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Viscous MHD solutions with no transfer of energy through the spectrum

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Exact space periodic solutions to the viscous magnetohydrodynamics equations with arbitrary vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are derived. A complete classification of the $\mathbf{p}_j$-periodic MHD solutions with pairwise non-interacting Fourier modes is obtained. The solutions have no transfer of energy through the spectrum and are smooth for all values of the time variable $t \geq 0$.

Keywords: Space periodic MHD solutions, Fourier modes, Beltrami flows, equipartition solutions.

1 Introduction

In this paper, we study the space periodic solutions with arbitrary vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ for the system of viscous MHD equations [1]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho\mu}\,\mathrm{curl}\,\mathbf{B} \times \mathbf{B} + \nu\Delta\mathbf{V}, \tag{1.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{V} \times \mathbf{B}) + \eta\Delta\mathbf{B}, \tag{1.2}$$

$$\mathrm{div}\,\mathbf{V} = 0, \quad \mathrm{div}\,\mathbf{B} = 0. \tag{1.3}$$

Here $\mathbf{V}(t, \mathbf{x})$ is the fluid velocity vector field, $\mathbf{B}(t, \mathbf{x})$ is magnetic field and $p(t, \mathbf{x})$ is the pressure, vector $\mathbf{x} = (x_1, x_2, x_3)$; the density ρ, kinematic viscosity ν, permeability μ and resistivity η are constant. For the Navier-Stokes equations

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{V}, \quad \operatorname{div} \mathbf{V} = 0, \tag{1.4}$$

the standard 2π-periodic solutions were considered in [2-4].

In papers [5, 6], exact solutions to the Navier-Stokes equations and viscous magnetohydrodynamics equations (MHD) were derived in the form of the time-dependent Beltrami flows. For these solutions, the fluid velocity $\mathbf{V}$, pressure p and magnetic field $\mathbf{B}$ have the form

$$\mathbf{V}(t, \mathbf{x}) = e^{-\alpha^2 \nu t} \int \int_{S^2} [\sin(\alpha \mathbf{k} \cdot \mathbf{x}) \mathbf{T}(\mathbf{k}) + \cos(\alpha \mathbf{k} \cdot \mathbf{x}) \mathbf{k} \times \mathbf{T}(\mathbf{k})] \, \mathrm{d}\sigma, \tag{1.5}$$

$$p(t, \mathbf{x}) = C_1 - \rho \mathbf{V}^2(t, \mathbf{x})/2, \quad \mathbf{B}(t, \mathbf{x}) = C_2 \exp(\alpha^2(\nu - \eta)t) \mathbf{V}(t, \mathbf{x}),$$

where C_1, C_2 are arbitrary constants and $\alpha \neq 0$ is an arbitrary parameter. Here the integral is taken with respect to an arbitrary measure $\mathrm{d}\sigma$ on the 2-dimensional unit sphere S^2: $\mathbf{k}^2 = 1$ and $\mathbf{T}(\mathbf{k})$ is an arbitrary smooth vector field tangent to the unit sphere, $\mathbf{T}(\mathbf{k}) \cdot \mathbf{k} = 0$. For $C_2 = 0$, formula (1.5) gives exact solutions to the Navier-Stokes equations (1.4). The generic solutions (1.5) depend on all four variables t, x_1, x_2, x_3 and have no geometrical symmetries. Similar NSE solutions were independently derived by Majda and Bertozzi in [7]. The idea of constructing the NSE and MHD exact solutions by using the Beltrami flows was first developed by Trkal [8] and Tasso [9].

For the special vector fields $\mathbf{T}(\mathbf{k})$ and the Euclidean measure $\mathrm{d}\sigma$, solutions (1.5) have the soliton-like properties [6, 10]. If the measure $\mathrm{d}\sigma$ has the form $\mathrm{d}\sigma = \delta(\mathbf{k}_1) + \cdots + \delta(\mathbf{k}_n)$, the formulae (1.5) give the exact solutions [6]:

$$\mathbf{V}(t, \mathbf{x}) = e^{-\alpha^2 \nu t} \sum_{i=1}^{n} [\sin(\alpha \mathbf{k}_i \cdot \mathbf{x}) \mathbf{T}(\mathbf{k}_i) + \cos(\alpha \mathbf{k}_i \cdot \mathbf{x}) \mathbf{k}_i \times \mathbf{T}(\mathbf{k}_i)] \,. \tag{1.6}$$

Here $\mathbf{k}_i^2 = 1$ and $T(\mathbf{k}_i) \cdot \mathbf{k}_i = 0$. For generic vectors $\mathbf{k}_1, \cdots, \mathbf{k}_n$ and $n > 3$, solutions (1.6) are quasi-periodic functions of $\mathbf{x}$. For $n = 3$ and $\mathbf{k}_1^2 = \mathbf{k}_2^2 = \mathbf{k}_3^2 = 1$, solutions (1.6) are periodic in $\mathbb{R}^3$ with vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ satisfying equations $\mathbf{k}_i \cdot \mathbf{p}_j = 2\pi \alpha^{-1} \delta_{ij}$.

In this paper, we solve the problem of complete classification of space periodic MHD solutions with non-interacting Fourier modes and arbitrary vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. In spite of the solutions have no transfer of energy through the spectrum they can describe a very complex dynamics of plasma with streamlines dense in 3-D domains. We show that for $\nu \neq \eta$ there are four infinite families of solutions with pairwise non-interacting modes. The simplest two of these families of exact solutions are classically known [11]. For $\nu = \eta$, there are six families. The solutions correspond to the infinite series of invariant submanifolds for the viscous MHD equations.

2 Dynamical system for space periodic solutions and its symmetries

I. We study MHD solutions that satisfy the periodicity conditions

$$\mathbf{B}(t, \mathbf{x} + \mathbf{p}_i) = \mathbf{B}(t, \mathbf{x}),\ \mathbf{V}(t, \mathbf{x} + \mathbf{p}_i) = \mathbf{V}(t, \mathbf{x}),\ \nabla p(t, \mathbf{x} + \mathbf{p}_i) = \nabla p(t, \mathbf{x}), \tag{2.1}$$

where $i = 1, 2, 3$ and vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are linearly independent. The integral combinations $n_1\mathbf{p} + n_2\mathbf{p}_2 + n_3\mathbf{p}_3$, $n_i \in \mathbb{Z}$, form a lattice of periods Λ. Let vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are defined by the equations

$$\mathbf{k}_i \cdot \mathbf{p}_j = 2\pi\delta_{ij}. \tag{2.2}$$

The integral combinations $m_1\mathbf{k} + m_2\mathbf{k}_2 + m_3\mathbf{k}_3$, $m_i \in \mathbb{Z}$, form the reciprocal lattice Λ^*. Vectors $\mathbf{k}_i$ are $\mathbf{k}_i = \lambda\mathbf{p}_j \times \mathbf{p}_k$ where $\lambda = 2\pi[\mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)]^{-1}$ and indices i, j, k form a cyclic permutation. We present the periodic solutions (2.1) in the form of Fourier series

$$\mathbf{B}(t, \mathbf{x}) = \sum_{\mathbf{k} \in \Lambda^*} \mathbf{B}_{\mathbf{k}}(t) \exp(i\mathbf{k} \cdot \mathbf{x}), \quad \mathbf{V}(t, \mathbf{x}) = \sum_{\mathbf{k} \in \Lambda^*} \mathbf{V}_{\mathbf{k}}(t) \exp(i\mathbf{k} \cdot \mathbf{x}), \tag{2.3}$$

$$\nabla p(t, \mathbf{x}) = \mathbf{p}_0(t) + i \sum_{\mathbf{k} \in \Lambda^*} p_{\mathbf{k}}(t)\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{x}),$$

where summation is taken over all vectors $\mathbf{k}$ of the reciprocal lattice Λ^*. For real functions $\mathbf{B}(t, \mathbf{x})$, $\mathbf{V}(t, \mathbf{x})$ and $p(t, \mathbf{x})$, the Fourier components $\mathbf{B}_{\mathbf{k}}, \mathbf{V}_{\mathbf{k}} \in \mathbb{C}^3$, and $p_{\mathbf{k}} \in \mathbb{C}$ satisfy the relations

$$\mathbf{B}_{-\mathbf{k}} = \overline{\mathbf{B}_{\mathbf{k}}}, \quad \mathbf{V}_{-\mathbf{k}} = \overline{\mathbf{V}_{\mathbf{k}}}, \quad p_{-\mathbf{k}} = \overline{p_{\mathbf{k}}}. \tag{2.4}$$

Vectors $\mathbf{B}_0(t)$, $\mathbf{V}_0(t)$ and $\mathbf{p}_0(t)$ are real. The incompressibility equations (1.3) imply

$$\mathbf{k}\cdot\mathbf{B_k}=0,\quad \mathbf{k}\cdot\mathbf{V_k}=0. \tag{2.5}$$

Substituting formulae (2.3) into equations (1.1) - (1.2), we obtain an infinite-dimensional dynamical system for $\mathbf{n}\neq 0$

$$\dot{\mathbf{V}}_\mathbf{n}+i\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{V_k}\cdot\mathbf{m})\mathbf{V_m}+\frac{ip_\mathbf{n}}{\rho}\mathbf{n}+\mathbf{n}^2\nu\mathbf{V_n}-\frac{i}{\rho\mu}\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{k}\times\mathbf{B_k})\times\mathbf{B_m}=0, \tag{2.6}$$

$$\dot{\mathbf{B}}_\mathbf{n}+\mathbf{n}^2\eta\mathbf{B_n}-i\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}\mathbf{V_k}\times\mathbf{B_m}=0. \tag{2.7}$$

For $\mathbf{n}=0$, using equations (2.5) we obtain

$$\dot{\mathbf{V}}_0+\rho^{-1}\mathbf{p}_0=0,\quad \dot{\mathbf{B}}_0=0. \tag{2.8}$$

For $\mathbf{n}\neq 0$, all functions $p_\mathbf{n}$ can be excluded from equations (2.6). Indeed, projecting equation (2.6) onto vector $\mathbf{n}$ and using equations (2.5), we obtain

$$p_\mathbf{n}=-\frac{\rho}{\mathbf{n}^2}\mathbf{n}\cdot\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{V_k}\cdot\mathbf{m})\mathbf{V_m}+\frac{1}{\mu\mathbf{n}^2}\mathbf{n}\cdot\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{k}\times\mathbf{B_k})\times\mathbf{B_m}. \tag{2.9}$$

However, vector $\mathbf{p}_0(t)$ cannot be excluded from the MHD dynamical system as it clearly follows from equations (2.8). This property is closely connected with the existence of a large group of symmetries of system (2.6) - (2.8).

Substituting formulae (2.9) into equations (2.6), we obtain the equivalent form:

$$\dot{\mathbf{V}}_\mathbf{n}=-\mathbf{n}^2\nu\mathbf{V_n}+\frac{i}{\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}\left[(\mathbf{V_k}\cdot\mathbf{m})\mathbf{V_m}-\gamma(\mathbf{k}\times\mathbf{B_k})\times\mathbf{B_m}\right]\right), \tag{2.10}$$

where we use formula $\mathbf{X}-(\mathbf{n}\cdot\mathbf{X})\mathbf{n}/\mathbf{n}^2=-\mathbf{n}\times(\mathbf{n}\times\mathbf{X})/\mathbf{n}^2$ and denote $\gamma=1/(\rho\mu)$.

II. Using the identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \tag{2.11}$$

and equations (2.5), we transform equations (2.10) and (2.7) into the dynamical system

$$\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} + \frac{i}{2\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{k}-\mathbf{m})\times[\mathbf{V}_{\mathbf{k}}\times\mathbf{V}_{\mathbf{m}}-\gamma\mathbf{B}_{\mathbf{k}}\times\mathbf{B}_{\mathbf{m}}]\right),$$

$$\dot{\mathbf{B}}_{\mathbf{n}} = -\mathbf{n}^2\eta\mathbf{B}_{\mathbf{n}} + i\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}\mathbf{V}_{\mathbf{k}}\times\mathbf{B}_{\mathbf{m}}, \qquad (2.12)$$

where all vectors $\mathbf{k}, \mathbf{m}, \mathbf{n}$ belong to the reciprocal lattice Λ^*. In view of Proposition 2 below, we assume $\mathbf{V}_0(t) = 0$ in equations (2.12).

The dynamical systems (2.12) for periodic solutions with two different triples of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ *are equivalent if and only if the corresponding lattices* Λ *are connected by an orthogonal transformation. The moduli space of non-equivalent systems (2.12) has dimension 6.*

Indeed, if the two systems (2.12) are equivalent, then their critical points $\mathbf{V}_{\mathbf{n}} = 0$, $\mathbf{B}_{\mathbf{n}} = 0$ have equal eigenvalues. These egenvalues are

$$\lambda_{1\mathbf{n}} = -\mathbf{n}^2\nu, \quad \lambda_{2\mathbf{n}} = -\mathbf{n}^2\eta \qquad (2.13)$$

for all vectors $\mathbf{n} \in \Lambda^*$. It is evident that the set of numbers $-\mathbf{n}^2$ for $\mathbf{n} \in \Lambda^*$ defines the lattice Λ^* up to an arbitrary rotation and inversion. Since the lattice Λ^* is defined by its basis $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, the moduli space M of non-equivalent dynamical systems (2.12) has dimension 6.

REMARK 1. The multiplicities of the eigenvalues (2.13) are defined by the cardinality of solutions to the equation

$$\mathbf{n}^2 = \sum_{i,j=1}^{3} n_i n_j \mathbf{k}_i \cdot \mathbf{k}_j = C, \quad n_i \in \mathbb{Z}. \qquad (2.14)$$

If the scalar products $\mathbf{k}_i \cdot \mathbf{k}_j$ are rationally independent then equation (2.14) has either two integral solutions (n_1, n_2, n_3) and $(-n_1, -n_2, -n_3)$ or none. Hence on the invariant submanifold defined by the constraints (2.4) all eigenvalues (2.13) have multiplicity 1. Let us show however that for a dense set of matrices $K_{ij} = \mathbf{k}_i \cdot \mathbf{k}_j$ the multiplicities of eigenvalues (2.13) or the number of solutions to equation (2.14) can be arbitrarily large when $C \longrightarrow \infty$. Indeed, suppose that matrix K_{ij} is proportional to a rational matrix: $\mathbf{k}_i \cdot \mathbf{k}_j = br_{ij}/q$ where $r_{ij}, q \in \mathbb{Z}$, $b \in \mathbb{R}$. This amounts to the condition

$$\mathbf{p}_i \cdot \mathbf{p}_j = b^{-1}(2\pi)^2(R^{-1})_{ij}, \quad R_{ij} = r_{ij}/q. \qquad (2.15)$$

Let P_N be the set of $(2N+1)^3$ points $\mathbf{n} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_3\mathbf{k}_3 \in \Lambda^*$ where $-N \le n_i \le N$. On the P_N, we have

$$b^{-1}\mathbf{n}^2 = b^{-1}\sum_{i,j=1}^{3} n_i n_j \mathbf{k}_i \cdot \mathbf{k}_j = \frac{r}{q} \le \frac{1}{q}\sum_{i,j=1}^{3} |r_{ij}| N^2 .$$

Hence the rational-valued function $b^{-1}\mathbf{n}^2 = r/q$ has at most $\sum |r_{ij}| N^2$ values on the set P_N of $(2N+1)^3$ points. Then by Dirichlet's principle there are at least $K_N = [(2N+1)^3 / \sum |r_{ij}| N^2]$ points of P_N where function $b^{-1}\mathbf{n}^2$ has the same value. As $K_N \approx cN \longrightarrow \infty$ for $N \longrightarrow \infty$, the equation (2.14) does have arbitrarily many solutions as $C \longrightarrow \infty$. Evidently this is true for a dense set of periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ satisfying condition (2.15).

REMARK 2. The above arguments are applicable also to any vector equations that contain the Laplace operator of velocity $\mathbf{V}(t, \mathbf{x})$, for example to the Navier-Stokes equations, the Stokes equations and the vector diffusion equations.

Proposition 1 *If matrix $K_{ij} = \mathbf{k}_i \cdot \mathbf{k}_j$ has the form $K_{ij} = br_{ij}/q$, where $r_{ij}, q \in \mathbb{Z}$, then the number of solutions $\mathbf{n} \in \Lambda^*$ to equation (2.14) which do not belong to a finite number ℓ of lines and to a finite number m of planes becomes arbitrarily large when $C \longrightarrow \infty$.*

Proof. Let P_N^* be the set P_N without points of the ℓ lines L and the m planes P. An intersection $L \cap P_N$ has at most $2N+1$ points and $P \cap P_N$ has at most $(2N+1)^2$ points. Hence the set P_N^* has at least M_N points, $M_N = (2N+1)^3 - \ell(2N+1) - m(2N+1)^2$ and the above proof works because $K_N^* = [M_N / \sum |r_{ij}| N^2] \approx cN \longrightarrow \infty$ when $C \longrightarrow \infty$. ☐

III. Let $Q \in O(3)$ be any element of the holohedry $H(\Lambda^*)$ (group of orthogonal transformations that preserve the lattice Λ^*), and $\mathbf{S}(t) \in \mathbb{R}^3$ be an arbitrary smooth vector function of t. The holohedry $H(\Lambda^*)$ always contains at least two elements $Q = 1$ and $Q = -1$.

The MHD dynamical system (2.6) - (2.8) has an infinite-dimensional Lie group G of symmetries

$$\tilde{\mathbf{V}}_{\mathbf{k}}(t) = \exp(i\mathbf{k}\cdot\mathbf{S}(t)) Q \mathbf{V}_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{V}}_0(t) = Q\mathbf{V}_0(t) - \dot{\mathbf{S}}(t), \tag{2.16}$$

$$\tilde{\mathbf{B}}_{\mathbf{k}}(t) = \sigma \exp(i\mathbf{k}\cdot\mathbf{S}(t)) Q \mathbf{B}_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{B}}_0(t) = \sigma Q \mathbf{B}_0(t), \quad \sigma = \pm 1, \tag{2.17}$$

$$\tilde{p}_{\mathbf{k}}(t) = \exp(i\mathbf{k}\cdot\mathbf{S}(t)) p_{Q^{-1}(\mathbf{k})}(t), \quad \tilde{\mathbf{p}}_0(t) = Q\mathbf{p}_0(t) + \rho\ddot{\mathbf{S}}(t). \tag{2.18}$$

The Lie group G is a semidirect product of the holohedry $H(\Lambda^*)$ and the abelian Lie group $\mathbf{A}_0$ of vector-valued functions $\mathbf{S}(t)$ and the group $\mathbb{Z}_2$, $G = H(\Lambda^*)\dot{\times}\mathbf{A}_0 \times \mathbb{Z}_2$.

The proof follows by a straightforward verification and can be obtained also from the Lie group analysis [12] of the non-periodic MHD equations.

Substituting formulae (2.16) - (2.18) into Fourier series (2.3), we arrive at the symmetries

$$\tilde{\mathbf{V}}(t, \mathbf{x}) = Q\mathbf{V}(t, Q^{-1}[\mathbf{x} + \mathbf{S}(t)]) - \dot{\mathbf{S}}(t), \tag{2.19}$$

$$\tilde{\mathbf{B}}(t, \mathbf{x}) = \sigma Q\mathbf{B}(t, Q^{-1}[\mathbf{x} + \mathbf{S}(t)]),\ \nabla\tilde{p}(t, \mathbf{x}) = Q\nabla p(t, Q^{-1}[\mathbf{x} + \mathbf{S}(t)]) + \rho\ddot{\mathbf{S}}(t).$$

A direct substitution to the equations (1.1) - (1.3) proves that transforms (2.19) with any matrix $Q \in O(3)$ are symmetries of the MHD equations for the general non-periodic case.

Remark 3. Transforms (2.16) - (2.18) and (2.19) have a clear physical meaning: the invariance of the viscous MHD equations (1.1) - (1.3) under transforms into an accelerated or non-inertial frame of reference. This is a generalization of the well-known Galilean invariance of the MHD equations where $\mathbf{S}(t) = \mathbf{u}t$, $\mathbf{u} = \text{const}$ and $\rho\ddot{\mathbf{S}}(t) = 0$. *An important point is that the transforms (2.16) - (2.18) give new solutions in the standard inertial frame of reference because the form of the MHD equations is preserved by them.* The following proposition gives a useful application of transforms (2.16) - (2.18).

Proposition 2 *Any smooth periodic solution to the MHD equations (2.6) - (2.8) with $\mathbf{V}_0(t) \neq 0$ can be transformed by symmetries (2.16) - (2.18) into a solution with $\tilde{\mathbf{V}}_0(t) = 0$, $\tilde{\mathbf{p}}_0(t) = 0$.*

Indeed, let us apply transform (2.16) - (2.18) where function $\mathbf{S}(t)$ satisfies the equations $\ddot{\mathbf{S}}(t) = -\rho^{-1}\mathbf{p}_0(t)$, $\dot{\mathbf{S}}(0) = \mathbf{V}_0(0)$, $\mathbf{S}(0) = 0$, and $Q = 1$. Then we get from (2.18) $\tilde{\mathbf{p}}_0(t) = 0$. Hence equations (2.8) imply $\dot{\tilde{\mathbf{V}}}_0(t) = 0$, $\tilde{\mathbf{V}}_0(t) = \tilde{\mathbf{V}}_0(0)$ and the second of equations (2.16) gives $\tilde{\mathbf{V}}_0(0) = \mathbf{V}_0(0) - \dot{\mathbf{S}}(0) = 0$. $\Box$

IV. For dynamical systems (2.12) with $\mathbf{V}_0 = \mathbf{p}_0 = 0$, symmetries (2.16) - (2.18) reduce to the transforms $\tilde{\mathbf{V}}_{\mathbf{k}} = \exp(i\mathbf{k} \cdot \mathbf{z})Q\mathbf{V}_{Q^{-1}(\mathbf{k})}$, $\tilde{\mathbf{B}}_{\mathbf{k}} = \sigma \exp(i\mathbf{k} \cdot \mathbf{z})Q\mathbf{B}_{Q^{-1}(\mathbf{k})}$, and $\tilde{p}_{\mathbf{k}} = \exp(i\mathbf{k} \cdot \mathbf{z})p_{Q^{-1}(\mathbf{k})}$. Here arbitrary vector $\mathbf{z}$ is defined up to the periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and thus belongs to the torus $\mathbb{T}^3$. Hence we obtain that dynamical systems (2.12) have the Lie groups of symmetries $G = H(\Lambda^*)\dot{\times}\mathbb{T}^3 \times \mathbb{Z}_2$.

3 Non-interacting Fourier modes

The interaction of the **k**- and **m**-modes is defined by the following terms in equations (2.12)

$$\mathbf{Z}_{\mathbf{km}} = (\mathbf{k}+\mathbf{m}) \times ((\mathbf{k}-\mathbf{m}) \times [\mathbf{V}_{\mathbf{k}} \times \mathbf{V}_{\mathbf{m}} - \gamma \mathbf{B}_{\mathbf{k}} \times \mathbf{B}_{\mathbf{m}}]), \qquad (3.1)$$

$$\mathbf{S}_{\mathbf{km}} = (\mathbf{k}+\mathbf{m}) \times [\mathbf{V}_{\mathbf{k}} \times \mathbf{B}_{\mathbf{m}} + \mathbf{V}_{\mathbf{m}} \times \mathbf{B}_{\mathbf{k}}]. \qquad (3.2)$$

The **k**- and **m**-modes do not interact if $\mathbf{Z}_{\mathbf{km}} = 0$ and $\mathbf{S}_{\mathbf{km}} = 0$.

Lemma 1 *Two modes* $\mathbf{k}$ *and* $\mathbf{m}$ *do not interact if and only if one of the following four conditions is met:*
1) The wave vectors $\mathbf{k}$ *and* $\mathbf{m}$ *are parallel.*
2) The vectors $\mathbf{V}_{\mathbf{k}}$*,* $\mathbf{V}_{\mathbf{m}}$*,* $\mathbf{B}_{\mathbf{k}}$ *and* $\mathbf{B}_{\mathbf{m}}$ *have the form*

$$\mathbf{V}_{\mathbf{k}} = v_{\mathbf{k}}\mathbf{e}, \quad \mathbf{V}_{\mathbf{m}} = v_{\mathbf{m}}\mathbf{e}, \quad \mathbf{B}_{\mathbf{k}} = a_{\mathbf{k}}\mathbf{e}, \quad \mathbf{B}_{\mathbf{m}} = a_{\mathbf{m}}\mathbf{e}, \qquad (3.3)$$

where $\mathbf{e}$ *is a unit vector orthogonal to the vectors* $\mathbf{k}$ *and* $\mathbf{m}$*.*
3) The Fourier components have the form

$$\mathbf{V}_{\mathbf{k}} = C_{\mathbf{k}}(\tau\mathbf{e} + \lambda\mathbf{k} \times \mathbf{e}), \quad \mathbf{V}_{\mathbf{m}} = C_{\mathbf{m}}(\tau\mathbf{e} + \lambda\mathbf{m} \times \mathbf{e}), \qquad (3.4)$$

$$\mathbf{B}_{\mathbf{k}} = C_{\mathbf{k}}(a\mathbf{e} + b\mathbf{k} \times \mathbf{e}), \quad \mathbf{B}_{\mathbf{m}} = C_{\mathbf{m}}(a\mathbf{e} + b\mathbf{m} \times \mathbf{e}),$$

where vectors $\mathbf{k}$ *and* $\mathbf{m}$ *have equal norms,* $\mathbf{k}^2 = \mathbf{m}^2$ *and coefficients* $C_{\mathbf{k}}(t)$*,* $C_{\mathbf{m}}(t)$*,* $\tau(t)$*,* $\lambda(t)$*,* $a(t)$ *and* $b(t)$ *are some functions.*
4) The Fourier components are

$$\mathbf{V}_{\mathbf{k}} = \pm\sqrt{\gamma}\mathbf{B}_{\mathbf{k}} + \tau b_{\mathbf{k}}\mathbf{e}, \quad \mathbf{V}_{\mathbf{m}} = \pm\sqrt{\gamma}\mathbf{B}_{\mathbf{m}} + \tau b_{\mathbf{m}}\mathbf{e}, \qquad (3.5)$$

$$\mathbf{B}_{\mathbf{k}} = a_{\mathbf{k}}\mathbf{e} + b_{\mathbf{k}}\mathbf{k} \times \mathbf{e}, \quad \mathbf{B}_{\mathbf{m}} = a_{\mathbf{m}}\mathbf{e} + b_{\mathbf{m}}\mathbf{m} \times \mathbf{e},$$

where coefficients $a_{\mathbf{k}}(t)$*,* $b_{\mathbf{k}}(t)$*,* $a_{\mathbf{m}}(t)$*,* $b_{\mathbf{m}}(t)$ *and* $\tau(t)$ *are some functions and* $\gamma = 1/(\rho\mu) = \mathrm{const}$*.*

Proof. Since vectors $\mathbf{V}_{\mathbf{k}}$, $\mathbf{B}_{\mathbf{k}}$ are orthogonal to **k** and $\mathbf{V}_{\mathbf{m}}$, $\mathbf{B}_{\mathbf{m}}$ are orthogonal to **m**, they have the form

$$\mathbf{V}_{\mathbf{k}}(t) = v_{\mathbf{k}}(t)\mathbf{e} + u_{\mathbf{k}}(t)\mathbf{k} \times \mathbf{e}, \quad \mathbf{V}_{\mathbf{m}}(t) = v_{\mathbf{m}}(t)\mathbf{e} + u_{\mathbf{m}}(t)\mathbf{m} \times \mathbf{e},$$

$$\mathbf{B_k}(t) = a_\mathbf{k}(t)\mathbf{e} + b_\mathbf{k}(t)\mathbf{k}\times\mathbf{e}, \quad \mathbf{B_m}(t) = a_\mathbf{m}(t)\mathbf{e} + b_\mathbf{m}(t)\mathbf{m}\times\mathbf{e},$$

where $\mathbf{e}\cdot\mathbf{k} = \mathbf{e}\cdot\mathbf{m} = 0$, $\mathbf{e}^2 = 1$. We have $(\mathbf{k}\times\mathbf{e})\times(\mathbf{m}\times\mathbf{e}) = z_\mathbf{km}\mathbf{e}$, where $z_\mathbf{km} = (\mathbf{k}\times\mathbf{m})\cdot\mathbf{e}$. Hence we find for the interaction terms (3.1), (3.2):

$$\mathbf{Z_{km}} = z_\mathbf{km}(\mathbf{m}^2 - \mathbf{k}^2)(u_\mathbf{k}u_\mathbf{m} - \gamma b_\mathbf{k}b_\mathbf{m})\mathbf{e} + \tag{3.6}$$

$$z_\mathbf{km}[v_\mathbf{k}u_\mathbf{m} - v_\mathbf{m}u_\mathbf{k} - \gamma(a_\mathbf{k}b_\mathbf{m} - a_\mathbf{m}b_\mathbf{k})](\mathbf{k}+\mathbf{m})\times\mathbf{e},$$

$$\mathbf{S_{km}} = z_\mathbf{km}[u_\mathbf{k}b_\mathbf{m} - u_\mathbf{m}b_\mathbf{k}](\mathbf{k}+\mathbf{m})\times\mathbf{e} + (v_\mathbf{k}b_\mathbf{m} - v_\mathbf{m}b_\mathbf{k} + u_\mathbf{k}a_\mathbf{m} - u_\mathbf{m}a_\mathbf{k})\mathbf{k}\times\mathbf{m}. \tag{3.7}$$

It is evident that $\mathbf{Z_{km}} = \mathbf{S_{km}} = 0$ if the wave vectors $\mathbf{k}$ and $\mathbf{m}$ are parallel. If they are not then $z_\mathbf{km} \neq 0$ and formula (3.7) yields

$$u_\mathbf{k}b_\mathbf{m} - u_\mathbf{m}b_\mathbf{k} = 0. \tag{3.8}$$

There are three different cases: *(a)* $b_\mathbf{k} = 0$, $b_\mathbf{m} = 0$; *(b)* $b_\mathbf{k} \neq 0$, $b_\mathbf{m} \neq 0$; *(c)* $b_\mathbf{k} = 0$, $b_\mathbf{m} \neq 0$.

(a) If $b_\mathbf{k} = b_\mathbf{m} = 0$ then $a_\mathbf{k} \neq 0$ and $a_\mathbf{m} \neq 0$ and equation $\mathbf{S_{km}} = 0$ (3.7) yields $u_\mathbf{k}a_\mathbf{m} = u_\mathbf{m}a_\mathbf{k}$. Hence $u_\mathbf{k} = \lambda a_\mathbf{k}$, $u_\mathbf{m} = \lambda a_\mathbf{m}$. Equation $\mathbf{Z_{km}} = 0$ (3.6) gives $\lambda^2(\mathbf{m}^2 - \mathbf{k}^2) = 0$, $\lambda(v_\mathbf{k}a_\mathbf{m} - v_\mathbf{m}a_\mathbf{k}) = 0$. For $\lambda = 0$ we get solutions (3.3). For $\lambda \neq 0$ we have $\mathbf{k}^2 = \mathbf{m}^2$ and $v_\mathbf{k} = \tau a_\mathbf{k}$, $v_\mathbf{m} = \tau a_\mathbf{m}$. Hence we obtain solutions

$$\mathbf{V_k} = a_\mathbf{k}(\tau\mathbf{e} + \lambda\mathbf{k}\times\mathbf{e}),\ \mathbf{V_m} = a_\mathbf{m}(\tau\mathbf{e} + \lambda\mathbf{m}\times\mathbf{e}),\ \mathbf{B_k} = a_\mathbf{k}\mathbf{e},\ \mathbf{B_m} = a_\mathbf{m}\mathbf{e}.$$

These solutions belong to the solutions (3.4) as a special case for $b(t) = 0$.

(b) For $b_\mathbf{k} \neq 0$, $b_\mathbf{m} \neq 0$ we get from (3.8):

$$u_\mathbf{k} = \lambda b_\mathbf{k}, \quad u_\mathbf{m} = \lambda b_\mathbf{m}, \tag{3.9}$$

and equation $\mathbf{S_{km}} = 0$ (3.7) implies $(v_\mathbf{k} - \lambda a_\mathbf{k})b_\mathbf{m} = (v_\mathbf{m} - \lambda a_\mathbf{m})b_\mathbf{k}$. Hence we find

$$v_\mathbf{k} = \lambda a_\mathbf{k} + \tau b_\mathbf{k}, \quad v_\mathbf{m} = \lambda a_\mathbf{m} + \tau b_\mathbf{m}, \tag{3.10}$$

where $\lambda(t)$ and $\tau(t)$ are some functions. Substituting equalities (3.9), (3.10) into formula (3.6), we see that the condition $\mathbf{Z_{km}} = 0$ yields

$$(\lambda^2-\gamma)(\mathbf{m}^2-\mathbf{k}^2)=0, \quad (\lambda^2-\gamma)(a_{\mathbf{k}}b_{\mathbf{m}}-a_{\mathbf{m}}b_{\mathbf{k}})=0. \tag{3.11}$$

For $\lambda=\pm\sqrt{\gamma}=$ const, $b_{\mathbf{k}}\neq 0$, $b_{\mathbf{m}}\neq 0$, we obtain solutions (3.5). For $\lambda^2\neq\gamma$ equations (3.11) imply $\mathbf{k}^2=\mathbf{m}^2$, $\quad a_{\mathbf{k}}=\beta b_{\mathbf{k}}$, $\quad a_{\mathbf{m}}=\beta b_{\mathbf{m}}$, and we get solutions

$$\mathbf{V_k}=\lambda\mathbf{B_k}+\kappa C_{\mathbf{k}}\mathbf{e}, \quad \mathbf{V_m}=\lambda\mathbf{B_m}+\kappa C_{\mathbf{m}}\mathbf{e},$$

$$\mathbf{B_k}=C_{\mathbf{k}}(a\mathbf{e}+b\mathbf{k}\times\mathbf{e}), \quad \mathbf{B_m}=C_{\mathbf{m}}(a\mathbf{e}+b\mathbf{m}\times\mathbf{e}),$$

where $\kappa=b\tau$, $b\neq 0$ and $a=\beta b$. These solutions are equivalent to the solutions (3.4) after changing notations.

(c) For $b_{\mathbf{k}}=0$, $b_{\mathbf{m}}\neq 0$, equation (3.8) gives $u_{\mathbf{k}}=0$. Hence equations $\mathbf{Z_{km}}=0$ and $\mathbf{S_{km}}=0$ yield $v_{\mathbf{k}}u_{\mathbf{m}}=\gamma a_{\mathbf{k}}b_{\mathbf{m}}$, $\quad v_{\mathbf{k}}b_{\mathbf{m}}=a_{\mathbf{k}}u_{\mathbf{m}}$. Hence we get $u_{\mathbf{m}}=\pm\sqrt{\gamma}b_{\mathbf{m}}$, $v_{\mathbf{k}}=\pm\sqrt{\gamma}a_{\mathbf{k}}$ and solutions take the form

$$\mathbf{V_k}=\pm\sqrt{\gamma}a_{\mathbf{k}}\mathbf{e}, \quad \mathbf{V_m}=v_{\mathbf{m}}\mathbf{e}\pm\sqrt{\gamma}b_{\mathbf{m}}\mathbf{m}\times\mathbf{e},$$

$$\mathbf{B_k}=a_{\mathbf{k}}\mathbf{e}, \quad \mathbf{B_m}=a_{\mathbf{m}}\mathbf{e}+b_{\mathbf{m}}\mathbf{m}\times\mathbf{e}.$$

These solutions belong to the solutions (3.5) as a special case for $b_{\mathbf{k}}(t)=0$ where $a_{\mathbf{k}}(t)$, $a_{\mathbf{m}}(t)$, $b_{\mathbf{m}}(t)$ and $\tau(t)$ are arbitrary functions. ☐

Let us consider a set S of modes where any two $\mathbf{k}$- and $\mathbf{m}$-modes do not interact. For the real MHD solutions $\mathbf{V}(t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x})$, the set S contains along with any $\mathbf{k}$-mode also the $(-\mathbf{k})$-mode satisfying the equations (2.4). For the set S of non-interacting modes, the dynamical system (2.12) takes the form

$$\dot{\mathbf{V}}_{\mathbf{n}}=-\mathbf{n}^2\nu\mathbf{V_n}, \quad \dot{\mathbf{B}}_{\mathbf{n}}=-\mathbf{n}^2\eta\mathbf{B_n}, \tag{3.12}$$

$$\mathbf{V_n}(t)=\exp(-\mathbf{n}^2\nu t)\mathbf{V_n}(0), \quad \mathbf{B_n}(t)=\exp(-\mathbf{n}^2\eta t)\mathbf{B_n}(0). \tag{3.13}$$

Hence we see that equalities (3.5) with $\sqrt{\gamma}=$ const are possible for all t only if $\nu=\eta$. Thus for $\nu\neq\eta$ only three cases 1), 2), 3) of Lemma 1 realize.

Corollary 1 *For $\nu\neq\eta$, if the $\mathbf{k}$- and $\mathbf{m}$-modes do not interact and the wave vectors $\mathbf{k}$ and $\mathbf{m}$ are not parallel then*

$$(\mathbf{k}+\mathbf{m})\cdot(\mathbf{V_k}\times\mathbf{V_m})=0, \quad (\mathbf{k}+\mathbf{m})\cdot(\mathbf{B_k}\times\mathbf{B_m})=0. \tag{3.14}$$

Proof. Lemma 1 implies that if $\mathbf{k}^2 \neq \mathbf{m}^2$ then vectors $\mathbf{V_k}$, $\mathbf{V_m}$, $\mathbf{B_k}$, $\mathbf{B_m}$ have form (3.3) and hence $\mathbf{V_k} \times \mathbf{V_m} = 0$, $\mathbf{B_k} \times \mathbf{B_m} = 0$; if $\mathbf{k}^2 = \mathbf{m}^2$ then the vectors have form (3.4) and hence

$$\mathbf{V_k} \times \mathbf{V_m} = C_\mathbf{k} C_\mathbf{m} \tau\lambda(\mathbf{m}-\mathbf{k}) + \lambda^2 z_\mathbf{km}\mathbf{e}, \quad \mathbf{B_k} \times \mathbf{B_m} = C_\mathbf{k} C_\mathbf{m} ab(\mathbf{m}-\mathbf{k}) + b^2 z_\mathbf{km}\mathbf{e},$$

and equations (3.14) follow. □

Lemma 2 *For $\nu \neq \eta$, if the $\mathbf{k}$- and $\mathbf{m}$-modes with $\mathbf{k}^2 = \mathbf{m}^2 = N$ do not interact then the same modes with the new Fourier components*

$$\tilde{\mathbf{V}}_\mathbf{k} = \zeta\mathbf{V_k} - i\theta\mathbf{k} \times \mathbf{V_k}, \quad \tilde{\mathbf{V}}_\mathbf{m} = \zeta\mathbf{V_m} - i\theta\mathbf{m} \times \mathbf{V_m}, \tag{3.15}$$

$$\tilde{\mathbf{B}}_\mathbf{k} = \alpha\mathbf{B_k} - i\beta\mathbf{k} \times \mathbf{B_k}, \quad \tilde{\mathbf{B}}_\mathbf{m} = \alpha\mathbf{B_m} - i\beta\mathbf{m} \times \mathbf{B_m}$$

do not interact either. Here ζ, θ, α and β are real constants.

Indeed, for $\nu \neq \eta$ the non-interacting modes with $\mathbf{k}^2 = \mathbf{m}^2$ have form (3.4). These equations are invariant with respect to the transforms (3.15).

The transform (3.15) gives the linearly dependent vectors $\tilde{\mathbf{V}}_\mathbf{k}$ and $\mathbf{V_k}$ only if $\mathbf{k} \times \mathbf{V_k} = \lambda\mathbf{V_k}$. Cross-multiplying this equality with vector $\mathbf{k}$ we get $\mathbf{k} \times (\mathbf{k} \times \mathbf{V_k}) = -\mathbf{k}^2\mathbf{V_k} = \lambda\mathbf{k} \times \mathbf{V_k} = \lambda^2\mathbf{V_k}$. Hence $\lambda = \pm i|\mathbf{k}|$. Thus only vectors $\mathbf{V_k}$, $\mathbf{B_k}$ satisfying the equations

$$\mathbf{k} \times \mathbf{V_k} = \pm i|\mathbf{k}|\mathbf{V_k}, \tag{3.16}$$

$$\mathbf{k} \times \mathbf{B_k} = \pm i|\mathbf{k}|\mathbf{B_k}, \tag{3.17}$$

span 1-dimensional invariant subspaces for the transforms (3.15). For such complex vectors $\mathbf{V_k}$, $\mathbf{B_k}$ we have $\mathbf{V_k} \cdot \mathbf{V_k} = 0$, $\mathbf{B_k} \cdot \mathbf{B_k} = 0$.

The vectors

$$\mathbf{V}_{\mathbf{k}\pm} = \mathbf{A_k} \mp \frac{i}{|\mathbf{k}|}\mathbf{k} \times \mathbf{A_k}, \quad \mathbf{B}_{\mathbf{k}\pm} = \mathbf{C_k} \mp \frac{i}{|\mathbf{k}|}\mathbf{k} \times \mathbf{C_k} \tag{3.18}$$

represent all solutions to the equations (3.16), (3.17), where $\mathbf{A_k}$ and $\mathbf{C_k}$ are real vectors orthogonal to $\mathbf{k}$: $\mathbf{A_k} \cdot \mathbf{k} = 0$, $\mathbf{A}_{-\mathbf{k}} = \mathbf{A_k}$, $\mathbf{C_k} \cdot \mathbf{k} = 0$, $\mathbf{C}_{-\mathbf{k}} = \mathbf{C_k}$.

Theorem 1 *For the MHD equations (1.1) - (1.3) with $\nu \neq \eta$, the $\mathbf{k}$-modes of a set S do not interact pairwise if and only if one of the following four conditions are met:*
1) All wave vectors $\mathbf{k} \in S$ are parallel;

2) All wave vectors $\mathbf{k}$ *lie in one plane* L *and the Fourier components* $\mathbf{V_k}$, $\mathbf{B_k}$ *are orthogonal to* L*;*
3) The vectors $\mathbf{k}$ *belong to a circumference* $\mathbf{k}\cdot\mathbf{e}_1 = 0$, $\mathbf{k}^2 = N$ *and vectors* $\mathbf{V_k}$, $\mathbf{B_k}$ *have the form*

$$\mathbf{V_k} = C_\mathbf{k}(\tau\mathbf{e}_1 + i\lambda\mathbf{k}\times\mathbf{e}_1), \quad \mathbf{B_k} = C_\mathbf{k}(\alpha\mathbf{e}_1 + i\beta\mathbf{k}\times\mathbf{e}_1), \tag{3.19}$$

where $\mathbf{e}_1 \in \Lambda$ *is a given vector and* $\tau(t)$, $\lambda(t)$, $\alpha(t)$, $\beta(t)$ *are some real functions,* $C_{-\mathbf{k}}(t) = \overline{C_\mathbf{k}(t)}$*;*
4) The vectors $\mathbf{k}$ *belong to a sphere* $\mathbf{k}^2 = N$ *and vectors* $\mathbf{V_k}$, $\mathbf{B_k}$ *satisfy the equations*

$$\mathbf{k}\times\mathbf{V_k} = \pm i\sqrt{N}\mathbf{V_k}, \quad \mathbf{B_k} = C_0\exp(N(\nu-\eta)t)\mathbf{V_k}, \tag{3.20}$$

with the same sign for all wave vectors $\mathbf{k}\in S$, $C_0 = \overline{C_0} = \text{const}$.

Proof. The wave vectors $\mathbf{k}$ of the set S belong either to a straight line (then they are parallel) or to a plane or there are at least three linearly independent wave vectors $\mathbf{k}, \mathbf{m}, \mathbf{p} \in S$.
1) If the wave vectors $\mathbf{k}\in S$ are parallel then Lemma 1 implies that all $\mathbf{k}$-modes do not interact pairwise.
2) - 3) Let all wave vectors $\mathbf{k}$ of the set S belong to a plane L and let $\mathbf{e}$ be a unit vector orthogonal to L: $\mathbf{e}\cdot\mathbf{k} = 0$, $\mathbf{e}^2 = 1$. Let $\mathbf{k}$ and $\mathbf{m}$ be some non-parallel wave vectors in S. Lemma 1 proves that vectors $\mathbf{V_k}$, $\mathbf{V_m}$, $\mathbf{B_k}$ and $\mathbf{B_m}$ have form (3.4). Any other wave vector $\mathbf{p}\in S$ is non-parallel either to $\mathbf{k}$ or to $\mathbf{m}$. Hence by Lemma 1 $\mathbf{V_p} = C_\mathbf{p}(\tau\mathbf{e}+\lambda\mathbf{p}\times\mathbf{e})$, $\mathbf{B_p} = C_\mathbf{p}(a\mathbf{e}+b\mathbf{p}\times\mathbf{e})$. If $\lambda = 0$, $b = 0$ then the Fourier components $\mathbf{V_k} = v_\mathbf{k}\mathbf{e}$, $\mathbf{B_k} = a_\mathbf{k}\mathbf{e}$ for all wave vectors $\mathbf{k}\in S$, that proves the case *2)*. If either $\lambda \neq 0$ or $b \neq 0$ then Lemma 1 gives $\mathbf{k}^2 = \mathbf{m}^2 = \mathbf{p}^2 = N$. Hence all wave vectors $\mathbf{k}$ of the set S lie on the circumference $\mathbf{k}\cdot\mathbf{e} = 0$, $\mathbf{k}^2 = N$. The formulae (3.4) for the non-interacting modes with vectors $\mathbf{k}$, $-\mathbf{k}$, $\mathbf{m}$, $-\mathbf{m}$ are compatible with the conditions (2.4) only if λ/τ and b/a are purely imaginary. Hence equations (3.4) are reduced to $\mathbf{V_k} = C_\mathbf{k}(\tau\mathbf{e} + i\lambda\mathbf{k}\times\mathbf{e})$, $\mathbf{B_k} = C_\mathbf{k}(\alpha\mathbf{e} + i\beta\mathbf{k}\times\mathbf{e})$ with some real functions $\tau(t)$, $\lambda(t)$, $\alpha(t)$, $\beta(t)$. The unit vector $\mathbf{e}$ satisfying the equations $\mathbf{e}\cdot\mathbf{k} = 0$, $\mathbf{e}\cdot\mathbf{m} = 0$ is proportional to the vector $\mathbf{e}_1 = \lambda\mathbf{k}\times\mathbf{m} \in \Lambda$. Hence formula (3.19) and the case *3)* follow.

4) Let the set S contain some three linearly independent wave vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$. By Lemma 1 vectors $\mathbf{V_k}$, $\mathbf{V_m}$, $\mathbf{B_k}$, $\mathbf{B_m}$ have form (3.4) where $\mathbf{k}\cdot\mathbf{e} = 0$, $\mathbf{m}\cdot\mathbf{e} = 0$. It is easy to verify that vectors (3.4) satisfy two equations

$$(\mathbf{k}+\mathbf{m})\times[(\mathbf{k}\times\mathbf{V_k})\times\mathbf{V_m} + (\mathbf{m}\times\mathbf{V_m})\times\mathbf{V_k}] = 0, \tag{3.21}$$

$$(\mathbf{k}+\mathbf{m})\times[(\mathbf{k}\times\mathbf{B_k})\times\mathbf{B_m}+(\mathbf{m}\times\mathbf{B_m})\times\mathbf{B_k}]=0. \qquad (3.22)$$

If in (3.4) either $\lambda \neq 0$ or $b \neq 0$ then Lemma 1 gives $\mathbf{k}^2=\mathbf{m}^2$. If $\lambda=b=0$ then $\mathbf{V_k}=v_\mathbf{k}\mathbf{e}$, $\mathbf{V_m}=v_\mathbf{m}\mathbf{e}$, $\mathbf{B_k}=a_\mathbf{k}\mathbf{e}$, $\mathbf{B_m}=a_\mathbf{m}\mathbf{e}$. Hence the linear independence of the vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$ implies $\mathbf{V_k}\cdot\mathbf{p}\neq 0$, $\mathbf{V_m}\cdot\mathbf{p}\neq 0$ and Lemma 1 for the pairs $\mathbf{p}$, $\mathbf{k}$ and $\mathbf{p}$, $\mathbf{m}$ yields $\mathbf{p}^2=\mathbf{k}^2$ and $\mathbf{p}^2=\mathbf{m}^2$. Thus for $\lambda, b\neq 0$ and for $\lambda=b=0$ we have $\mathbf{k}^2=\mathbf{m}^2$. Hence we get $\mathbf{k}^2=\mathbf{m}^2=\mathbf{p}^2=N$. Since any vector $\mathbf{q}\in S$ is linearly independent with some two of the vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$, we get $\mathbf{q}^2=N$ for all vectors $\mathbf{q}\in S$.

Hence Lemma 2 is applicable to the set of modes S and gives the pairwise non-interacting $\mathbf{k}$-modes with the new Fourier components $\tilde{\mathbf{V}}_\mathbf{k}$, $\tilde{\mathbf{B}}_\mathbf{k}$ (3.15).

Let us prove by contradiction that all vectors $\mathbf{V_k}$ satisfy equations (3.16) with the same sign and all vectors $\mathbf{B_k}$ satisfy equations (3.17) with the same sign. The proof is the same for the Fourier components $\mathbf{V_k}$ and $\mathbf{B_k}$; we present below the proof for vectors $\mathbf{B_k}$.

If for some vector $\mathbf{B_k}$ equation (3.17) does not hold then we consider two wave vectors $\mathbf{m},\mathbf{p}\in S$ that form a linearly independent triple $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$. Since the modes $\mathbf{k}$, $-\mathbf{k}$, $\mathbf{m}$, $-\mathbf{m}$ do not interact, Lemma 1 and conditions (2.4) yield $\mathbf{B_k}=C_\mathbf{k}(\alpha\mathbf{e}+i\beta\mathbf{k}\times\mathbf{e})$, $\mathbf{B_m}=C_\mathbf{m}(\alpha\mathbf{e}+i\beta\mathbf{m}\times\mathbf{e})$ where $\mathbf{e}\cdot\mathbf{k}=\mathbf{e}\cdot\mathbf{m}=0$ and $\alpha(t)$, $\beta(t)$ are some real functions. For this vector $\mathbf{B_k}$, equations (3.17) are equivalent to the equalities $\alpha=\mp\beta\sqrt{N}$, $\alpha^2-\beta^2N=0$. Hence if vector $\mathbf{B_k}$ does not satisfy equations (3.17) we have $(\alpha^2-\beta^2N)\neq 0$. Applying transform (3.15), $\tilde{\mathbf{B}}_\mathbf{q}=\alpha\mathbf{B_q}-i\beta\mathbf{q}\times\mathbf{B_q}$, to the vectors $\mathbf{B_k}$, $\mathbf{B_{-k}}$, $\mathbf{B_m}$ and $\mathbf{B_{-m}}$, we obtain due to Lemma 2 the non-interacting $\mathbf{k}$-, $-\mathbf{k}$-, $\mathbf{m}$- and $-\mathbf{m}$-modes with the Fourier components

$$\tilde{\mathbf{B}}_\mathbf{k}=\lambda C_\mathbf{k}\mathbf{e}, \quad \tilde{\mathbf{B}}_{-\mathbf{k}}=\lambda\overline{C_\mathbf{k}}\mathbf{e}, \quad \tilde{\mathbf{B}}_\mathbf{m}=\lambda C_\mathbf{m}\mathbf{e}, \quad \tilde{\mathbf{B}}_{-\mathbf{m}}=\lambda\overline{C_\mathbf{m}}\mathbf{e}, \qquad (3.23)$$

where $\lambda=\alpha^2-\beta^2N\neq 0$. The vector $\tilde{\mathbf{B}}_\mathbf{p}=\alpha\mathbf{B_p}-i\beta\mathbf{p}\times\mathbf{B_p}\neq 0$ because $\mathbf{p}^2=N$ and $\alpha^2-\beta^2N\neq 0$. Since the wave vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$ are linearly independent and the equations $\mathbf{e}\cdot\mathbf{k}=0$, $\mathbf{e}\cdot\mathbf{m}=0$, $\tilde{\mathbf{B}}_\mathbf{p}\cdot\mathbf{p}=0$ hold, we have $\mathbf{e}\times\tilde{\mathbf{B}}_\mathbf{p}=\mathbf{U}\neq 0$. The formulae (3.23) imply that the four vectors

$$\tilde{\mathbf{B}}_\mathbf{k}\times\tilde{\mathbf{B}}_\mathbf{p}, \quad \tilde{\mathbf{B}}_{-\mathbf{k}}\times\tilde{\mathbf{B}}_\mathbf{p}, \quad \tilde{\mathbf{B}}_\mathbf{m}\times\tilde{\mathbf{B}}_\mathbf{p}, \quad \tilde{\mathbf{B}}_{-\mathbf{m}}\times\tilde{\mathbf{B}}_\mathbf{p}$$

are proportional to the vector $\mathbf{U}$. Applying the second of equations (3.14) to the four pairs of non-interacting modes $(\mathbf{k},\mathbf{p})$, $(-\mathbf{k},\mathbf{p})$, $(\mathbf{m},\mathbf{p})$, $(-\mathbf{m},\mathbf{p})$ we get

$$(\mathbf{k}+\mathbf{p})\cdot\mathbf{U}=0, \quad (-\mathbf{k}+\mathbf{p})\cdot\mathbf{U}=0, \quad (\mathbf{m}+\mathbf{p})\cdot\mathbf{U}=0, \quad (-\mathbf{m}+\mathbf{p})\cdot\mathbf{U}=0.$$

Hence the vector $\mathbf{U} \neq 0$ is orthogonal to the three linearly independent vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$, a contradiction. Hence any vector $\mathbf{B_k}$ satisfies one of the two equations (3.17).

Suppose that there are two wave vectors $\mathbf{k}, \mathbf{m} \in S$ for which the signs in the equations (3.17) are different. Let $\mathbf{p} \in S$ be any vector that forms a linearly independent triple $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$. With no loss of generality, let the signs in the equations (3.17) for the $\mathbf{k}$-, $\mathbf{m}$-, $\mathbf{p}$-modes be $+, -, +$. Therefore we have

$$(\mathbf{k}+\mathbf{m}) \times [(\mathbf{k} \times \mathbf{B_k}) \times \mathbf{B_m} + (\mathbf{m} \times \mathbf{B_m}) \times \mathbf{B_k}] = 2i\sqrt{N}(\mathbf{k}+\mathbf{m}) \times (\mathbf{B_k} \times \mathbf{B_m}),$$

$$(\mathbf{p}+\mathbf{m}) \times [(\mathbf{p} \times \mathbf{B_p}) \times \mathbf{B_m} + (\mathbf{m} \times \mathbf{B_m}) \times \mathbf{B_p}] = 2i\sqrt{N}(\mathbf{p}+\mathbf{m}) \times (\mathbf{B_p} \times \mathbf{B_m}).$$

Hence equations (3.22) for the non-interacting modes yield

$$(\mathbf{k}+\mathbf{m}) \times (\mathbf{B_k} \times \mathbf{B_m}) = 0, \quad (\mathbf{p}+\mathbf{m}) \times (\mathbf{B_p} \times \mathbf{B_m}) = 0. \tag{3.24}$$

Equations (3.14) give

$$(\mathbf{k}+\mathbf{m}) \cdot (\mathbf{B_k} \times \mathbf{B_m}) = 0, \quad (\mathbf{p}+\mathbf{m}) \cdot (\mathbf{B_p} \times \mathbf{B_m}) = 0. \tag{3.25}$$

The equations (3.24) and (3.25) imply $\mathbf{B_k} \times \mathbf{B_m} = 0$ and $\mathbf{B_p} \times \mathbf{B_m} = 0$. Hence the vectors $\mathbf{B_k}$, $\mathbf{B_m}$, $\mathbf{B_p}$ are proportional to a vector $\mathbf{U}_1$. Hence equations (2.5) yield that the vector $\mathbf{U}_1$ is orthogonal to the three linearly independent vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$, a contradiction. Hence equations (3.17) for all wave vectors $\mathbf{q} \in S$ have the same sign.

Using these results and equations (3.4) we obtain that the Fourier components for the $\mathbf{k}$- and $\mathbf{m}$-modes have the form

$$\mathbf{V_k} = D_{\mathbf{k}}(\mathbf{e} + i\varepsilon\alpha\mathbf{k} \times \mathbf{e}), \quad \mathbf{V_m} = D_{\mathbf{m}}(\mathbf{e} + i\varepsilon\alpha\mathbf{m} \times \mathbf{e}), \tag{3.26}$$

$$\mathbf{B_k} = CD_{\mathbf{k}}(\mathbf{e} + i\sigma\alpha\mathbf{k} \times \mathbf{e}), \quad \mathbf{B_m} = CD_{\mathbf{m}}(\mathbf{e} + i\sigma\alpha\mathbf{m} \times \mathbf{e}),$$

where $\alpha = 1/\sqrt{N}$, $\varepsilon = \pm 1$ and $\sigma = \pm 1$. The same relations are true for the $\mathbf{k}$-, $-\mathbf{k}$-, $\mathbf{m}$- and $-\mathbf{m}$-modes. Hence using equations (2.4) we obtain $D_{-\mathbf{k}} = \overline{D_{\mathbf{k}}}$, $D_{-\mathbf{m}} = \overline{D_{\mathbf{m}}}$, $C = \overline{C}$. For the non-interacting modes we find using equations (3.13): $C = C_0 \exp(N(\nu - \eta)t)$.

Let us prove by contradiction that the case $\sigma = -\varepsilon$ is not possible if the set S contains at least three linearly independent modes $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$. Indeed, equations (3.4) and (3.13) yield that the Fourier components $\mathbf{V_k}$, $\tilde{\mathbf{B}}_{\mathbf{k}} = \mathbf{B_k} + C\mathbf{V_k}$ also do not interact, where $C = C_0 \exp(N(\nu - \eta)t)$. For the case (3.26) and $\sigma = -\varepsilon$ we have

$$\tilde{\mathbf{B}}_{\mathbf{k}} = 2CD_{\mathbf{k}}\mathbf{e}, \quad \tilde{\mathbf{B}}_{-\mathbf{k}} = 2C\overline{D_{\mathbf{k}}}\mathbf{e}, \quad \tilde{\mathbf{B}}_{\mathbf{m}} = 2CD_{\mathbf{m}}\mathbf{e}, \quad \tilde{\mathbf{B}}_{-\mathbf{m}} = 2C\overline{D_{\mathbf{m}}}\mathbf{e}. \tag{3.27}$$

Let $\mathbf{p} \in S$ be a mode that forms a linearly independent triple $\mathbf{k}$, $\mathbf{m}$, $\mathbf{p}$ and $\tilde{\mathbf{B}}_{\mathbf{p}} = \mathbf{B_p} + C\mathbf{V_p}$. Equations (3.27) coincide with equations (3.23). Hence the same proof by contradiction implies that the case $\sigma = -\varepsilon$ is not possible for the non-interacting modes.

For $\sigma = \varepsilon$, equations (3.26) imply that $\mathbf{B_k} = C_0 \exp(N(\nu - \eta)t)\mathbf{V_k}$ for all modes $\mathbf{k} \in S$ and therefore equations (3.20) and case *4)* follow. $\square$

4 Exact space periodic solutions

Theorem 2 *For the viscous MHD equations (1.1) - (1.3) with $\nu \neq \eta$, there exists only four classes of space periodic solutions with pairwise non-interacting Fourier modes:*

1) The two families (for the sign $+$ and $-$) of exact solutions:

$$\mathbf{V}_{N\pm}(t, \mathbf{x}) = \exp(-N\nu t) \sum_{\mathbf{k} \in \Lambda^*} [\mathbf{A_k} \cos(\mathbf{k} \cdot \mathbf{x}) \pm \frac{1}{\sqrt{N}} \mathbf{k} \times \mathbf{A_k} \sin(\mathbf{k} \cdot \mathbf{x})], \tag{4.1}$$

$$\mathbf{B}_{N\pm}(t, \mathbf{x}) = C_0 \exp(N(\nu - \eta)t)\mathbf{V}_{N\pm}(t, \mathbf{x}), \quad p(t, \mathbf{x}) = C - \rho \mathbf{V}^2_{N\pm}/2,$$

where vectors $\mathbf{k} \in \Lambda^*$ satisfy the equation $\mathbf{k}^2 = N$ and arbitrary real vectors $\mathbf{A_k}$ conform the equations $\mathbf{A_k} \cdot \mathbf{k} = 0$. The exact solutions (4.1) form a linear space that can have an arbitrarily large dimension.

2) The exact solutions:

$$\mathbf{V}_{N\mathbf{e}}(t, \mathbf{x}) = e^{-N\nu t}(\tau \mathbf{U} + \lambda \operatorname{curl} \mathbf{U}), \quad \mathbf{B}_{N\mathbf{e}}(t, \mathbf{x}) = e^{-N\eta t}(\alpha \mathbf{U} + \beta \operatorname{curl} \mathbf{U}), \tag{4.2}$$

$$p(t, \mathbf{x}) = C + \left[\frac{e^{-2N\eta t}}{2\mu}(\beta^2 N - \alpha^2) - \frac{\rho e^{-2N\nu t}}{2}(\lambda^2 N - \tau^2)\right] \mathbf{e}^2 f_N^2 - \frac{\rho}{2}\mathbf{V}^2_{N\mathbf{e}},$$

where α, β, τ, λ are arbitrary reals and vector field U has the form

$$\mathbf{U}(t,\mathbf{x}) = f_N(t,\mathbf{x})\mathbf{e}, \quad f_N = \sum_{\mathbf{k}} [a_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{x}) - b_{\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{x})]. \tag{4.3}$$

Here vectors $\mathbf{k} \in \Lambda^*$ *and constants* $a_{\mathbf{k}}$, $b_{\mathbf{k}}$ *satisfy the equations*

$$\mathbf{k}\cdot\mathbf{e} = 0, \quad \mathbf{k}^2 = N, \quad a_{-\mathbf{k}} = a_{\mathbf{k}}, \quad b_{-\mathbf{k}} = -b_{\mathbf{k}}. \tag{4.4}$$

3) The convergent series defined for any vector $\mathbf{n} \in \Lambda^*$*:*

$$\mathbf{V}_{\mathbf{n}}(t,\mathbf{x}) = \sum_{k=1}^{\infty} \exp(-k^2 L\nu t) \left[\mathbf{A}_{k\mathbf{n}} \cos(k\mathbf{n}\cdot\mathbf{x}) + \mathbf{B}_{k\mathbf{n}} \sin(k\mathbf{n}\cdot\mathbf{x})\right], \tag{4.5}$$

$$\mathbf{B}_{\mathbf{n}}(t,\mathbf{x}) = \sum_{k=1}^{\infty} \exp(-k^2 L\eta t) \left[\mathbf{C}_{k\mathbf{n}} \cos(k\mathbf{n}\cdot\mathbf{x}) + \mathbf{D}_{k\mathbf{n}} \sin(k\mathbf{n}\cdot\mathbf{x})\right],$$

where vectors $\mathbf{A}_{k\mathbf{n}}$, $\mathbf{B}_{k\mathbf{n}}$, $\mathbf{C}_{k\mathbf{n}}$ *and* $\mathbf{D}_{k\mathbf{n}}$ *are orthogonal to the vector* $\mathbf{n}$, $L = \mathbf{n}^2$ *and pressure* $p(t,\mathbf{x}) = C - \mathbf{B}^2(t,\mathbf{x})/(2\mu)$.

4) The convergent series defined for any two non-parallel vectors $\mathbf{n},\mathbf{m} \in \Lambda^*$*:*

$$\mathbf{V}_{\mathbf{n},\mathbf{m}}(t,\mathbf{x}) = \sum_{k,\ell=-\infty}^{\infty} e^{-(k\mathbf{n}+\ell\mathbf{m})^2 \nu t} \times \tag{4.6}$$

$$[a_{k\ell} \cos((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x}) + b_{k\ell} \sin((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x})]\mathbf{n}\times\mathbf{m},$$

$$\mathbf{B}_{\mathbf{n},\mathbf{m}}(t,\mathbf{x}) = \sum_{k,\ell=-\infty}^{\infty} e^{-(k\mathbf{n}+\ell\mathbf{m})^2 \nu t} \times$$

$$[c_{k\ell} \cos((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x}) + d_{k\ell} \sin((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x})]\mathbf{n}\times\mathbf{m},$$

where k,ℓ *are arbitrary integers and* $a_{k\ell}$, $b_{k\ell}$, $c_{k\ell}$ *and* $d_{k\ell}$ *are arbitrary constants that define the convergent Fourier series (4.6); the pressure is* $p(t,\mathbf{x}) = C - \mathbf{B}^2(t,\mathbf{x})/(2\mu)$.

Proof. For the sub-case *4)* of Theorem 1 the formulae (3.18) and (3.13) after substituting into the Fourier series (2.3) give the exact solutions (4.1). The solutions (4.1) satisfy the Beltrami equation

$$\operatorname{curl} \mathbf{V}_{N\pm} = \mp\sqrt{N}\mathbf{V}_{N\pm}. \tag{4.7}$$

Let $r_3(N)$ be the number of integral solutions to the equation $\sum_{ij} n_i n_j \mathbf{k}_i \cdot \mathbf{k}_j = N$. Proposition 1 proves that the number $r_3(N)$ can be arbitrarily large if the rationality condition (2.15) is met. Each pair of $\mathbf{k}$- and $(-\mathbf{k})$-modes defines a 2-dimensional family of exact solutions (4.1). Hence the linear space $S_{N\pm}$ of exact solutions (4.1) has dimension $r_3(N)+1$. The known identity

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \operatorname{curl}\mathbf{V}\times\mathbf{V} + \operatorname{grad}(\mathbf{V}^2/2) \tag{4.8}$$

and equation (4.7) yield $(\mathbf{V}_{N\pm}\cdot\nabla)\mathbf{V}_{N\pm} = \operatorname{grad}(\mathbf{V}^2_{N\pm}/2)$. Hence equations (1.1) and (4.7) imply for the pressure $p = C - \rho\mathbf{V}^2_{N\pm}/2$.

The sub-case *3)* of Theorem 1 for $C_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ in (3.19) gives the exact solutions (4.2) - (4.3). Equations (4.4) imply the formulae $\mathbf{e}\cdot\operatorname{grad} f_N = 0$, $\Delta f_N = -N f_N$, $\operatorname{div}\mathbf{U} = 0$, $\Delta\mathbf{U} = -N\mathbf{U}$. Therefore using identity $\operatorname{curl}\operatorname{curl}\mathbf{V} = \operatorname{grad}\operatorname{div}\mathbf{V} - \Delta\mathbf{V}$, we obtain $\operatorname{curl}\mathbf{V}_{N\mathbf{e}} = \exp(-N\nu t)(\lambda N\mathbf{U} + \tau\operatorname{curl}\mathbf{U})$. Hence applying identity (2.11) we get

$$e^{2N\nu t}\operatorname{curl}\mathbf{V}_{N\mathbf{e}}\times\mathbf{V}_{N\mathbf{e}} = (\tau^2 - \lambda^2 N)\operatorname{curl}\mathbf{U}\times\mathbf{U} = \operatorname{grad}\left[(\lambda^2 N - \tau^2)\mathbf{e}^2 f_N^2\right]/2.$$

Hence (4.8) yields $(\mathbf{V}_{N\mathbf{e}}\cdot\nabla)\mathbf{V}_{N\mathbf{e}} = \operatorname{grad}\left[e^{-2N\nu t}(\lambda^2 N - \tau^2)\mathbf{e}^2 f_N^2 + \mathbf{V}^2_{N\mathbf{e}}\right]/2$. Analogously we find

$$e^{2N\eta t}\operatorname{curl}\mathbf{B}_{N\mathbf{e}}\times\mathbf{B}_{N\mathbf{e}} = (\alpha^2 - \beta^2 N)\operatorname{curl}\mathbf{U}\times\mathbf{U} = \operatorname{grad}\left[(\beta^2 N - \alpha^2)\mathbf{e}^2 f_N^2\right]/2,$$

Hence formula for the pressure (4.2) follows from equations (1.1) and (3.13).

The exact solutions corresponding to the sub-cases *1)* and *2)* of Theorem 1 take the form (4.5) and (4.6) respectively; the solutions are classically known [11]. For these solutions we have $(\mathbf{V}\cdot\nabla)\mathbf{V} = 0$ and $(\mathbf{B}\cdot\nabla)\mathbf{B} = 0$. Hence applying identity (4.8) for $\mathbf{B}(t,\mathbf{x})$ we find $\operatorname{curl}\mathbf{B}\times\mathbf{B} = -\operatorname{grad}(\mathbf{B}^2/2)$ and hence equations (1.1) imply for the pressure $p(t,\mathbf{x}) = C - \mathbf{B}^2(t,\mathbf{x})/(2\mu)$. $\square$

REMARK 4. Solutions (4.1) for the simplest case $N = 1$ and $(\mathbf{k}_i)_j = \delta_{ij}$ and after the change of time $\mathrm{d}\tau/\mathrm{d}t = \exp(-N\nu t)$ turn into the ABC-flows [13, 14]:

$$\dot{x}_1 = A\sin x_3 + C\cos x_2, \quad \dot{x}_2 = B\sin x_1 + A\cos x_3, \quad \dot{x}_3 = C\sin x_2 + B\cos x_1,$$

where $\mathbf{x}(t)$ describes the trajectory of a fluid particle (Lagrangian description). The corresponding vectors $\mathbf{k}$ and $\mathbf{A}_{\mathbf{k}}$ in (4.1) are: $\mathbf{k}_1 = (1,0,0)$, $\mathbf{A}_{\mathbf{k}_1} = (0,0,B)$, $\mathbf{k}_2 = (0,1,0)$, $\mathbf{A}_{\mathbf{k}_2} = (C,0,0)$, $\mathbf{k}_3 = (0,0,1)$, $\mathbf{A}_{\mathbf{k}_3} = (0,A,0)$ and the minus sign is chosen in (4.1). Hence the stationary periodic solutions (4.1) ($\nu = 0$, $C_0 = 0$) for an arbitrary N and arbitrary vector periods $\mathbf{p}_1$,

$\mathbf{p}_2$, $\mathbf{p}_3$ form an infinite family of generalizations of the ABC-flows of dimensions $r_3(N)$ that can be arbitrarily large. As is known [13, 14], the generic trajectories of the ABC-flows are dense in 3-D domains. Hence the generic solutions (4.1) also describe a complex dynamics of fluid with dense trajectories. Therefore exact solutions (4.1) can be used, as well as the ABC-flows, to model Lagrangian turbulence of fluid inspite of the absence of the Fourier modes interaction and cascades of energy.

5 Space periodic solutions for $\nu = \eta$

Theorem 3 *For the MHD equations (1.1) - (1.3) with $\nu = \eta$, the $\mathbf{k}$-modes of a set S do not interact pairwise if and only if one of the following six conditions are met: the four of Theorem 2 and the two additional conditions:*
5) The wave vectors $\mathbf{k}$ belong to a plane $\mathbf{k} \cdot \mathbf{e} = 0$, vectors $\mathbf{B_k}$ and $\mathbf{V_k}$ have the form

$$\mathbf{B_k} = a_{\mathbf{k}}\mathbf{e} + i\lambda b_{\mathbf{k}}\mathbf{k} \times \mathbf{e}, \quad \mathbf{V_k} = \frac{\varepsilon}{\sqrt{\rho\mu}}\mathbf{B_k} + \tau b_{\mathbf{k}}\mathbf{e}, \tag{5.1}$$

where $\mathbf{e} \in \Lambda$ is a given vector, $\varepsilon = \pm 1$, τ, λ are arbitrary reals and complex functions $a_{\mathbf{k}}(t)$, $b_{\mathbf{k}}(t)$ satisfy the equations $a_{-\mathbf{k}}(t) = \overline{a_{\mathbf{k}}(t)}$, $b_{-\mathbf{k}} = \overline{b_{\mathbf{k}}(t)}$;
6) The Fourier components $\mathbf{B_k}$ and $\mathbf{V_k}$ are linked by the equipartition equation ($\varepsilon = \pm 1$):

$$\mathbf{V_k} = \frac{\varepsilon}{\sqrt{\rho\mu}}\mathbf{B_k}. \tag{5.2}$$

The proof is analogous to that of Theorem 2 and uses condition *4)* of Lemma 1 that is realizable only at $\nu = \eta$ in view of equations (3.13).

For the case *5)*, the space periodic solutions belong to the family of vector fields

$$\mathbf{V} = \pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B} + \tau f\mathbf{e}, \quad \mathbf{B} = g\mathbf{e} + \lambda \operatorname{grad} f \times \mathbf{e}, \tag{5.3}$$

where τ and λ are arbitrary constants and f, g are some functions satisfying the equations

$$\operatorname{grad} f \cdot \mathbf{e} = 0, \quad \operatorname{grad} g \cdot \mathbf{e} = 0. \tag{5.4}$$

For the vector fields (5.3), we have: $\mathbf{V} \times \mathbf{B} = \lambda\tau\mathbf{e}^2 \operatorname{grad} f^2/2$, and

$$\operatorname{curl}\mathbf{V}\times\mathbf{V}-\frac{1}{\rho\mu}\operatorname{curl}\mathbf{B}\times\mathbf{B}=\mp\frac{\tau\mathbf{e}^2}{\sqrt{\rho\mu}}\operatorname{grad}(fg)-\frac{\tau^2\mathbf{e}^2}{2}\operatorname{grad}f^2.$$

Substituting these formulae, we find that equations (1.1) - (1.3) for $\nu=\eta$ are reduced to the two diffusion equations

$$\frac{\partial f}{\partial t}=\nu\Delta f,\quad \frac{\partial g}{\partial t}=\nu\Delta g, \tag{5.5}$$

and pressure p takes the form

$$p=C_1-\frac{\rho\mathbf{V}^2}{2}\pm\tau\mathbf{e}^2\sqrt{\frac{\rho}{\mu}}fg+\frac{\rho\tau^2\mathbf{e}^2}{2}f^2. \tag{5.6}$$

Let $\mathbf{k},\mathbf{m}\in\Lambda^*$ be two linearly independent vectors and vector $\mathbf{e}=\mathbf{k}\times\mathbf{m}$. Let functions f and g be the convergent series

$$f(t,\mathbf{x})=\sum_{p,q=-\infty}^{\infty}f_{pq}\exp(-\nu(p\mathbf{k}+q\mathbf{m})^2t+i(p\mathbf{k}+q\mathbf{m})\cdot\mathbf{x}), \tag{5.7}$$

$$g(t,\mathbf{x})=\sum_{p,q=-\infty}^{\infty}g_{pq}\exp(-\nu(p\mathbf{k}+q\mathbf{m})^2t+i(p\mathbf{k}+q\mathbf{m})\cdot\mathbf{x}),$$

where $f_{-p-q}=\overline{f_{pq}}$ and $g_{-p-q}=\overline{g_{pq}}$. Functions (5.7) clearly are periodic and satisfy equations (5.4) and (5.5). Hence the corresponding vector fields (5.3) are the MHD solutions with non-interacting Fourier modes.

For the case *6)*, the space periodic solutions belong to the family of unsteady equipartition solutions [15]:

$$\mathbf{V}(t,\mathbf{x})=\pm\frac{1}{\sqrt{\rho\mu}}\mathbf{B}(t,\mathbf{x}), \tag{5.8}$$

for which the densities of the kinetic and magnetic energies $\rho\mathbf{V}^2/2$ and $\mathbf{B}^2/(2\mu)$ are equal. For solutions (5.8), the MHD equations (1.1) - (1.3) are equivalent to the system $\partial\mathbf{B}/\partial t=\eta\Delta\mathbf{B}$, $\operatorname{div}\mathbf{B}=0$. Periodic solutions to these equations are the convergent series

$$\mathbf{B}(t,\mathbf{x})=i\sum_{\mathbf{n}}\exp(-\eta\mathbf{n}^2t+i\mathbf{n}\cdot\mathbf{x})\mathbf{n}\times\mathbf{A}_{\mathbf{n}}, \tag{5.9}$$

where $\mathbf{A}_{\mathbf{n}}\in\mathbb{C}^3$ are arbitrary vectors satisfying equations $\mathbf{A}_{-\mathbf{n}}=\overline{\mathbf{A}_{\mathbf{n}}}$ and $|\mathbf{A}_{\mathbf{n}}|<C/\mathbf{n}^3$.

6 Conclusions

We have derived a complete classification of space periodic MHD solutions with pairwise non-interacting Fourier modes. The classification is independent of the vector periods $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. However the dimensions D of invariant submanifolds of solutions (4.1) depend on the periods in a drastic way: D can be arbitrarily large if the rationality condition (2.15) is met and $D = 2$ for the opposite case. For $\nu \neq \eta$ there are four infinite series of invariant submanifolds for the MHD dynamical systems (2.12) on which all solutions are smooth and exist for all moments of time $t \geq 0$. The wave vectors $\mathbf{k} \in \Lambda^*$ for them belong to the following four families of sets:

(1) the spheres $S^2 \cap \Lambda^*$: $\mathbf{k}^2 = \alpha^2$,
(2) the circumferences $S^1 \cap \Lambda^*$: $\mathbf{k} \cdot \mathbf{e} = 0$, $\mathbf{k}^2 = \alpha^2$,
(3) the straight lines $L^1 \cap \Lambda^*$: $\mathbf{k} = \lambda \mathbf{n}$,
(4) the planes $P^2 \cap \Lambda^*$: $\mathbf{k} \cdot \mathbf{e} = 0$.

Here $\mathbf{e} \in \Lambda$, and $\mathbf{n} \in \Lambda^*$. The families (3) and (4) correspond to the classically known exact solutions (4.5) and (4.6), see [11]. For $\nu = \eta$, there are five infinite series of such invariant submanifolds and two submanifolds of equipartition solutions $\mathbf{V}(t, \mathbf{x}) = \pm \mathbf{B}(t, \mathbf{x})/\sqrt{\rho\mu}$. The direct and inverse cascades do not work for all these solutions since there is no interaction between their Fourier modes. However, dynamics of plasma is very complex for the exact periodic solutions (4.1) and (5.8), (5.9). The generic trajectories of fluid particles for solutions (4.1) and (5.8), (5.9) are dense in 3-D domains, as for the ABC-flows [13, 14] that belong to the solutions (4.1) for the simplest case $N = 1$. The generic exact solutions (4.1) and (5.8), (5.9) depend on all four variables t, x, y, z and have no geometrical symmetries.

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