frac{\partial(P^m)}{\partial P}\big[n \cdot Q^{n-1}\big] \\
&= \; m \sum_{k=1}^{n} \underbrace{Q^{k-1}P^{m-1}}_{\overset{\text{F.1}}{\overset{\hbar}{=}} \; P^{m-1}Q^{k-1} + \sum_{r=1}^{m-1}\binom{m-1}{r}\binom{k-1}{r} r!(i\hbar)^r P^{m-1-r}Q^{k-1-r}} Q^{n-k} \\
&- \; n \sum_{l=1}^{m} P^{l-1} \underbrace{Q^{n-1}P^{m-l}}_{\overset{\text{F.1}}{\overset{\hbar}{=}} \; P^{m-l}Q^{n-1} + \sum_{s=1}^{n-1}\binom{n-1}{s}\binom{m-l}{s} s!(i\hbar)^s P^{m-l-s}Q^{n-1-s}}
\end{aligned}$$

$$\begin{aligned}
\overset{\hbar}{=} \quad & \underbrace{m \sum_{k=1}^{n} P^{m-1}Q^{n-1} - n \sum_{l=1}^{m} P^{m-1}Q^{n-1}}_{=0} \\
+ \quad & m \sum_{r=1}^{m-1}\sum_{k=1}^{n} \binom{m-1}{r}\binom{k-1}{r} r!(i\hbar)^r P^{m-1-r}Q^{n-1-r} \\
- \quad & n \sum_{s=1}^{n-1}\sum_{l=1}^{m} \binom{n-1}{s}\binom{m-l}{s} s!(i\hbar)^s P^{m-1-s}Q^{n-1-s} \quad .
\end{aligned}$$

Here, both sum ranges for r and s may be cut off at $\min(m-1, n-1)$ since for higher indices, the corresponding binomial coefficients are zero anyway. Collecting term thus yields

$$\sum_{t=1}^{\min(n,m)-1} \Big(m \cdot \binom{m-1}{t} \sum_{k=1}^{n} \binom{k-1}{t} - n \cdot \binom{n-1}{s} \sum_{l=1}^{m} \binom{m-l}{t}\Big) \; t!(i\hbar)^t P^{m-1-t}Q^{n-1-t}$$

which vanishes because the well-known properties of binomial coefficients

$$\sum_{j=0}^{J} \binom{j}{N} \overset{(*)}{=} \binom{J+1}{N+1} \qquad \text{and} \qquad \binom{J}{N} \underset{(*)}{=} \frac{J}{N} \cdot \binom{J-1}{N-1}$$

yield

$$
\begin{aligned}
m \cdot \binom{m-1}{t} \sum_{k=1}^{n} \binom{k-1}{t} &\overset{(*)}{=} m \cdot \binom{m-1}{t} \cdot \binom{n}{t+1} \\
&\underset{(*)}{=} m \cdot \binom{m-1}{t} \cdot \frac{n}{t+1} \cdot \binom{n-1}{t}
\end{aligned}
$$

$$
\begin{aligned}
n \cdot \binom{n-1}{t} \sum_{l=1}^{m} \binom{m-l}{t} &\overset{(*)}{=} n \cdot \binom{n-1}{t} \cdot \binom{m}{t+1} \\
&\underset{(*)}{=} n \cdot \binom{n-1}{t} \cdot \frac{m}{t+1} \cdot \binom{m-1}{t} \ . \quad \square
\end{aligned}
$$