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Algebraic Structure of Discrete Zero Curvature Equations and Master Symmetries of Discrete Evolution Equations

Wen-Xiu Ma[†1] and Benno Fuchssteiner[‡2]

[†]Department of Mathematics, City University of Hong Kong,
Kowloon, Hong Kong, China

[‡]Department of Mathematics, University of Paderborn, D-33098 Paderborn, Germany

Abstract

An algebraic structure related to discrete zero curvature equations is established. It is used to give an approach for generating master symmetries of the first degree for systems of discrete evolution equations and an answer to why there exist such master symmetries. The key of the theory is to generate nonisospectral flows ($\lambda_t = \lambda^l,\ l \geq 0$) from the discrete spectral problem associated with a given system of discrete evolution equations. Three examples are given.

I Introduction

The theory of integrable systems has various aspects, although the term "integrable" is somewhat ambiguous, especially for systems of partial differential equations. Symmetries are one of those important aspects and have a deep mathematical and physical background. When any special character, for example the Lax pair, hasn't been found for a given system of continuous or discrete equations, among the most efficient ways is to consider its symmetries in order to obtain exact solutions. It is through symmetries that Russian scientists et al. developed some theories for testing the integrability of systems of evolution equations, both continuous and discrete, and classified many types of systems of nonlinear equations that possess higher differential or differential-difference degree symmetries (for example, see [1] [2]). Usually an integrable system of equations is referred as to a system possessing infinitely many symmetries [3] [4]. Moreover these symmetries form nice and interesting algebraic structures [3] [4].

For a given system of evolution equations $u_t = K(u)$, both continuous and discrete, a vector field $\sigma(u)$ is called its symmetry if $\sigma(u)$ satisfies its linearized system

$$\frac{d\sigma(u)}{dt} = K'[\sigma],\ \text{i.e.,}\ \frac{\partial \sigma}{\partial t} = [K, \sigma] := K'[\sigma] - \sigma'[K], \tag{1}$$

[1]Email: mawx@cityu.edu.hk
[2]Email: benno@uni-paderborn.de

where the prime means the Gateaux derivative. Starting from a Lie-point symmetry, we can often construct the corresponding explicit group-invariant solutions. A symmetry σ may, of course, depend explicitly on the evolution variable t. If a symmetry σ of the system $u_t = K(u)$ not depending explicitly on t is a polynomial in t, i.e.,

$$\sigma(t,u) = \sum_{i=0}^{n} \frac{t^i}{i!}\rho_i(u), \ \ n \geq 1, \tag{2}$$

then we have

$$\rho_i = [K, \rho_{i-1}], \ \ 1 \leq i \leq n, \tag{3}$$

and

$$(\mathrm{ad}_K)^{n+1}\rho_0 = 0, \ \ \text{where } (\mathrm{ad}_K)\rho_0 = [K, \rho_0]. \tag{4}$$

Therefore the symmetry (2) is totally determined by a vector field ρ_0 satisfying (4). This kind of vector field ρ_0 has been discussed in considerable detail and is called a master symmetry of degree n of $u_t = K(u)$ by one of the authors (BF) in Ref. [5].

The appearance of first degree master symmetries gives a common character for integrable systems of continuous evolution equations, both in $1+1$ dimensions and in $2+1$ dimensions, for example, the KdV equation and the KP equation. The resulting symmetries are sometimes called τ-symmetries (for more information, see [6] for example) and usually constitute centerless Virasoro algebras together with time-independent symmetries [7] [8] [9]. Moreover these τ-symmetries may be generated by use of zero curvature equations or Lax equations [10] and the corresponding master symmetry flows may also be solved by the inverse scattering method [11] [12]. In the case of systems of discrete evolution equations, there exist some similar results. For example, many systems of discrete evolution equations have τ-symmetries and centerless Virasoro symmetry algebras [13] [14] [15], and the inverse scattering method may still be applied in solving themselves and their master symmetry flows [16] [17] [18] [19]. So far, however, to the best of our knowledge, there hasn't been a systematic mathematical theory to explain why there exist τ-symmetries for systems of discrete evolution equations and how we can construct those τ-symmetries when they exist, from the point of discrete zero curvature equations.

Throughout this paper, "master symmetries" is used to express the first degree master symmetries that generate τ-symmetries. Our purpose is to give an algebraic explanation of the first question above and to provide a procedure to generate those master symmetries for a given lattice hierarchy. The discrete zero curvature equation is our basic tool to give rise to our answer and procedure. The Volterra lattice hierarchy, the Toda lattice hierarchy and a sub-KP lattice hierarchy are chosen and analyzed as some illustrative examples, which have one dependent variable, two dependent variables and three dependent variables, respectively.

Let us now describe our notation. Assume that $u = (u_1, \cdots, u_q)^T$ where $u_i = u_i(t, n)$, $1 \le i \le q$, are real functions defined over $\mathbb{R} \times \mathrm{Z}$ (in the case of the complex function, the discussion is similar), and let $\mathcal{B}$ denote all real functions $P[u] = P(t, n, u)$ which are C^∞-differentiable with respect to t and n, and C^∞-Gateaux differentiable with respect to u. We always write E as a shift operator and

$$(E^m x)(n) = x^{(m)}(n) = x(m+n), \ \text{where } x : \mathrm{Z} \to \mathbb{R}, \ m, n \in \mathrm{Z} \, . \tag{5}$$

Note that $x^{(m)}$ here doesn't mean the m-th derivative. Set $\mathcal{B}^r = \{(P_1, \cdots, P_r)^T | \, P_i \in \mathcal{B}, \ 1 \le i \le r\}$, and denote by $\mathcal{V}^r$ all matrix operators $\Phi = (\Phi_{ij})_{r\times r}$ where the entries $\Phi_{ij} = \Phi_{ij}(t, n, u) \in \mathcal{B}$, and by $\widetilde{\mathcal{V}}^r$, all matrix operators depending on a parameter λ: $U = (U_{ij})_{r\times r}$, where the entries $U_{ij} = U_{ij}(t, n, u, \lambda) \in \mathcal{B}$ for all λ, being C^∞-differentiable with respect to λ.

We will need a multiplication operator

$$[n] : \mathcal{B} \to \mathcal{B}, \ P[u] \mapsto [n]P[u], \ ([n]P[u])(m) = m(P[u])(m), \tag{6}$$

which is often involved in the construction of master symmetries. This avoids an unclear expression $nP[u]$, which may also mean $(nP[u])(m) = n(P[u])(m)$. For example, it is absolutely clear that $([n]P[u])(m) = mu(m-1) + mu(m)$, when $P[u] = E^{-1}u + u$. We also need a difference operator $\Delta = E - E^{-1}$, whose inverse operator may be defined by

$$(\Delta^{-1}u)(n) = ((E - E^{-1})^{-1}u)(n) := \frac{1}{2}\Big(\sum_{k=-\infty}^{-1} u(n+1+2k) - \sum_{k=1}^{\infty} u(n-1+2k)\Big), \tag{7}$$

where u is required to be rapidly vanishing at the infinity. Moreover we define

$$(\Delta^{-1}\alpha) = (1/2)\alpha[n], \ \text{i.e.,} \ (\Delta^{-1}\alpha)(n) = (1/2)\alpha n, \ \alpha = \text{const.} \tag{8}$$

Obviously we can find that

$$(E-1)^{-1} = \Delta^{-1}(1 + E^{-1}), \ (1 - E^{-1})^{-1} = \Delta^{-1}(E+1), \tag{9}$$

and thus

$$(E-1)^{-1}\alpha = \alpha[n], \ (1 - E^{-1})^{-1}\alpha = \alpha[n], \ \alpha = \text{const.}, \tag{10}$$

which may also be viewed as a definition of two inverse operators $(E-1)^{-1}$ and $(1-E^{-1})^{-1}$. Note that here we have used the operator $[n]$ so that two functions $(E-1)^{-1}\alpha$ and $(1 - E^{-1})^{-1}\alpha$ have the other clear expressions. The operators Δ^{-1}, $(E-1)^{-1}$ and $(1-E^{-1})^{-1}$ often appears in the expressions of master symmetries and thus master symmetries are usually nonlocal vector fields belonging to $\mathcal{B}^q$.

In order to carefully analyze algebraic structures related to symmetries, we specify the definition of the Gateaux derivative $X'[S]$ of any vector valued function $X \in \mathcal{B}^r$ at a direction $S \in \mathcal{B}^q$ as follows

$$X'[S] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X(u + \varepsilon S), \tag{11}$$

which implies that X' is an operator from $\mathcal{B}^q$ to $\mathcal{B}^r$, and need the following two product operations

$$[K, S] = K'[S] - S'[K],\ \ K, S \in \mathcal{B}^q, \tag{12}$$

$$[\![f, g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda),\ \ f, g \in C^\infty(\mathbb{R}), \tag{13}$$

where $C^\infty(\mathbb{R})$ denotes the space of smooth functions defined over $\mathbb{R}$. It is known that $(\mathcal{B}^q, [\cdot,\cdot])$ and $(C^\infty(\mathbb{R}), [\![\cdot,\cdot]\!])$ are all Lie algebras.

We now assume that $U \in \widetilde{\mathcal{V}}^r$ and the Gateaux derivative operator U' is injective throughout the paper. Let us consider the discrete spectral problem

$$\begin{cases} E\phi = U\phi = U(n, u, \lambda)\phi, \\ \phi_t = V\phi = V(n, u, \lambda)\phi, \end{cases} \tag{14}$$

where $V \in \widetilde{\mathcal{V}}^r$. Its adjoint system reads as

$$\begin{cases} E^{-1}\psi = U\psi = U(n, u, \lambda)\psi, \\ \psi_t = (EV)\psi = (EV(n, u, \lambda))\psi. \end{cases}$$

Their integrability conditions are given by the following discrete zero curvature equation

$$U_t = (EV)U - UV. \tag{15}$$

If the operator equation (15) is equivalent to a system of discrete evolution equations $u_t = K(n, u),\ K \in \mathcal{B}^q$, then it is called a discrete zero curvature representation of $u_t = K(n, u)$. Evidently

$$U_t = U'[u_t] + f(\lambda)U_\lambda,\ \text{if}\ \lambda_t = f(\lambda),$$

where $U_\lambda = \frac{\partial U}{\partial \lambda}$. Therefore a system of discrete evolution equations $u_t = K(n, u),\ K \in \mathcal{B}^q$, is the integrability condition of (14) with the evolution law $\lambda_t = f(\lambda)$ if and only if

$$U'[K] + fU_\lambda = (EV)U - UV. \tag{16}$$

Note that the injective property of U' is indispensable in deriving zero curvature representations of systems of evolution equations. The equation (16) exposes an essential relation between a system of discrete evolution equations and its discrete zero curvature representation. It will play an important role in the context of our construction of master symmetries.

The paper is divided into five sections. The next section will be devoted to a general algebraic structure related to discrete zero curvature equations. Then the third section will establish an approach for constructing master symmetries by the use of discrete zero curvature representations, along with an explanation of why there exist master symmetries for systems of discrete evolution equations. In the fourth section, we will go on to illustrate our approach by three concrete examples of lattice hierarchies. Finally, the fifth section provides a conclusion and some remarks.

II Basic algebraic structure

We aim to discuss Lie algebraic structures of symmetries including master symmetries by using zero curvature equations. It is natural to ask what algebraic structure exists, related to zero curvature equations. To answer this question, we first plan to expose a Lie algebraic structure for the space $(\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$.

Let (K, V, f), $(S, W, g) \in (\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$, in other words, K, S are vector fields, V, W are $r \times r$ matrix operators and f, g are smooth functions. We introduce their product

$$[\![(K, V, f), (S, W, g)]\!] = ([K, S], [\![V, W]\!], [\![f, g]\!]), \tag{17}$$

where $[K, S], [\![f, g]\!]$ are defined by (12), (13), respectively, and $[\![V, W]\!]$ is defined by

$$[\![V, W]\!] = V'[S] - W'[K] + [V, W] + gV_\lambda - fW_\lambda, \tag{18}$$

where $[V, W] = VW - WV$. The same product as (18) has been introduced for the continuous case in [20].

Theorem 1 (Lie algebra) *The space* $((\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R})), [\![\cdot, \cdot]\!])$ *is a Lie algebra, the product* $[\![\cdot, \cdot]\!]$ *being defined by (17), i.e.,*

$$[\![(K, V, f), (S, W, g)]\!] = ([K, S], [\![V, W]\!], [\![f, g]\!])$$

where

$$\begin{cases} [K, S] = K'[S] - S'[K], \\ [\![V, W]\!] = V'[S] - W'[K] + [V, W] + gV_\lambda - fW_\lambda, \\ [\![f, g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda). \end{cases}$$

The proof of the theorem will be given in Appendix A. Upon looking at the product a little bit more carefully, we can find that the Lie algebra $((\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R})), [\![\cdot,\cdot]\!])$ has a Lie sub-algebra $((\mathcal{B}^q, \widetilde{\mathcal{V}}^r, 0), [\![\cdot,\cdot]\!])$, for which everything corresponds to the isospectral case. Moreover the center of an element of this Lie sub-algebra is often Abelian.

The above theorem exposes that a Lie algebraic structure hidden in the back of vector fields, Lax operators and spectral evolution laws. Usually we just touch Lie algebraic structures of vector fields while discussing symmetries. If we analyze symmetries from the point of zero curvature equations, it is natural that we need to find and handle Lie algebraic structure for all triples $(K, V, f) \in (\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$ where K, V and f are related to each other by zero curvature equations. In other words, we need to observe how two triples $(K, V, f), (S, W, g)$ that appear in zero curvature equations connect with each other. The following theorem tells us that such a kind of connection can be reflected by the Lie algebraic operation of $(\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$ in Theorem 1. Its proof can be found in Appendix B.

Theorem 2 (Algebraic structure of representations) *Let* $V, W \in \widetilde{\mathcal{V}}^r$, $K, S \in \mathcal{B}^q$ *and* $f, g \in C^\infty(\mathbb{R})$. *If two equalities*

$$(EV)U - UV = U'[K] + fU_\lambda, \tag{19}$$

$$(EW)U - UW = U'[S] + gU_\lambda, \tag{20}$$

hold, then we have a third equality

$$(E[\![V, W]\!])U - U[\![V, W]\!] = U'[T] + [\![f, g]\!]U_\lambda, \quad T = [K, S], \tag{21}$$

where $[\![V, W]\!]$, $[K, S]$ *and* $[\![f, g]\!]$ *are defined by (18), (12) and (13), respectively.*

According to this theorem, we can easily find that if a system $u_t = K(n, u)$ is isospectral, i.e., $\lambda_t = f = 0$, then the product system $u_t = [K, S]$ for any $S \in \mathcal{B}^q$ can be viewed to be still isospectral because we have $[\![f, g]\!] = [\![0, g]\!] = 0$, where g is the evolution law corresponding $u_t = S(n, u)$. Actually the above theorem gives a discrete zero curvature representation for a product system $u_t = [K, S]$, which possesses the same order matrix operators as ones for the original systems $u_t = K(n, u)$ and $u_t = S(n, u)$ (see [20] for the continuous case). Combining two theorems above can show the following.

Corollary 1 *The space defined by*

$$\{(K, V, f) \in (\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R})) |\, U'[K] + fU_\lambda = (EV)U - UV\}$$

is a Lie sub-algebra of $(\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$ *under the Lie product (17).*

This corollary tells us a Lie algebraic structure about zero curvature equations, wich will help us to establish Lie algebraic structures of symmetries including master symmetries.

However, for zero curvature representations, some interesting problems remain to be solved. For example, assuming that two initial systems $u_t = K(n,u)$ and $u_t = S(n,u)$ have zero curvature representations possessing different order matrix operators, we want to know whether there exist any zero curvature representations for the product system $u_t = [K,S]$ and what structures the resulting zero curvature representations possess if the answer is yes. It is likely to be helpful in solving this problem to use the Kronecker product as in [22].

III Lax operators and master symmetries

Assume that we already have a hierarchy of isospectral integrable systems of discrete evolution equations of the form

$$u_t = K_k = \Phi^k K_0,\ \Phi \in \mathcal{V}^q,\ K_0 \in \mathcal{B}^q,\ k \geq 0. \tag{22}$$

or of the form

$$u_t = K_k = JG_k = MG_{k-1},\ J, M \in \mathcal{V}^q,\ G_{k-1} \in \mathcal{B}^q,\ k \geq 0. \tag{23}$$

associated with a discrete spectral problem

$$E\phi = U\phi,\ \phi = (\phi_1, \cdots, \phi_r)^T. \tag{24}$$

The second form (23) occurs more often than the first form (22), although it is simpler to deal with the first form (22). Generally speaking, the operator Φ above is a hereditary symmetry operator (see [23] for definition) determined by the spectral problem (24) and J, M constitute a bi-Hamiltonian pair [24] [25]. If we choose $\Phi = MJ^{-1}$ when J is invertible, then the form (23) may be changed into the form (22). Usually Φ involves nonlocal operators, for example, Δ^{-1}, but J, M often involves only local operators. Our examples are all local Hamiltonian systems.

III.1 Structures of Lax operators

For a given $X \in \mathcal{B}^q$ or $G \in \mathcal{B}^q$, let us introduce an operator equation of $\Omega \in \widetilde{\mathcal{V}}^r$:

$$(E\Omega(X))U - U\Omega(X) = U'[\Phi X] - \lambda U'[X], \tag{25}$$

in the case of (22), or an operator equation of $\Omega_J \in \widetilde{\mathcal{V}}^r$:

$$(E\Omega_J(G))U - U\Omega_J(G) = U'[MG] - \lambda U'[JG], \tag{26}$$

in the case of (23). We call them the characteristic operator equations of U. The introduction of the operator equation (25) (or (26)) is an important step in our manipulation. Obviously, we can choose $\Omega_J(G) = \Omega(JG)$ when $\Phi = MJ^{-1}$. We demand that (25) (or (26)) has solutions, and $\Omega = \Omega(X)$ (or $\Omega_J(G)$) is a particular solution at X (or at G). Usually (25) (or (26)) has infinitely many solutions once one solution exists, because we can construct others $\Omega(X) + fV$ for any $f \in C^\infty(\mathbb{R})$ when $V \in \mathcal{V}^r \otimes C[\lambda, \lambda^{-1}]$ solves the stationary discrete zero curvature equation $(EV)U - UV = 0$. The existence of solutions of $(EV)U - UV = 0$ may result from the existence of an isospectral hierarchy associated with $E\phi = U\phi$.

Theorem 3 (Structure of Lax operators) *Let two matrices $V_0, W_0 \in \widetilde{\mathcal{V}}^r$ and two vector fields $K_0, \rho_0 \in \mathcal{B}^q$ (or $\rho_0 = J\gamma_0,\ \gamma_0 \in \mathcal{B}^q$) satisfy*

$$(EV_0)U - UV_0 = U'[K_0], \tag{27}$$

$$(EW_0)U - UW_0 = U'[\rho_0] + \lambda U_\lambda. \tag{28}$$

If we define $\rho_l,\ l \geq 1$, $V_k,\ k \geq 1$, and $W_l,\ l \geq 1$, as follows

$$\rho_l = \Phi^l \rho_0,\ l \geq 1\ (\mathit{or}\ \rho_l = J\gamma_l = M\gamma_{l-1},\ \gamma_l \in \mathcal{B}^q,\ l \geq 1), \tag{29}$$

$$V_k = \lambda^k V_0 + \sum_{i=1}^{k} \lambda^{k-i} \Omega(K_{i-1})\ (\mathit{or}\ \Omega_J(G_{i-1})),\ k \geq 1, \tag{30}$$

$$W_l = \lambda^l W_0 + \sum_{j=1}^{l} \lambda^{l-j} \Omega(\rho_{j-1})\ (\mathit{or}\ \Omega_J(\gamma_{j-1})),\ l \geq 1, \tag{31}$$

then $V_k, W_l,\ k, l \geq 0$, satisfy

$$(EV_k)U - UV_k = U'[K_k],\ (EW_l)U - UW_l = U'[\rho_l] + \lambda^{l+1} U_\lambda,\ k, l \geq 0. \tag{32}$$

Therefore for any $k, l \geq 0$, the systems of discrete evolution equations $u_t = K_k$ and $u_t = \rho_l$ possess the isospectral ($\lambda_t = 0$) and nonisospectral ($\lambda_t = \lambda^{l+1}$) discrete zero curvature representations

$$U_t = (EV_k)U - UV_k,\ U_t = (EW_l)U - UW_l,$$

respectively.

The theorem shows that the Lax operators associated with two hierarchies of interesting vector fields can be constructed simply by a unified form. Its proof is left to Appendix C. We are successful, thanks to introducing a characteristic operator equation. The difficulty is now transferred to seeking a solution to the characteristic operator equation. However this can automatically be solved on basis of the structure of Lax operators of isospectral hierarchies, which will be seen in the next section B.

III.2 A method for constructing master symmetries

Now we focus our attention on the construction problem of master symmetries. Theorem 3 already shows the structure of Lax operators associated with the isospectral and nonisospectral hierarchies (refer to [26] for the continuous case). When an isospectral hierarchy (22) (or (23)) is known, the theorem also provides us with a method to construct a nonisospectral hierarchy associated with the discrete spectral problem (24) by solving an initial discrete zero curvature equation (28) and solving a characteristic operator equation (25) (or (26)).

However, a solution to (25) (or (26)) may easily be generated by observing the resulting Lax operators. In fact, we have

$$\Omega(K_k) \text{ (or } \Omega_J(G_k)) = V_{k+1} - \lambda V_k. \tag{33}$$

This may be checked, say, for the case of (22), as follows

$$V_{k+1} - \lambda V_k = \left(\lambda^{k+1} V_0 + \sum_{i=1}^{k+1} \lambda^{k-i+1} \Omega(K_{i-1})\right) - \lambda \left(\lambda^k V_0 + \sum_{i=1}^{k} \lambda^{k-i} \Omega(K_{i-1})\right) = \Omega(K_k),$$

by using (30). Now by the first equality of (32), we may compute the following

$$\begin{aligned} & (E\Omega(K_k))U - U\Omega(K_k) \\ = \ & (EV_{k+1} - \lambda EV_k)U - U(V_{k+1} - \lambda V_k) \\ = \ & ((EV_{k+1})U - UV_{k+1}) - \lambda((EV_k)U - UV_k) \\ = \ & U'[K_{k+1}] - \lambda U'[K_k] = U'[\Phi K_k] - \lambda U'[K_k], \end{aligned}$$

for example, for the case of (22). Therefore we see that a possible solution $\Omega(X)$ to (25) (or $\Omega_J(G)$ to (26)) may be generated by replacing the element K_k (or G_k) in the equality (33) with X (or G).

The Lax operator matrices V_{k+1} and V_k are known, when the isospectral hierarchy has already been found. Thus we don't have to directly solve the characteristic operator equations and then the whole process of construction of the nonisospectral hierarchy becomes *an easy task*: finding ρ_0, W_0 to satisfy (28) and computing $V_{k+1} - \lambda V_k$ to find a solution to (25) (or (26)).

The nonisospectral hierarchy (29) is exactly the master symmetries that we need to find. The reasons are that the product systems between the isospectral hierarchy and the nonisospectral hierarchy are still isospectral by Theorem 2 or as we said before in Section II, and that usually all systems of the isospectral hierarchy commute with each other. Therefore it is because there exists a nonisospectral hierarchy that there exist master symmetries for isospectral systems of discrete evolution equations derived from a given discrete spectral problem.

In the next section, we shall in detail illustrate our construction process by three concrete examples and establish the corresponding centerless Virasoro symmetry algebras.

IV Applications

We illustrate only by three examples how to apply the method in the last section to construct master symmetries for various lattice hierarchies.

To make the process clearer, we introduce a conception for a given discrete spectral problem $E\phi = U\phi$, which has an injective Gateaux derivative U'. That is a uniqueness property similar to the one in the continuous case [27]:

if $(EV)U - UV = U'[K]$**,** $V \in \mathcal{V}^r \otimes C[\lambda, \lambda^{-1}]$, $K \in \mathcal{B}^q$**, and** $V|_{u=0} = 0$**, then** $V = 0$ and further $K = 0$ by the injective property of U'. It means that if an isospectral ($\lambda_t = 0$) Lax operator V equals zero at $u = 0$, then so does V itself. Actually, this property corresponds to the uniqueness of an integrable hierarchy associated with a spectral problem $E\phi = U\phi$. That is to say, when initial conditions and constants of inverse difference operators are fixed (for example, as in (7) and (8)), the associated isospectral hierarchy is uniquely determined. Most of discrete spectral problems share the uniqueness property. The following three spectral problems are exactly examples that share such a property.

IV.1 The Volterra lattice hierarchy

Let us first consider the following discrete spectral problem [15]:

$$E\phi = U\phi,\ U = \begin{pmatrix} 1 & u \\ \lambda^{-1} & 0 \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{34}$$

The corresponding isospectral integrable lattice hierarchy reads as

$$u_t = K_k = \Phi^k K_0 = u(a_{k+1}^{(1)} - a_{k+1}^{(-1)}),\ K_0 = u(u^{(-1)} - u^{(1)}),\ k \geq 0. \tag{35}$$

Here the matrix $V = \sum_{i\geq 0} \begin{pmatrix} a_i & uc_{i+1}^{(1)} \\ c_i & -a_i \end{pmatrix} \lambda^{-i}$ solves the stationary discrete zero curvature equation $(EV)U - UV = 0$, where we choose the initial conditions

$$a_0 = \frac{1}{2},\ c_0 = 0,\ a_1 = -u,\ c_1 = 1$$

and the hereditary operator Φ is given by

$$\Phi = u(1 + E^{-1})(-u^{(1)}E^2 + u)(E - 1)^{-1}u^{-1}, \tag{36}$$

where $(E-1)^{-1}$ is determined by (9). It is worth pointing out that each system in (35) is local and polynomially dependent on u, although the hereditary operator Φ has nonlocal and non-polynomially dependent features.

The first discrete evolution equation is the Volterra lattice equation [28]

$$(u(n))_t = u(n)(u(n-1) - u(n+1)),$$

which is significantly generalized by Bogoyavlensky [29]. The associated Lax operators are as follows

$$V_k = (\lambda^{k+1}V)_{\geq 1} + \begin{pmatrix} a_{k+1} & 0 \\ c_{k+1} & a_{k+1}^{(-1)} \end{pmatrix}, \ k \geq 0, \tag{37}$$

where $(P)_{\geq 1}$ denotes the selection of the terms with degrees of λ no less than 1. In particular, the initial isospectral Lax operator reads as

$$V_0 = \begin{pmatrix} \frac{1}{2}\lambda - u & \lambda u \\ 1 & -\frac{1}{2}\lambda - u^{(-1)} \end{pmatrix}. \tag{38}$$

The result until here can be obtained from (34) by using a powerful method in [30].

We easily obtain the corresponding quantities in the nonisospectral ($\lambda_t = \lambda$) initial discrete zero curvature equation (28):

$$\rho_0 = u, \ W_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{39}$$

and a solution to the characteristic operator equation (25) by (33):

$$\Omega(X) = \begin{pmatrix} \Omega_{11}(X) & \Omega_{12}(X) \\ \Omega_{21}(X) & \Omega_{22}(X) \end{pmatrix}, \tag{40}$$

where $\Omega_{ij}(X)$, $i,j = 1,2$, are given by

$$\begin{aligned}
\Omega_{11}(X) &= (E-1)^{-1}(-u^{(1)}E^2 + u)(E-1)^{-1}u^{-1}X \\
\Omega_{12}(X) &= \lambda u E(E-1)^{-1}u^{-1}X \\
\Omega_{21}(X) &= (E-1)^{-1}u^{-1}X \\
\Omega_{22}(X) &= -\lambda(E-1)^{-1}u^{-1}X + E^{-1}(E-1)^{-1}(-u^{(1)}E^2 + u)(E-1)^{-1}u^{-1}X.
\end{aligned}$$

Now by Theorem 3, we obtain a hierarchy of nonisospectral discrete evolution equations $u_t = \rho_l = \Phi^l \rho_0, \ l \geq 0$, associated with the spectral problem (34).

Let us now consider how to compute the corresponding symmetry algebra. The idea below can be applied to other cases. We first make the following computation at $u = 0$:

$$K_k|_{u=0} = 0, \ \rho_l|_{u=0} = \Phi^l \rho_0|_{u=0} = 0, \ k,l \geq 0,$$

$$V_k|_{u=0} = \lambda^k \begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix}, \ k \ge 0,$$
$$W_l|_{u=0} = \lambda^l \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix} + (1-\delta_{l0})\lambda^{l-1} \begin{pmatrix} 0 & 0 \\ [n] & -\lambda[n] \end{pmatrix}, \ l \ge 0,$$
$$V_{k\lambda}|_{u=0} = k\lambda^{k-1} \begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix} + \lambda^k \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix}, \ k \ge 0,$$
$$W_{l\lambda}|_{u=0} = l\lambda^{l-1} \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix} + (1-\delta_{l0}) \begin{pmatrix} 0 & 0 \\ (l-1)\lambda^{l-2}[n] & -l\lambda^{l-1}[n] \end{pmatrix}, \ l \ge 0,$$

where $V_k,\ W_l,\ k,l \ge 0$, are given as in Theorem 3 and δ_{l0} represents the Kronecker symbol. While computing $W_l|_{u=0}$, we need to note that $\Omega(\rho_0)|_{u=0} \ne 0$, but $\Omega(\rho_l)|_{u=0} = 0,\ l \ge 1$. The other two examples below have a similar character, too. Now we can find by the definition (18) of the product of two Lax operators that

$$\begin{cases} [\![V_k, V_l]\!]|_{u=0} = 0,\ k,l \ge 0, \\ [\![V_k, W_l]\!]|_{u=0} = (k+1)V_{k+l}|_{u=0},\ k,l \ge 0, \\ [\![W_k, W_l]\!]|_{u=0} = (k-l)W_{k+l}|_{u=0},\ k,l \ge 0. \end{cases} \tag{41}$$

For example, we can compute that

$$\begin{aligned} [\![V_k, W_l]\!]|_{u=0} &= [V_k|_{u=0}, W_l|_{u=0}] + \lambda^{l+1} V_{k\lambda}|_{u=0} \\ &= \lambda^{k+l} \left[\begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix}, \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix} + (1-\delta_{l0})\lambda^{-1} \begin{pmatrix} 0 & 0 \\ [n]-\lambda[n] & \end{pmatrix} \right] \\ &\quad + \lambda^{l+1} \left(k\lambda^{k-1} \begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix} + \lambda^k \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix} \right) \\ &= \lambda^{k+l} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k\lambda^{k+l} \begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix} + \lambda^{k+l+1} \begin{pmatrix} \frac12 & 0 \\ 0 & -\frac12 \end{pmatrix} \\ &= (k+1)\lambda^{k+l} \begin{pmatrix} \frac12\lambda & 0 \\ 1 & -\frac12\lambda \end{pmatrix} = (k+1)V_{k+l}|_{u=0}. \end{aligned}$$

Because $[\![V_k, V_l]\!], [\![V_k, W_l]\!] - (k+1)V_{k+l}, [\![W_k, W_l]\!] - (k-l)W_{k+l},\ k,l \ge 0$, are all isospectral ($\lambda_t = 0$) Lax operators belonging to $\mathcal{V}^2 \otimes C[\lambda, \lambda^{-1}]$ by Theorem 2, based upon (41) we obtain a Lax operator algebra by the uniqueness property of the spectral problem (34)

$$\begin{cases} [\![V_k, V_l]\!] = 0,\ k,l \ge 0, \\ [\![V_k, W_l]\!] = (k+1)V_{k+l},\ k,l \ge 0, \\ [\![W_k, W_l]\!] = (k-l)W_{k+l},\ k,l \ge 0. \end{cases} \tag{42}$$

Further, due to the injective property of U', we finally obtain a vector field algebra of the isospectral hierarchy and the nonisospectral hierarchy

$$\begin{cases} [K_k, K_l] = 0,\ k, l \geq 0, \\ [K_k, \rho_l] = (k+1)K_{k+l},\ k, l \geq 0, \\ [\rho_k, \rho_l] = (k-l)\rho_{k+l},\ k, l \geq 0. \end{cases} \tag{43}$$

This implies that $\rho_l,\ l \geq 0$, are all master symmetries of each lattice equation $u_t = K_{k_0}$ in the isospectral hierarchy, and the symmetries

$$K_k,\ k \geq 0,\ \text{and}\ \tau_l^{(k_0)} = t[K_{k_0}, \rho_l] + \rho_l,\ l \geq 0,$$

constitute a symmetry algebra of the Virasoro type possessing the same commutator relations as (43).

IV.2 The Toda lattice hierarchy

Second, let us consider the discrete spectral problem [30]:

$$E\phi = U\phi,\ U = \begin{pmatrix} 0 & 1 \\ -v & \lambda - p \end{pmatrix},\ u = \begin{pmatrix} p \\ v \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{44}$$

The corresponding isospectral integrable Toda lattice hierarchy [31] reads as

$$u_t = K_k = \Phi^k K_0 = \begin{pmatrix} a_{k+2} - a_{k+2}^{(1)} \\ v(b_{k+2}^{(1)} - b_{k+2}) \end{pmatrix},\ K_0 = \begin{pmatrix} v - v^{(1)} \\ v(p - p^{(-1)}) \end{pmatrix},\ k \geq 0. \tag{45}$$

Here $V = \sum_{i\geq 0} \begin{pmatrix} a_i & b_i \\ -vb_i^{(1)} & -a_i \end{pmatrix} \lambda^{-i}$ solves $(EV)U - UV = 0$, in which we choose

$$a_0 = \frac{1}{2},\ b_0 = 0,\ a_1 = 0,\ b_1 = -1,$$

and the hereditary operator Φ is determined by

$$\Phi = \begin{pmatrix} p & (v^{(1)}E^2 - v)(E-1)^{-1}v^{-1} \\ v(E^{-1}+1) & v(pE - p^{(-1)})(E-1)^{-1}v^{-1} \end{pmatrix}. \tag{46}$$

The first system of discrete evolution equations is the Toda lattice [32]

$$\begin{cases} (p(n))_t = v(n) - v(n+1), \\ (v(n))_t = v(n)(p(n) - p(n-1)), \end{cases}$$

up to a transform of dependent variables. The lattice hierarchy above has a local tri-Hamiltonian structure

$$u_t = K_k = J\frac{\delta H_{k+2}}{\delta u} = M\frac{\delta H_{k+1}}{\delta u} = N\frac{\delta H_k}{\delta u},\ k \ge 0,$$

where the Hamiltonian operators J, M, N and the conserved quantities H_k defined by

$$\begin{aligned}
&J = \begin{pmatrix} 0 & (1-E)v \\ v(E^{-1}-1) & 0 \end{pmatrix},\\
&M = J\Phi^\dagger = -\Phi J = \begin{pmatrix} Ev - vE^{-1} & p(E-1)v \\ v(1-E^{-1})p & v(E-E^{-1})v \end{pmatrix},\\
&N = M\Phi^\dagger = -\Phi M\\
&= \begin{pmatrix} p(vE^{-1}-Ev) + (vE^{-1}-Ev)p & p^2(1-E)v + (vE^{-1}-Ev)(1+E)v \\ v(E^{-1}+1)(vE^{-1}-Ev) + v(E^{-1}-1)p^2 & 2v(E^{-1}p - pE)v \end{pmatrix},\\
&H_0 = p + \frac{1}{2}\ln v,\ H_k = -\frac{b_{k+1}}{k},\ k \ge 1,
\end{aligned}$$

where $\Phi^\dagger$ denotes the conjugate operator of Φ. Note that this tri-Hamiltonian structure may be established through a trace identity [30]. The corresponding Lax operators read as

$$V_k = (\lambda^{k+1}V)_+ + \begin{pmatrix} b_{k+2} & 0 \\ 0 & 0 \end{pmatrix},\ k \ge 0, \tag{47}$$

where the subscript $+$ denotes selecting the non-negative part. Hence in particular

$$V_0 = \begin{pmatrix} \frac{1}{2}\lambda - p^{(-1)} & -1 \\ v & -\frac{1}{2}\lambda \end{pmatrix}. \tag{48}$$

It is easy to find the corresponding quantities in the nonisospectral ($\lambda_t = \lambda$) initial discrete zero curvature equation (28):

$$\rho_0 = \begin{pmatrix} p \\ 2v \end{pmatrix},\ W_0 = \begin{pmatrix} [n]-1 & 0 \\ 0 & [n] \end{pmatrix}, \tag{49}$$

where $[n]$ is the multiplication operator defined by (6), and a solution to the characteristic operator equation (25) by (33):

$$\Omega(X) = \begin{pmatrix} \Omega_{11}(X) & \Omega_{12}(X) \\ \Omega_{21}(X) & \Omega_{22}(X) \end{pmatrix},\ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{50}$$

where $\Omega_{ij}(X)$, $i,j = 1,2$, are given by

$$\begin{aligned}
\Omega_{11}(X) &= E^{-1}(E-1)^{-1}X_1 + (p^{(-1)} - \lambda)(E-1)^{-1}v^{-1}X_2,\\
\Omega_{12}(X) &= (E-1)^{-1}v^{-1}X_2,\\
\Omega_{21}(X) &= vE(E-1)^{-1}v^{-1}X_2,\\
\Omega_{22}(X) &= (E-1)^{-1}X_1.
\end{aligned}$$

In this way, we obtain a hierarchy of nonisospectral systems of discrete evolution equations $\rho_l = \Phi^l \rho_o,\ l \geq 0$, associated with the spectral problem (44).

In order to construct a vector field algebra, we make a similar computation at $u = 0$:

$$
\begin{aligned}
&K_k|_{u=0} = 0,\ \rho_l|_{u=0} = \Phi^l \rho_0|_{u=0} = 0,\ k, l \geq 0, \\
&V_k|_{u=0} = \lambda^k \begin{pmatrix} \frac{1}{2}\lambda & -1 \\ 0 & -\frac{1}{2}\lambda \end{pmatrix},\ k \geq 0, \\
&W_l|_{u=0} = \lambda^l \begin{pmatrix} [n]-1 & 0 \\ 0 & [n] \end{pmatrix} + (1-\delta_{l0}) \begin{pmatrix} -2\lambda[n] & 2[n] \\ 0 & 0 \end{pmatrix},\ l \geq 0, \\
&V_{k\lambda}|_{u=0} = k\lambda^{k-1} \begin{pmatrix} \frac{1}{2}\lambda & -1 \\ 0 & -\frac{1}{2}\lambda \end{pmatrix} + \lambda^k \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},\ k \geq 0, \\
&W_{l\lambda}|_{u=0} = l\lambda^{l-1} \begin{pmatrix} [n]-1 & 0 \\ 0 & [n] \end{pmatrix} + (1-\delta_{l0}) \begin{pmatrix} -2l\lambda^{l-1}[n] & 2(l-1)\lambda^{l-2}[n] \\ 0 & 0 \end{pmatrix},\ l \geq 0.
\end{aligned}
$$

Now we can find through the product definition of $[\![\cdot,\cdot]\!]$ in (18) that

$$
\begin{cases}
[\![V_k, V_l]\!]|_{u=0} = 0,\ k, l \geq 0, \\
[\![V_k, W_l]\!]|_{u=0} = (k+1)V_{k+l}|_{u=0},\ k, l \geq 0, \\
[\![W_k, W_l]\!]|_{u=0} = (k-l)W_{k+l}|_{u=0},\ k, l \geq 0.
\end{cases} \tag{51}
$$

A similar argument yields a Lax operator algebra by the uniqueness property of the spectral problem (44)

$$
\begin{cases}
[\![V_k, V_l]\!] = 0,\ k, l \geq 0, \\
[\![V_k, W_l]\!] = (k+1)V_{k+l},\ k, l \geq 0, \\
[\![W_k, W_l]\!] = (k-l)W_{k+l},\ k, l \geq 0.
\end{cases} \tag{52}
$$

And then because of the injective property of U', we obtain a semi-product Lie algebra of the isospectral hierarchy and the nonisospectral hierarchy

$$
\begin{cases}
[K_k, K_l] = 0,\ k, l \geq 0, \\
[K_k, \rho_l] = (k+1)K_{k+l},\ k, l \geq 0, \\
[\rho_k, \rho_l] = (k-l)\rho_{k+l},\ k, l \geq 0,
\end{cases} \tag{53}
$$

which gives rise to a symmetry algebra of the Virasoro type for the isospectral Toda hierarchy (45).

IV.3 A sub-KP lattice hierarchy

Let us finally consider the discrete spectral problem [33]:

$$E\phi = U\phi,\ U = \begin{pmatrix} 0 & 1 & 0 \\ b-\lambda & a & 1 \\ c & 0 & 0 \end{pmatrix},\ u = \begin{pmatrix} a \\ b \\ c \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{54}$$

which is equivalent to $(-E^2+b+aE+E^{-1}c)\phi_1 = \lambda\phi_1$, a sub-KP discrete spectral problem [34]. The corresponding isospectral integrable lattice hierarchy reads as

$$u_t = K_k = JG_k = MG_{k-1},\ k \geq 0, \tag{55}$$

where a Hamiltonian pair J, M and G_{-1}, G_0, G_1 are defined by

$$\begin{aligned}
J &= \begin{pmatrix} E-E^{-1} & 0 & 0 \\ 0 & 0 & (E^{-1}-1)c \\ 0 & -c(E-1) & 0 \end{pmatrix}, \\
M &= \begin{pmatrix} Eb-bE^{-1}+a\Delta_+\Delta^{-1}\Delta_-a & EcE-E^{-1}c & -a\Delta_+\Delta^{-1}\Delta_-c \\ cE-E^{-1}cE^{-1} & E^{-1}ac-acE & -b\Delta_-c \\ c\Delta_+-\Delta^{-1}\Delta_-a & -c\Delta_+b & c[\Delta_+\Delta^{-1}\Delta_- - \Delta_- - \Delta_+]c \end{pmatrix}, \\
G_{-1} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ G_0 = \begin{pmatrix} c \\ b \\ a \end{pmatrix},\ G_1 = \begin{pmatrix} c(Eb+b) \\ b^2+ac+E^{-1}ac \\ a(Eb+b)-Ec-E^{-1}c \end{pmatrix},
\end{aligned}$$

where Δ_+, Δ_- are the difference operators: $\Delta_+ = E-1,\ \Delta_- = 1-E^{-1}$. The first nonlinear system of discrete evolution equations is

$$\begin{cases} (a(n))_t = c(n+1) - c(n-1), \\ (b(n))_t = a(n-1)c(n-1) - a(n)c(n), \\ (c(n))_t = c(n)(b(n) - b(n+1)). \end{cases}$$

We easily find the corresponding quantities in (27) and (28):

$$\begin{aligned}
K_0 &= \begin{pmatrix} (E-E^{-1})c \\ (E^{-1}-1)ac \\ c(1-E)b \end{pmatrix},\ V_0 = \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ -E^{-1}ac & E^{-1}c & \lambda-b \end{pmatrix}, \\
\rho_0 &= J\gamma_0 = M\gamma_{-1} = J\begin{pmatrix} \frac{1}{2}\Delta^{-1}a \\ -\frac{3}{2}[n] \\ -c^{-1}\Delta_-^{-1}b \end{pmatrix} = M\begin{pmatrix} 0 \\ 0 \\ -([n]+\frac{3}{2})c^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a \\ b \\ \frac{3}{2}c \end{pmatrix}, \\
W_0 &= \begin{pmatrix} \frac{1}{2}[n] & 0 & 0 \\ 0 & \frac{1}{2}([n]+1) & 0 \\ 0 & 0 & \frac{1}{2}([n]+2) \end{pmatrix}.
\end{aligned}$$

We can also obtain a solution to the characteristic operator equation (26) by (33):

$$\Omega_J(G) = \begin{pmatrix} \Omega_{11}(G) & \Omega_{12}(G) & \Omega_{13}(G) \\ \Omega_{21}(G) & \Omega_{22}(G) & \Omega_{23}(G) \\ \Omega_{31}(G) & \Omega_{32}(G) & \Omega_{33}(G) \end{pmatrix}, \quad G = \begin{pmatrix} G_{(1)} \\ G_{(2)} \\ G_{(3)} \end{pmatrix}, \tag{56}$$

where $\Omega_{ij}(G)$, $i, j = 1, 2, 3$, are determined by

$$\begin{aligned}
\Omega_{11}(G) &= -(E^2 + E)^{-1}(cG_{(3)} + EaG_{(1)}) \\
\Omega_{12}(G) &= E^{-1}G_{(1)} \\
\Omega_{13}(G) &= G_{(2)} \\
\Omega_{21}(G) &= cEG_{(2)} + (b - \lambda)G_{(1)} \\
\Omega_{22}(G) &= -(E + 1)^{-1}(cG_{(3)} + EaG_{(1)}) + aG_{(1)} \\
\Omega_{23}(G) &= G_{(1)}, \\
\Omega_{31}(G) &= E^{-1}cE^{-1}G_{(1)} - E^{-1}acG_{(2)}, \\
\Omega_{32}(G) &= E^{-1}cG_{(2)} \\
\Omega_{33}(G) &= -E(E + 1)^{-1}(cG_{(3)} + EaG_{(1)}) + \Delta_+ aG_{(1)} - (b - \lambda)G_{(2)}.
\end{aligned} \tag{57}$$

By Theorem 3, we get a hierarchy of nonisospectral systems of discrete evolution equations $u_t = \rho_l = \Phi^l \rho_0,\ l \geq 0$, associated with the spectral problem (54).

In order to generate a vector field algebra of the isospectral hierarchy and the non-isospectral hierarchy, we need the following quantities, which may be directly worked out:

$$\begin{aligned}
&K_k|_{u=0} = 0,\ \rho_0|_{u=0} = J\gamma_0|_{u=0} = 0,\ \rho_l|_{u=0} = J\gamma_l|_{u=0} = M\gamma_{l-1}|_{u=0} = 0,\ k \geq 0,\ l \geq 1, \\
&V_k|_{u=0} = \lambda^k \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\
&W_l|_{u=0} = \lambda^l \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2}([n] + 1) & 0 \\ 0 & 0 & \frac{1}{2}([n] + 2) \end{pmatrix} + (1 - \delta_{l0})\lambda^{l-1} \begin{pmatrix} 0 & 0 & -\frac{3}{2}[n] \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}\lambda[n] \end{pmatrix}, \\
&V_{k\lambda}|_{u=0} = k\lambda^{k-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \lambda^k \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&W_{l\lambda}|_{u=0} = l\lambda^{l-1} \begin{pmatrix} \frac{1}{2}[n] & 0 & 0 \\ 0 & \frac{1}{2}([n] + 1) & 0 \\ 0 & 0 & \frac{1}{2}([n] + 2) \end{pmatrix} + (1 - \delta_{l0}) \begin{pmatrix} 0 & 0 & -\frac{3}{2}(l - 1)\lambda^{l-2}[n] \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}l\lambda^{l-1}[n] \end{pmatrix}.
\end{aligned}$$

Now we easily find according to the product definition of $[\![\cdot,\cdot]\!]$ that

$$\begin{cases} [\![V_k, V_l]\!]|_{u=0} = 0, \ k,l \geq 0, \\ [\![V_k, W_l]\!]|_{u=0} = (k+1)V_{k+l}|_{u=0}, \ k,l \geq 0, \\ [\![W_k, W_l]\!]|_{u=0} = (k-l)W_{k+l}|_{u=0}, \ k,l \geq 0. \end{cases}$$

The same deduction leads to a Lax operator algebra

$$\begin{cases} [\![V_k, V_l]\!] = 0, \ k,l \geq 0, \\ [\![V_k, W_l]\!] = (k+1)V_{k+l}, \ k,l \geq 0, \\ [\![W_k, W_l]\!] = (k-l)W_{k+l}, \ k,l \geq 0. \end{cases} \tag{58}$$

and further a vector field algebra

$$\begin{cases} [K_k, K_l] = 0, \ k,l \geq 0, \\ [K_k, \rho_l] = (k+1)K_{k+l}, \ k,l \geq 0, \ , \\ [\rho_k, \rho_l] = (k-l)\rho_{k+l}, \ k,l \geq 0. \end{cases} \tag{59}$$

which may generate a master symmetry algebra possessing the same algebraic structure as (59).

V Conclusion and remarks

We have established an algebraic structure related to discrete zero curvature equations and further introduced a simple but systematic approach for constructing master symmetries of the first degree for isospectral lattice hierarchies associated with discrete spectral problems. The resulting algebraic structures also leads to an explanation of why there exist master symmetries of the first degree. Some complicated calculation in our construction is saved by using a characteristic operator equation (25) (or (26)) and a uniqueness property of discrete spectral problems. The crucial step is the construction of the corresponding nonisospectral lattice hierarchies, which can be found by solving an initial nonisospectral discrete zero curvature equation. Three lattice hierarchies are shown as illustrative examples and the corresponding master symmetry algebras of the centerless Virasoro type are exhibited. Some of the results in this paper have been reported at SIDE II, UK [35].

It is worth noting that three examples described in the last section possess the same commutator relations between their isospectral and nonisospectral vector fields. In general, we have $[K_k, \rho_l] = (k+\gamma)K_{k+l}, \ \gamma = const.$, but the other two equalities of the whole Virasoro algebra don't change. This is also a common phenomenon for continuous integrable hierarchies [36] [37]. Furthermore we may add a nonisospectral master symmetry with $\lambda_t = 1$ to the whole Virasoro symmetry algebra but this often requires additional checking. For example, a nonisospectral master symmetry with $\lambda_t = 1$ of the sub-KP lattice hierarchy (55) is $\rho_{-1} = J\gamma_{-1} = (0, 1, 0)^T$. On the other hand, similar to the theory in [37], we may also choose an operator solution $\Omega(X)$ (or $\Omega_J(G)$) satisfying $\Omega(X)|_{X=0} = 0$ (or $\Omega_J(G)|_{G=0} = 0$) (all three examples in the last section have this property) and then we only need to compute $[\![V_0, W_0]\!]|_{u=0}$ so as to give Lax operator algebras at $u = 0$ and finally give Lax operator algebras generally.

In our discussion, in fact, we haven't used the hereditary property of the recursion operator Φ (or the bi-Hamiltonian property of J and M), while we construct Virasoro symmetry algebras for integrable lattice hierarchies, and thus it can also be applied to lattice hierarchies that possess non-hereditary recursion operators. The advantage of our scheme is to fully utilize discrete zero curvature equations so that the whole process to generate master symmetries of the first degree becomes an easy task. There were also an algorithm implemented in MuPAD [38] and other direct tricks [13] [14] [15] [39] to compute master symmetries of first degree for systems of discrete evolution equations. However our theory focuses on seeking an answer to the existence and structure problem of master symmetries of the first degree.

We should mention that there exists a large variety of other theories or methods to discuss integrable properties of systems of nonlinear discrete equations, which include Hamiltonian theory [40] [41], Bäcklund-Darboux transformation [42] [43], R-matrix method [34] [44], symmetry reduction [45] etc. Moreover we can consider the time discretization problem [46] and periodic initial and boundary value problems of time discretizations [47] for symmetry flows of systems of discrete evolution equations. The resulting difference equations and mappings should be useful in discussing the integrability of the underlying systems of discrete evolution equations themselves. We are also curious about the following natural problem: Are there any higher degree master symmetries for systems of discrete evolution equations that don't depend explicitly on the evolution variable? If the answer is yes, can we establish any relations between those higher degree master symmetries and discrete zero curvature equations as we did for the first degree master symmetries?

Appendix A: Proof of Theorem 1

Let $(K_i, V_i, f_i) \in (\mathcal{B}^q, \widetilde{\mathcal{V}}^r, C^\infty(\mathbb{R}))$, $1 \le i \le 3$. Because the bilinearity and the skew-symmetry of the product (17) are self-evident and we already know that the products defined by (12) and (13) are Lie products, we only need to prove the following Jacobi identity:

$$[\![[\![V_1, V_2]\!], V_3]\!] + \mathrm{cycle}(1, 2, 3) = 0. \tag{A.1}$$

Let us first compute by (18) that

$$
\begin{aligned}
& [\![[\![V_1,V_2]\!],V_3]\!] = ([\![V_1,V_2]\!])'[K_3] - V_3'[[K_1,K_2]] + [[\![V_1,V_2]\!],V_3] + f_3[\![V_1,V_2]\!]_\lambda - [\![f_1,f_2]\!]V_{3\lambda} \\
= \; & (V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] + [V_1,V_2]'[K_3] + f_2(V_{1\lambda})'[K_3] - f_1(V_{2\lambda})'[K_3] \\
& -V_3'[[K_1,K_2]] + [V_1'[K_2],V_3] - [V_2'[K_1],V_3] + [[V_1,V_2],V_3] + f_2[V_{1\lambda},V_3] - f_1[V_{2\lambda},V_3] \\
& +f_3(V_1'[K_2])_\lambda - f_3(V_2'[K_1])_\lambda + f_3[V_1,V_2]_\lambda + f_{2\lambda}f_3V_{1\lambda} + f_2f_3V_{1\lambda\lambda} \\
& -f_{1\lambda}f_3V_{2\lambda} - f_1f_3V_{2\lambda\lambda} - [\![f_1,f_2]\!]V_{3\lambda}. \qquad \text{(A.2)}
\end{aligned}
$$

We need to use the following fundamental equalities

$$
\begin{aligned}
& (V_\lambda)'[K] = (V'[K])_\lambda, \ V \in \widetilde{\mathcal{V}}^r, \ K \in \mathcal{B}^q, \\
& [V,W]_\lambda = [V_\lambda,W] + [V,W_\lambda], \ V,W \in \widetilde{\mathcal{V}}^r, \\
& [V,W]'[K] = [V'[K],W] + [V,W'[K]], \ V,W \in \widetilde{\mathcal{V}}^r, \ K \in \mathcal{B}^q, \\
& V'[T] = (V'[K])'[S] - (V'[S])'[K], \ T = [K,S], \ V \in \widetilde{\mathcal{V}}^r, \ K,S \in \mathcal{B}^q,
\end{aligned}
$$

which may be shown by a direct computation and the last equality of which is a similar result as in [21]. Now we can go on to compute that

$$
\begin{aligned}
{\Delta_a}^{123} & := (V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] - V_3'[[K_1,K_2]] \\
& = (V_1'[K_2])'[K_3] - (V_2'[K_1])'[K_3] - (V_3'[K_1])'[K_2] + (V_3'[K_2])'[K_1], \\
{\Delta_b}^{123} & := [V_1,V_2]'[K_3] + [V_1'[K_2],V_3] - [V_2'[K_1],V_3] \\
& = [V_1'[K_3],V_2] - [V_2'[K_3],V_1] + [V_1'[K_2],V_3] - [V_2'[K_1],V_3], \\
{\Delta_c}^{123} & := f_2(V_{1\lambda})'[K_3] - f_1(V_{2\lambda})'[K_3] + f_3(V_1'[K_2])_\lambda - f_3(V_2'[K_1])_\lambda \\
& = f_2(V_{1\lambda})'[K_3] - f_1(V_{2\lambda})'[K_3] + f_3(V_{1\lambda})'[K_2] - f_3(V_{2\lambda})'[K_1], \\
{\Delta_d}^{123} & := f_2[V_{1\lambda},V_3] - f_1[V_{2\lambda},V_3] + f_3[V_1,V_2]_\lambda \\
& = f_2[V_{1\lambda},V_3] - f_1[V_{2\lambda},V_3] + f_3[V_{1\lambda},V_2] - f_3[V_{2\lambda},V_1], \\
{\Delta_e}^{123} & := f_{2\lambda}f_3V_{1\lambda} + f_2f_3V_{1\lambda\lambda} - f_{1\lambda}f_3V_{2\lambda} - f_1f_3V_{2\lambda\lambda} - [\![f_1,f_2]\!]V_{3\lambda}, \\
& = f_{2\lambda}f_3V_{1\lambda} + f_2f_3V_{1\lambda\lambda} - f_{1\lambda}f_3V_{2\lambda} - f_1f_3V_{2\lambda\lambda} - f_{1\lambda}f_2V_{3\lambda} + f_1f_{2\lambda}V_{3\lambda}.
\end{aligned}
$$

A direct check can result in that

$$
{\Delta_*}^{123} + \mathrm{cycle}(1,2,3) = 0, \ \text{where } * = a,b,c,d \text{ or } e.
$$

Noting (A.2), it follows therefore that

$$
\begin{aligned}
& [\![[\![V_1,V_2]\!],V_3]\!] + \mathrm{cycle}(1,2,3) \\
= \; & {\Delta_a}^{123} + {\Delta_b}^{123} + {\Delta_c}^{123} + {\Delta_d}^{123} + {\Delta_e}^{123} + [[V_1,V_2],V_3] + \mathrm{cycle}(1,2,3) = 0,
\end{aligned}
$$

which is exactly the Jacobi identity (A.1) and thus completes the proof.

Appendix B: Proof of Theorem 2

The proof is an application of the equalities (19) and (20) and the third equality

$$(U'[K])'[S]-(U'[S])'[K]=U'[T],\ T=[K,S], \tag{B.1}$$

which has been mentioned in the proof of the first theorem. We observe that

$$(\text{Eqn. (19)})'[S]-(\text{Eqn. (20)})'[K]+g(\text{Eqn. (19)})_\lambda-f(\text{Eqn. (20)})_\lambda.$$

The resulting equality reads as

$$\begin{aligned}
& (U'[K])'[S]-(U'[S])'[K]+[\![f,g]\!]U_\lambda \\
= & \ (EV'[S])U+(EV)U'[S]-U'[S]V-UV'[S] \\
& -(EW'[K])U-(EW)U'[K]+U'[K]W+UW'[K] \\
& +g(EV_\lambda)U+g(EV)U_\lambda-gU_\lambda V-gUV_\lambda \\
& -f(EW_\lambda)U-f(EW)U_\lambda+fU_\lambda W+fUW_\lambda.
\end{aligned} \tag{B.2}$$

On the other hand, we have immediately

$$\begin{aligned}
& (E[\![V,W]\!])U-U[\![V,W]\!] \\
= & \ (EV'[S])U-(EW'[K])U+(EV)(EW)U-(EW)(EV)U \\
& +g(EV_\lambda)U-f(EW_\lambda)U-UV'[S]+UW'[K] \\
& -UVW+UWV-gUV_\lambda+fUW_\lambda.
\end{aligned} \tag{B.3}$$

It follows therefore from (B.1), (B.2) and (B.3) that

$$\begin{aligned}
& (E[\![V,W]\!])U-U[\![V,W]\!]-U'[T]-[\![f,g]\!]U_\lambda \\
= & \ (E[\![V,W]\!])U-U[\![V,W]\!]-(U'[K])'[S]+(U'[S])'[K]-[\![f,g]\!]U_\lambda \\
= & \ (EV)\{(EW)U-V'[S]-gU_\lambda\}-(EW)\{(EV)U-U'[K]-fU_\lambda\} \\
& -UVW+UWV+gU_\lambda V-fU_\lambda W+U'[S]V-U'[K]W \\
= & \ (EV)UW-(EW)UV-UVW+UWV+gU_\lambda V-fU_\lambda W+U'[S]V-U'[K]W \\
= & \ \{(EV)U-UV-fU_\lambda-U'[K]\}W-\{(EW)U-UW-gU_\lambda-U'[S]\}V \\
= & \ 0,
\end{aligned}$$

which is what we need to prove.

Appendix C: Proof of Theorem 3

We prove two equalities in (32). The rest is obvious. We compute that

$$(EV_k)U-UV_k$$

$$
\begin{aligned}
&= \lambda^k[(EV_0)U - UV_0] + \sum_{i=1}^{k} \lambda^{k-i}[(E\Omega(K_{i-1}))U - U\Omega(K_{i-1})] \\
&= \lambda^k U'[K_0] + \sum_{i=1}^{k} \lambda^{k-i}\{U'[\Phi K_{i-1}] - \lambda U'[K_{i-1}]\} \\
&= \lambda^k U'[K_0] + \sum_{i=1}^{k} \lambda^{k-i}\{U'[K_i] - \lambda U'[K_{i-1}]\} \\
&= U'[K_k], \ k \geq 1; \\
& \quad (EW_l)U - UW_l \\
&= \lambda^l[(EW_0)U - UW_0] + \sum_{j=1}^{l} \lambda^{l-j}[(E\Omega(\rho_{j-1}))U - U\Omega(\rho_{j-1})] \\
&= \lambda^l\{U'[\rho_0] + \lambda U_\lambda\} + \sum_{j=1}^{l} \lambda^{l-j}\{U'[\Phi\rho_{j-1}] - \lambda U'[\rho_{j-1}]\} \\
&= \lambda^l\{U'[\rho_0] + \lambda U_\lambda\} + \sum_{j=1}^{l} \lambda^{l-j}\{U'[\rho_j] - \lambda U'[\rho_{j-1}]\} \\
&= U'[\rho_l] + \lambda^{l+1}U_\lambda, \ l \geq 1.
\end{aligned}
$$

Note that we have used the characteristic operator equation (25) but the situation in the case of (26) is conpletely similar. The proof is therefore finished.

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