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Exact MHD solutions with crystallographic symmetries and non-interacting Fourier modes

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Exact periodic MHD solutions with crystallographic symmetries and arbitrary vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are derived. Complete classification of periodic solutions with pairwise non-interacting Fourier modes is obtained.

Keywords: Periodic solutions; Crystallographic groups; Non-interacting Fourier modes; Viscous MHD; Reciprocal lattice

1 Introduction

The viscous magnetohydrodynamics equations have the form [1]:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho\mu}\operatorname{curl}\mathbf{B} \times \mathbf{B} + \nu\Delta\mathbf{V}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \eta\Delta\mathbf{B}, \tag{2}$$

$$\operatorname{div}\mathbf{V} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{3}$$

where $\mathbf{V}(t, \mathbf{x})$ is the fluid velocity vector field, $\mathbf{B}(t, \mathbf{x})$ is magnetic field and $p(t, \mathbf{x})$ is the pressure, vector $\mathbf{x} = (x_1, x_2, x_3)$. We suppose that the density ρ, kinematic viscosity ν, permuability μ and resistivity η are constant. The

Navier-Stokes equations (NSE) form a special case of the viscous MHD equations (1) - (3) for $\mathbf{B}(t, \mathbf{x}) = 0$. The theory of weak solutions to the NSE is developed by Leray [2]. Classical exact solutions to the NSE are presented by Batchelor [3].

The standard 2π-periodic solutions with orthogonal vector periods $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ were considered for the NSE in [4, 5, 6]. We study the more general class of periodic solutions with arbitrary linearly independent vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. The MHD equations are reduced to a dynamical system for Fourier components; we show that the dynamical systems for different triples of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ generically are not equivalent to each other. This is true also for any equations containing the Laplace operator of velocity $\mathbf{V}(t, \mathbf{x})$, for example for the vector diffusion equations.

We construct exact solutions with crystallographic symmetry groups G that have pure rotational point groups $\Gamma \subset SO(3)$. There are 52 such groups G among 219 nonisomorphic crystallographic groups in three dimensions. The point group Γ can be either cyclic C_n, or dihedral D_n, $n = 2, 3, 4, 6$, or tetrahedral T or octahedral group O. The constructed exact solutions depend on all four variables t, x_1, x_2, x_3.

We work out the problem of classification of periodic solutions with pairwise non-interacting Fourier modes. The solutions have no transfer of energy through the spectrum. The obtained classification is independent of the periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, however the resulting exact solutions have different properties. The classification is complete and hence proves that for any solution outside of the constructed four (for $\nu \neq \eta$) infinite series of invariant submanifolds there exists necessarily a non-zero interaction between its Fourier modes.

2 Exact MHD solutions with arbitrary periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$

I. The periodicity conditions have the form $(i = 1, 2, 3)$

$$\mathbf{V}(t, \mathbf{x}+\mathbf{p}_i) = \mathbf{V}(t, \mathbf{x}),\ \mathbf{B}(t, \mathbf{x}+\mathbf{p}_i) = \mathbf{B}(t, \mathbf{x}),\ \nabla p(t, \mathbf{x}+\mathbf{p}_i) = \nabla p(t, \mathbf{x}). \quad (4)$$

The integral linear combinations of vectors $\mathbf{p}_i$ form a lattice Λ. Consider vectors

$$\mathbf{k}_1 = \lambda \mathbf{p}_2 \times \mathbf{p}_3, \quad \mathbf{k}_2 = \lambda \mathbf{p}_3 \times \mathbf{p}_1, \quad \mathbf{k}_3 = \lambda \mathbf{p}_1 \times \mathbf{p}_2, \quad (5)$$

where $\lambda = 2\pi[\mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)]^{-1}$. It is evident that

$$\mathbf{k}_i \cdot \mathbf{p}_j = 2\pi\delta_{ij}, \tag{6}$$

and $\mathbf{p}_i = 2\pi\lambda^{-1}\mathbf{k}_j \times \mathbf{k}_\ell$. The vectors $\mathbf{k}_i$ (5) generate a reciprocal lattice Λ^*. Periodic solutions with periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are represented by Fourier series

$$\mathbf{V}(t,\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda^*} \mathbf{V}_\mathbf{k}(t)\exp(i\mathbf{k}\cdot\mathbf{x}), \quad \mathbf{B}(t,\mathbf{x}) = \sum_{\mathbf{k}\in\Lambda^*} \mathbf{B}_\mathbf{k}(t)\exp(i\mathbf{k}\cdot\mathbf{x}), \tag{7}$$

$$\nabla p(t,\mathbf{x}) = \mathbf{p}_0(t) + i\sum_{\mathbf{k}\in\Lambda^*} p_\mathbf{k}(t)\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x}),$$

where summation is taken over all vectors $\mathbf{k}$ of the reciprocal lattice Λ^*, $\mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_3\mathbf{k}_3$, $n_i \in \mathbb{Z}$. The incompressibility equations (3) yield

$$\mathbf{V}_\mathbf{k}\cdot\mathbf{k} = 0, \quad \mathbf{B}_\mathbf{k}\cdot\mathbf{k} = 0. \tag{8}$$

For real functions $\mathbf{V}(t,\mathbf{x})$, $\mathbf{B}(t,\mathbf{x})$ and $p(t,\mathbf{x})$, the Fourier components $\mathbf{V}_\mathbf{k}, \mathbf{B}_\mathbf{k} \in \mathbb{C}^3$, and $p_\mathbf{k} \in \mathbb{C}$ satisfy the equations $\mathbf{V}_{-\mathbf{k}} = \overline{\mathbf{V}_\mathbf{k}}$, $\mathbf{B}_{-\mathbf{k}} = \overline{\mathbf{B}_\mathbf{k}}$, $p_{-\mathbf{k}} = \overline{p_\mathbf{k}}$. Vectors $\mathbf{V}_0(t)$ and $\mathbf{p}_0(t)$ are real.

II. First we construct the periodic vector fields that satisfy the Beltrami equation

$$\operatorname{curl}\mathbf{V}(\mathbf{x}) = \alpha\mathbf{V}(\mathbf{x}), \tag{9}$$

where α is an arbitrary constant. Any Beltrami vector field (9) leads to the following exact MHD solutions [7, 8, 9]:

$$\mathbf{V}(t,\mathbf{x}) = \exp(-\alpha^2\nu t)\mathbf{V}(\mathbf{x}), \quad \mathbf{B}(t,\mathbf{x}) = C\exp(-\alpha^2\eta t)\mathbf{V}(\mathbf{x}), \tag{10}$$

where $p = C_1 - \rho\mathbf{V}^2/2$. Fourier series (7) yield $\operatorname{curl}\mathbf{V} = i\sum \mathbf{k}\times\mathbf{V}_\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})$. Hence equation (9) implies

$$\mathbf{k}\times\mathbf{V}_\mathbf{k} = -i\alpha\mathbf{V}_\mathbf{k}. \tag{11}$$

Cross-multiplying this equation with vector $\mathbf{k}$ and using $\mathbf{V}_\mathbf{k}\cdot\mathbf{k} = 0$, we find $\mathbf{k}^2 = \alpha^2$. Solutions to equation (11) have the form

$$\mathbf{V}_\mathbf{k} = \mathbf{A}_\mathbf{k} + \frac{i}{\alpha}\mathbf{k}\times\mathbf{A}_\mathbf{k}, \tag{12}$$

where $\mathbf{A}_\mathbf{k}$ are any real vectors obeying the equations $\mathbf{A}_\mathbf{k}\cdot\mathbf{k} = 0$, $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_\mathbf{k}$. Hence the periodic MHD solutions (10) have the form [8]

$$\mathbf{V}_\alpha(t,\mathbf{x}) = \exp(-\alpha^2\nu t) \sum_{\mathbf{k}^2=\alpha^2} [\mathbf{A}_\mathbf{k}\cos(\mathbf{k}\cdot\mathbf{x}) - \frac{1}{\alpha}(\mathbf{k}\times\mathbf{A}_\mathbf{k})\sin(\mathbf{k}\cdot\mathbf{x})], \quad (13)$$

$$\mathbf{B}_\alpha(t,\mathbf{x}) = C\exp(\alpha^2(\nu-\eta)t)\mathbf{V}_\alpha(t,\mathbf{x}), \quad p_\alpha(t,\mathbf{x}) = C_1 - \rho\mathbf{V}_\alpha^2(t,\mathbf{x})/2.$$

Here summation is taken over all vectors $\mathbf{k} \in \Lambda^*$ of the same norm $|\mathbf{k}| = |\alpha|$. The exact solutions (13) for the Navier-Stokes equations ($C = 0$) were independently derived in [8, 15, 16].

Remark 1. Solutions (13) depend upon the number of points of the reciprocal lattice Λ^* which belong to the sphere $S^2 : \mathbf{k}^2 = \alpha^2$. Let us show that this number can be arbitrarily large when $|\alpha| \longrightarrow \infty$. Indeed, suppose that the scalar products of the basis vectors $\mathbf{k}_i$ (5) are rational: $\mathbf{k}_i\cdot\mathbf{k}_j = r_{ij}/q$, $r_{ij}, q \in \mathbb{Z}$. This amounts to the rationality condition

$$(2\pi)^{-2}\mathbf{p}_i\cdot\mathbf{p}_j = (R^{-1})_{ij}, \quad R_{ij} = r_{ij}/q. \quad (14)$$

Let P_N be the set of $(2N+1)^3$ points $\mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_3\mathbf{k}_3 \in \Lambda^*$ defined by: $-N \le n_i \le N$. On the P_N, we have

$$\mathbf{k}^2 = \sum_{ij} n_i n_j \mathbf{k}_i\cdot\mathbf{k}_j = \frac{r}{q} \le \frac{1}{q}\sum_{ij}|r_{ij}|N^2.$$

Hence the rational-valued function $\mathbf{k}^2 = r/q$ has at most $\sum|r_{ij}|N^2$ values on the set P_N of $(2N+1)^3$ points. Then by Dirichlet's principle there are at least $k_N = [(2N+1)^3/\sum|r_{ij}|N^2]$ points of P_N where function $\mathbf{k}^2$ has the same value. As $k_N \longrightarrow \infty$ when $N \longrightarrow \infty$, the equation $\mathbf{k}^2 = \alpha^2$ does have arbitrarily many solutions as $|\alpha| \longrightarrow \infty$ and this is true for a dense set of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ that satisfy the rationality condition (14).

3 Exact MHD solutions with crystallographic symmetries

I. As is known [10, 11], a crystallographic (or space) group G is generated by 3 basis translations $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{p}_i$ and the generators of the point group $\Gamma \subset O(3)$ combined with nonprimitive translations $\mathbf{t}_R \in \mathbb{R}^3$, where $R \in \Gamma$. The group transforms are $\tilde{\mathbf{x}} = R\mathbf{x} + \mathbf{t}_R$. The nonprimitive translations $\mathbf{t}_R$ are defined up to a primitive translation $n_1\mathbf{p}_1 + n_2\mathbf{p}_2 + n_3\mathbf{p}_3$ that belongs to the lattice Λ.

We construct exact solutions with crystallographic symmetries by using solutions (11) - (13) with some special vectors $\mathbf{A}_{\mathbf{k}}$. For any orthogonal matrix $Q \in O(3)$ we have $Q(\mathbf{A} \times \mathbf{B}) = (\det Q) Q\mathbf{A} \times Q\mathbf{B}$. Hence equation (11) is invariant with respect to the orthogonal transformations Q only if $\det Q = 1$. Therefore we use only space groups with pure rotational point groups $\Gamma \subset SO(3)$. There are 219 non-isomorphic space groups in three dimensions. Analyzing the tables of these groups [10], one finds that 52 of them have rotational point groups $\Gamma \subset SO(3)$. Among these 52 space groups, there are 23 symmorphic groups which are the semi-direct products $\Gamma \dot{\times} \mathbb{Z}^3$ and have all translations $\mathbf{t}_R = 0$. The other 29 nonsymmorphic groups are extensions of $\mathbb{Z}^3$ by the corresponding groups Γ.

There are 10 point crystallographic groups $\Gamma \subset SO(3)$: 4 cyclic groups C_n, 4 dihedral groups D_n, $n = 2, 3, 4, 6$, the tetrahedral group T (rotations of a regular tetrahedron), and the octahedral group O (rotations of a cube). These groups have respectively n, $2n$, 12 and 24 elements and 1, $n+1$, 7 and 13 axes of rotations. The groups C_2, C_4, D_2, D_4, T and O leave invariant a lattice with orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the groups C_3, C_6, D_3, D_6 one with basis $\mathbf{e}_1$, $\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\sqrt{3}\mathbf{e}_2$, $\mathbf{e}_3$. If a vector $\mathbf{k} \in \Lambda^*$ is not parallel to any of the axes of rotations then all vectors $R\mathbf{k} \in \Lambda^*$ are different for $R \in \Gamma$.

II. Vector field $\mathbf{V}(t, \mathbf{x})$ is invariant with respect to a transform $\tilde{\mathbf{x}} = R\mathbf{x} + \mathbf{t}_R$ if $\mathbf{V}(t, R\mathbf{x} + \mathbf{t}_R) = R\mathbf{V}(t, \mathbf{x})$. Hence for the Fourier components (7) we have $\mathbf{V}_{R\mathbf{k}} = \exp(-iR\mathbf{k} \cdot \mathbf{t}_R) R\mathbf{V}_{\mathbf{k}}$. Substituting here equation (12) we find the invariance condition

$$\mathbf{A}_{R\mathbf{k}} = R\left(\cos(R\mathbf{k} \cdot \mathbf{t}_R)\mathbf{A}_{\mathbf{k}} + \frac{1}{\alpha}\sin(R\mathbf{k} \cdot \mathbf{t}_R)\mathbf{k} \times \mathbf{A}_{\mathbf{k}}\right). \tag{15}$$

Our construction goes as follows. Take a crystallographic group G with a point group $\Gamma \subset SO(3)$. The rotations $R \in \Gamma$ preserve both lattices Λ and Λ^* corresponding to the space group G. Take a generic vector $\mathbf{k} \in \Lambda^*$, $\mathbf{k}^2 = \alpha^2$, that is not parallel to the axes of rotations of Γ (notice that there are at most 13 such axes). Then all vectors $R\mathbf{k} \in \Lambda^*$ are different for $R \in \Gamma$. We define vectors $\mathbf{A}_{R\mathbf{k}}$ by formula (15). If there are other vectors $\mathbf{k}_1 \in \Lambda^*$, $\mathbf{k}_1^2 = \alpha^2$, different from $R\mathbf{k}$, we repeat the construction and so on. For the exceptional vectors $\mathbf{m} \in \Lambda^*$ that are parallel to one of the axes of rotations, we put $\mathbf{A}_{R\mathbf{m}} = 0$. The group axioms for the space groups G and equations (15) imply that such a vector field (13) is correctly defined and is invariant under the group G action. An important property of the constructed exact solutions (13) is that they depend on all four variables t, x_1, x_2, x_3.

For the 23 symmorphic space groups $G = \Gamma \dot{\times} \mathbb{Z}^3$ with $\mathbf{t}_R = 0$, the equation (15) takes a simple form

$$\mathbf{A}_{R\mathbf{k}} = R\mathbf{A}_{\mathbf{k}}. \tag{16}$$

Example 1. *(a)* Let lattice Λ^* has basis $\mathbf{e}_1$, $\frac{1}{2}\mathbf{e}_1 + \frac{1}{2}\sqrt{3}\mathbf{e}_2$, $\mathbf{e}_3$. The 12 vectors $\pm\mathbf{e}_1 \pm \mathbf{e}_3$, $\pm\frac{1}{2}\mathbf{e}_1 \pm \frac{1}{2}\sqrt{3}\mathbf{e}_2 \pm \mathbf{e}_3 \in \Lambda^*$ have the same norm $|\mathbf{k}| = \sqrt{2}$ and form an orbit of the dihedral group D_6 that is generated by a 60^o rotation R_1 around $\mathbf{e}_3$ and a 180^o rotation R_2 around $\mathbf{e}_1$. Hence for $\alpha = \pm\sqrt{2}$, the exact solutions (13), (16) are invariant under the crystallographic group $D_6 \dot{\times} \mathbb{Z}^3$.

(b) Let lattice Λ^* has orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{k}_1 = (1, 5, 6)$, $\mathbf{k}_2 = (2, 3, 7)$. We have $\mathbf{k}_1^2 = \mathbf{k}_2^2 = 62$. For the 24 rotations of a unit cube, the 48 vectors $R\mathbf{k}_1$, $R\mathbf{k}_2$ are different for all $R \in O$. Hence for $\alpha = \pm\sqrt{62}$, the exact solutions (13), (16) are invariant with respect to the crystallographic group $O \dot{\times} \mathbb{Z}^3$.

(c) Using the inclusions of the groups $C_2 \subset C_4 \subset D_4 \subset O$, $D_2 \subset D_4$, $T \subset O$, $C_3 \subset D_3 \subset D_6$, $C_6 \subset D_6$, we define restrictions of the examples *(a)* and *(b)* onto the subgroups Γ where vectors $\mathbf{A}_{R\mathbf{k}}$ are independent on different orbits of the group Γ. Thus we obtain solutions (13), (16) that are invariant under the symmorphic crystallographic groups $\Gamma \dot{\times} \mathbb{Z}^3$.

4 Moduli space for the periodic dynamical systems

I. Substituting formulae (7) into equations (1) - (2), we obtain the infinite-dimensional dynamical system ($\mathbf{n} \neq 0$)

$$\dot{\mathbf{V}}_{\mathbf{n}} + i \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} (\mathbf{V}_{\mathbf{k}} \cdot \mathbf{m}) \mathbf{V}_{\mathbf{m}} + \frac{i p_{\mathbf{n}}}{\rho} \mathbf{n} + \mathbf{n}^2 \nu \mathbf{V}_{\mathbf{n}} - \frac{i}{\rho\mu} \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} (\mathbf{k} \times \mathbf{B}_{\mathbf{k}}) \times \mathbf{B}_{\mathbf{m}} = 0, \tag{17}$$

$$\dot{\mathbf{B}}_{\mathbf{n}} + \mathbf{n}^2 \eta \mathbf{B}_{\mathbf{n}} - i\mathbf{n} \times \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} \mathbf{V}_{\mathbf{k}} \times \mathbf{B}_{\mathbf{m}} = 0, \tag{18}$$

$$\dot{\mathbf{V}}_0 + \rho^{-1} \mathbf{p}_0 = 0, \quad \dot{\mathbf{B}}_0 = 0. \tag{19}$$

Projecting equation (17) onto vector $\mathbf{n}$ and using equations (8), we obtain

$$p_{\mathbf{n}} = -\frac{\rho}{\mathbf{n}^2} \mathbf{n} \cdot \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} (\mathbf{V}_{\mathbf{k}} \cdot \mathbf{m}) \mathbf{V}_{\mathbf{m}} + \frac{1}{\mu \mathbf{n}^2} \mathbf{n} \cdot \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} (\mathbf{k} \times \mathbf{B}_{\mathbf{k}}) \times \mathbf{B}_{\mathbf{m}}. \tag{20}$$

Substituting formulae (20) into equations (17), we obtain ($\gamma = 1/(\rho\mu)$)

$$\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} + \frac{i}{\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}[(\mathbf{V}_{\mathbf{k}}\cdot\mathbf{m})\mathbf{V}_{\mathbf{m}} - \gamma(\mathbf{k}\times\mathbf{B}_{\mathbf{k}})\times\mathbf{B}_{\mathbf{m}}]\right), \tag{21}$$

where we use the formula $\mathbf{X} - (\mathbf{n}\cdot\mathbf{X})\mathbf{n}/\mathbf{n}^2 = -\mathbf{n}\times(\mathbf{n}\times\mathbf{X})/\mathbf{n}^2$.

Equations (19) show that vector $\mathbf{p}_0(t)$ can not be excluded. The arbitrary function $\mathbf{p}_0(t)$ causes a non-uniqueness of the Cauchy problem solutions and appearence of singularities. Therefore we suppose $\mathbf{p}_0(t) = 0$ and $\mathbf{V}_0(t) = 0$ in what follows.

II. Using the vector algebra identities, we transform equations (21) and (18) for $\mathbf{V}_0 = \mathbf{p}_0 = 0$ to the form

$$\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} + \frac{i}{2\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{k}-\mathbf{m})\times[\mathbf{V}_{\mathbf{k}}\times\mathbf{V}_{\mathbf{m}} - \gamma\mathbf{B}_{\mathbf{k}}\times\mathbf{B}_{\mathbf{m}}]\right),$$

$$\dot{\mathbf{B}}_{\mathbf{n}} = -\mathbf{n}^2\eta\mathbf{B}_{\mathbf{n}} + i\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}\mathbf{V}_{\mathbf{k}}\times\mathbf{B}_{\mathbf{m}}, \tag{22}$$

where all vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{n}\in\Lambda^*$. The dynamical systems (22) for two different triples of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ generically are not equivalent because they depend upon the reciprocal lattices Λ^*.

The dynamical systems (22) for periodic solutions with two different triples of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ *are equivalent if and only if the corresponding lattices* Λ *are connected by an orthogonal transformation.*

Indeed, if the two systems (22) are equivalent, then their critical points $\mathbf{V}_{\mathbf{n}} = 0$, $\mathbf{B}_{\mathbf{n}} = 0$ have equal eigenvalues. These egenvalues are $-\mathbf{n}^2\nu$, $-\mathbf{n}^2\eta$ for all vectors $\mathbf{n}\in\Lambda^*$. It is evident that the set of numbers $-\mathbf{n}^2$ for all $\mathbf{n}\in\Lambda^*$ defines the lattice Λ^* up to an arbitrary rotation and inversion. Since the lattice Λ^* is defined by its basis $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, the moduli space M of non-equivalent dynamical systems (22) has dimension 6. The evident invariance of systems (22) under the scale transforms $\tilde{\mathbf{V}}_{\mathbf{n}} = \lambda\mathbf{V}_{\mathbf{n}}$, $\tilde{\mathbf{B}}_{\mathbf{n}} = \lambda\mathbf{B}_{\mathbf{n}}$, $\tilde{\mathbf{n}} = \lambda\mathbf{n}$, $\tilde{t} = \lambda^{-2}t$ reduces the moduli space dimension to 5. The above arguments are applicable also to any vector equations that contain the Laplace operator of velocity $\mathbf{V}(t,\mathbf{x})$, for example to the Navier-Stokes equations, the Stokes equations and the vector diffusion equations.

5 Non-interacting Fourier modes for $\nu \neq \eta$

The interaction of the **k**- and **m**-modes in the dynamical systems (22) is defined by the following terms

$$\mathbf{Z_{km}} = (\mathbf{k}+\mathbf{m}) \times [(\mathbf{k}-\mathbf{m}) \times (\mathbf{V_k} \times \mathbf{V_m} - \gamma \mathbf{B_k} \times \mathbf{B_m})], \tag{23}$$

$$\mathbf{S_{km}} = (\mathbf{k}+\mathbf{m}) \times [\mathbf{V_k} \times \mathbf{B_m} + \mathbf{V_m} \times \mathbf{B_k}].$$

The **k**- and **m**-modes do not interact if $\mathbf{Z_{km}} = 0$ and $\mathbf{S_{km}} = 0$.

For the MHD dynamical systems (22) with $\nu \neq \eta$, the **k***-modes of a set S do not interact pairwise if and only if one of the following four conditions is met:*
(1) All wave vectors $\mathbf{k} \in S$ are parallel;
(2) All wave vectors $\mathbf{k} \in S$ lie in a plane L and Fourier components $\mathbf{V_k}$, $\mathbf{B_k}$ are orthogonal to L;
(3) The vectors **k** *belong to a circumference $\mathbf{k}\cdot\mathbf{e} = 0$, $\mathbf{k}^2 = \alpha^2$ and vectors $\mathbf{V_k}$, $\mathbf{B_k}$ have the form*

$$\mathbf{V_k} = C_{\mathbf{k}} e^{-\alpha^2 \nu t}(\tau \mathbf{e} + i\lambda \mathbf{k} \times \mathbf{e}), \quad \mathbf{B_k} = C_{\mathbf{k}} e^{-\alpha^2 \eta t}(a\mathbf{e} + ib\mathbf{k} \times \mathbf{e}),$$

where $\mathbf{e} \in \Lambda$ and τ, λ, a, b are some real constants, $C_{-\mathbf{k}} = \overline{C_{\mathbf{k}}}$;
(4) The vectors **k** *belong to a sphere $\mathbf{k}^2 = \alpha^2$ and vectors $\mathbf{V_k}$, $\mathbf{B_k}$ satisfy the equations*

$$\mathbf{k} \times \mathbf{V_k} = \pm i\alpha \mathbf{V_k}, \quad \mathbf{B_k} = C \exp(\alpha^2(\nu - \eta)t)\mathbf{V_k},$$

with the same sign for all wave vectors $\mathbf{k} \in S$.

The proof consists on an analysis of the vector equations $\mathbf{Z_{km}} = 0$ and $\mathbf{S_{km}} = 0$ (23) and is similar to the one published in [12] for the NSE.

For the four cases, the MHD dynamical systems (22) take the linear form $\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2 \nu \mathbf{V_n}$, $\dot{\mathbf{B}}_{\mathbf{n}} = -\mathbf{n}^2 \eta \mathbf{B_n}$ and are integrable. The corresponding exact solutions *(1)* depend on one variable, solutions *(2)* depend on two variables and are unidirectional, solutions *(3)* are translationally invariant and have the form

$$\mathbf{V}(t,\mathbf{x}) = e^{-\alpha^2 \nu t}(\tau \mathbf{U} + \lambda \,\mathrm{curl}\, \mathbf{U}), \quad \mathbf{B}(t,\mathbf{x}) = e^{-\alpha^2 \eta t}(a\mathbf{U} + b \,\mathrm{curl}\, \mathbf{U}), \tag{24}$$

$$\mathbf{U}(t,\mathbf{x}) = (\sum_{\mathbf{k}^2=\alpha^2} [a_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{x}) - b_{\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{x})])\mathbf{e}, \ \mathbf{k}\cdot\mathbf{e} = 0, \ a_{-\mathbf{k}} = a_{\mathbf{k}}, \ b_{-\mathbf{k}} = -b_{\mathbf{k}},$$

$$2p(t,\mathbf{x}) = C + \left[e^{-2\alpha^2\eta t}(\alpha^2 b^2 - a^2)\mu^{-1} - \rho e^{-2\alpha^2\nu t}(\alpha^2\lambda^2 - \tau^2)\right]\mathbf{U}^2 - \rho\mathbf{V}^2.$$

Solutions (24) form a periodic case of the ones of Sect. 5 of [13]. The most interesting are solutions of class *(4)* which depend upon all four variables t, x_1, x_2, x_3; they have explicit form (13). All these exact solutions have no transfer of energy through the spectrum. Since the obtained classification is complete, any other solution has non-zero interaction between Fourier modes.

6 Periodic solutions for $\nu = \eta$

For the MHD dynamical systems (22) with $\nu = \eta$, the $\mathbf{k}$-modes of a set S do not interact pairwise if and only if in addition to the above the following two conditions are met:

(5) The wave vectors $\mathbf{k}$ belong to a plane $\mathbf{k}\cdot\mathbf{e} = 0$, vectors $\mathbf{B_k}$ and $\mathbf{V_k}$ have the form

$$\mathbf{B_k} = a_{\mathbf{k}}\mathbf{e} + i\lambda b_{\mathbf{k}}\mathbf{k}\times\mathbf{e}, \quad \mathbf{V_k} = \frac{\varepsilon}{\sqrt{\rho\mu}}\mathbf{B_k} + \tau b_{\mathbf{k}}\mathbf{e},$$

where $\mathbf{e}\in\Lambda$, $\varepsilon = \pm 1$, τ, λ are arbitrary reals and complex functions $a_{\mathbf{k}}(t)$, $b_{\mathbf{k}}(t)$ satisfy the equations $a_{-\mathbf{k}}(t) = \overline{a_{\mathbf{k}}(t)}$, $b_{-\mathbf{k}} = \overline{b_{\mathbf{k}}(t)}$;

(6) The Fourier components $\mathbf{B_k}$ and $\mathbf{V_k}$ are linked by the equation $\mathbf{V_k} = \varepsilon\mathbf{B_k}/\sqrt{\rho\mu}$, where $\varepsilon = \pm 1$.

For the case *(5)*, periodic solutions belong to the family of vector fields

$$\mathbf{V} = \pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B} + \tau f\mathbf{e}, \quad \mathbf{B} = g\mathbf{e} + \lambda\,\mathrm{grad}\, f\times\mathbf{e}, \tag{25}$$

where f and g are some functions satisfying the equations

$$\mathrm{grad}\, f\cdot\mathbf{e} = 0, \quad \mathrm{grad}\, g\cdot\mathbf{e} = 0. \tag{26}$$

For vector fields (25), we have $\mathbf{V}\times\mathbf{B} = \lambda\tau\mathbf{e}^2\,\mathrm{grad}\, f^2/2$ and

$$\mathrm{curl}\,\mathbf{V}\times\mathbf{V} - \frac{1}{\rho\mu}\,\mathrm{curl}\,\mathbf{B}\times\mathbf{B} = \mp\frac{\tau\mathbf{e}^2}{\sqrt{\rho\mu}}\,\mathrm{grad}(fg) - \frac{\tau^2\mathbf{e}^2}{2}\,\mathrm{grad}\, f^2.$$

Substituting these formulae, we find that equations (1) - (3) for $\nu = \eta$ are reduced to the two diffusion equations

$$\partial f/\partial t = \nu \Delta f, \quad \partial g/\partial t = \nu \Delta g, \tag{27}$$

and pressure takes the form $p = C_1 - \rho \mathbf{V}^2/2 \pm \tau \mathbf{e}^2 \sqrt{\rho/\mu} fg + \rho \tau^2 \mathbf{e}^2 f^2/2$.

Let $\mathbf{k}, \mathbf{m} \in \Lambda^*$ be two linearly independent vectors. Equations (5) imply that vector $\mathbf{e} = 2\pi \lambda^{-1} \mathbf{k} \times \mathbf{m} \in \Lambda$. Let functions f and g be the convergent series $f(t, \mathbf{x}) = \sum_{pq} f_{pq} \exp(-\nu(p\mathbf{k} + q\mathbf{m})^2 t + i(p\mathbf{k} + q\mathbf{m}) \cdot \mathbf{x})$, $g(t, \mathbf{x}) = \sum_{pq} g_{pq} \exp(-\nu(p\mathbf{k} + q\mathbf{m})^2 t + i(p\mathbf{k} + q\mathbf{m}) \cdot \mathbf{x})$, where $f_{-p-q} = \overline{f_{pq}}$ and $g_{-p-q} = \overline{g_{pq}}$. These functions clearly are periodic and satisfy equations (26) and (27). Hence the corresponding vector fields (25) are the MHD solutions with non-interacting Fourier modes.

For the case *(6)*, periodic solutions belong to the family of unsteady equipartition solutions [13]:

$$\mathbf{V}(t, \mathbf{x}) = \pm \frac{1}{\sqrt{\rho\mu}} \mathbf{B}(t, \mathbf{x}), \tag{28}$$

for which the densities of the plasma kinetic and magnetic energies $\rho \mathbf{V}^2/2$ and $\mathbf{B}^2/(2\mu)$ are equal. For solutions (28), the MHD equations (1) - (3) are equivalent to the system $\partial \mathbf{B}/\partial t = \nu \Delta \mathbf{B}$, $\operatorname{div} \mathbf{B} = 0$. Their periodic solutions are the convergent series

$$\mathbf{B}(t, \mathbf{x}) = i \sum_{\mathbf{k} \in \Lambda^*} \exp(-\nu \mathbf{k}^2 t + i \mathbf{k} \cdot \mathbf{x}) \mathbf{k} \times \mathbf{A}_{\mathbf{k}}, \tag{29}$$

where summation is over all vectors $\mathbf{k} \in \Lambda^*$ and $\mathbf{A}_{\mathbf{k}} \in \mathbb{C}^3$ are arbitrary vectors satisfying equations $\mathbf{A}_{-\mathbf{k}} = \overline{\mathbf{A}_{\mathbf{k}}}$ and $|\mathbf{A}_{\mathbf{k}}| < C/\mathbf{k}^3$.

Periodic equipartition solutions (28) - (29) depend upon all four variables t, x_1, x_2, x_3 and have no transfer of energy through the spectrum.

7 Conclusions

We have constructed exact periodic MHD solutions (13), (15) that are invariant with respect to one of the 52 crystallographic groups G that have pure rotational point groups $\Gamma \subset SO(3)$ and depend on all four variables t, x_1, x_2, x_3. Here Γ is either a cyclic C_n or a dihedral group D_n, $n = 2, 3, 4, 6$, or the tetrahedral group T or the octahedral group O. The exact solutions with one of the 23 symmorphic crystallographic symmetry groups $\Gamma \dot{\times} \mathbb{Z}^3$ are defined by construction (13), (16). It is essential for our construction that the point crystallographic group Γ is pure rotational. The existence of periodic MHD solutions with other 167 crystallographic symmetry groups, where $\Gamma \subset O(3)$ but $\Gamma \not\subset SO(3)$, remains an open problem.

We have shown that dynamical systems (22) describing periodic solutions with different periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ generically are not equivalent to each other. The systems are equivalent if and only if their reciprocal lattices Λ^* are connected by an orthogonal transformation. This is a general result that is true also for any equations containing the Laplace operator of velocity $\mathbf{V}(t, \mathbf{x})$ - for example the Navier-Stokes equations, the Stokes equations and the vector diffusion equations.

The derived classification of solutions with pairwise non-interacting Fourier modes is independent on the periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ but the corresponding exact solutions have different dimensions of invariant submanifolds and different groups of symmetries. For $\nu \neq \eta$ there are four infinite series of invariant submanifolds for the MHD dynamical systems (22) on which all solutions are smooth and exist for all moments of time $t \geq 0$. The wave vectors $\mathbf{k} \in \Lambda^*$ for them belong to the following four families of sets:

(1) the spheres $S^2 \cap \Lambda^*$: $\mathbf{k}^2 = \alpha^2$,
(2) the circumferences $S^1 \cap \Lambda^*$: $\mathbf{k} \cdot \mathbf{e} = 0$, $\mathbf{k}^2 = \alpha^2$,
(3) the straight lines $L^1 \cap \Lambda^*$: $\mathbf{k} = \lambda \mathbf{n}$,
(4) the planes $P^2 \cap \Lambda^*$: $\mathbf{k} \cdot \mathbf{e} = 0$.

Here $\mathbf{e} \in \Lambda$, and $\mathbf{n} \in \Lambda^*$. For $\nu = \eta$, there are five infinite series of such invariant submanifolds and two submanifolds of equipartition solutions $\mathbf{V}(t, \mathbf{x}) = \pm \mathbf{B}(t, \mathbf{x})/\sqrt{\rho\mu}$. The direct and inverse cascades do not work for all these solutions since there is no interaction between their Fourier modes. These results imply that the stochastization of the solutions occurs very slowly if the initial data are in a small neighbourhood of the above invariant submanifolds. Hence the rate of stochastization is not uniform in the functional space of the MHD solutions.

The derived classification of solutions with no pairwise interaction between Fourier modes is complete and hence proves that for any solution outside of the above invariant submanifolds there exists necessarily a non-zero interaction between its Fourier modes.

The most complex dynamics of plasma is realized for the exact periodic solutions (13) and for the equipartition solutions (28) - (29) which depend on all four variables t, x, y, z and generically have no geometrical symmetries.

APPENDIX

Infinite-dimensional Lie groups of symmetries

Let $Q \in O(3)$ be any element of the holohedry $H(\Lambda^*)$ (group of orthogonal transformations that preserve the lattice Λ^*), and $\mathbf{S}(t) \in \mathbb{R}^3$ be an arbitrary smooth vector function of t. The holohedry $H(\Lambda^*)$ always contains at least two elements $Q = 1$ and $Q = -1$.

The MHD dynamical system (17) - (19) has an infinite-dimensional Lie group G of symmetries

$$\tilde{\mathbf{V}}_{\mathbf{k}}(t) = \exp(i\mathbf{k}\cdot\mathbf{S}(t))Q\mathbf{V}_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{V}}_0(t) = Q\mathbf{V}_0(t) - \dot{\mathbf{S}}(t), \qquad (30)$$

$$\tilde{\mathbf{B}}_{\mathbf{k}}(t) = \sigma\exp(i\mathbf{k}\cdot\mathbf{S}(t))Q\mathbf{B}_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{B}}_0(t) = \sigma Q\mathbf{B}_0(t), \quad \sigma = \pm 1,$$

$$\tilde{p}_{\mathbf{k}}(t) = \exp(i\mathbf{k}\cdot\mathbf{S}(t))p_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{p}}_0(t) = Q\mathbf{p}_0(t) + \rho\ddot{\mathbf{S}}(t).$$

The Lie group G is a semidirect product of the holohedry $H(\Lambda^*)$ and the abelian Lie group $\mathbf{A}_0$ of vector-valued functions $\mathbf{S}(t)$ and the group $\mathbb{Z}_2$, $G = H(\Lambda^*)\dot{\times}\mathbf{A}_0 \times \mathbb{Z}_2$.

The proof follows by a straightforward verification and can be obtained also from the Lie group analysis [14] of the non-periodic MHD equations. For $\mathbf{S}(t) = \mathbf{u}t$, $\mathbf{u} = \text{const}$, the transforms (30) are manifestations of the Galilean invariance of the viscous MHD equations.

For dynamical systems (22) with $\mathbf{V}_0 = \mathbf{p}_0 = 0$, the symmetries (30) reduce to the transforms

$$\tilde{\mathbf{V}}_{\mathbf{k}} = \exp(i\mathbf{k}\cdot\mathbf{z})Q\mathbf{V}_{Q^{-1}(\mathbf{k})}, \quad \tilde{\mathbf{B}}_{\mathbf{k}} = \sigma\exp(i\mathbf{k}\cdot\mathbf{z})Q\mathbf{B}_{Q^{-1}(\mathbf{k})},$$

and $\tilde{p}_{\mathbf{k}} = \exp(i\mathbf{k}\cdot\mathbf{z})p_{Q^{-1}(\mathbf{k})}$. Here arbitrary vector $\mathbf{z}$ is defined up to the periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and thus belongs to the torus $\mathbb{T}^3$. Hence we obtain that dynamical systems (22) have the Lie groups of symmetries $G = H(\Lambda^*)\dot{\times}\mathbb{T}^3 \times \mathbb{Z}_2$.

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