Symmetries and Integrability of Differential Equations, eds. Peter A. Clarkson and Frank W. Nijhoff, Cambridge University Press, 1999, pp.247–260

An Approach to Master Symmetries of Lattice Equations

Benno Fuchssteiner[†1] and Wen-Xiu Ma[‡2]
[†]Universität-GH Paderborn, D-33098 Paderborn, Germany
[‡]Dept. of Math., University of Manchester Institute of Science and Technology, Manchester M60 1QD, UK

Abstract

An approach to master symmetries of lattice equations is proposed by the use of discrete zero curvature equation. Its key is to generate non-isospectral flows from the discrete spectral problem associated with a given lattice equation. A Volterra-type lattice hierarchy and the Toda lattice hierarchy are analyzed as two illustrative examples.

1 Introduction

Symmetries are one of important aspects of soliton theory. When any integrable character hasn't been found for a given equation, among the most efficient ways is to consider its symmetries. It is through symmetries that Russian scientists et al. classified many integrable equations including lattice equations [1] [2]. They gave some specific description for the integrability of nonlinear equations in terms of symmetries, and showed that if an equation possesses higher differential-difference degree symmetries, then it is subject to certain conditions, for example, the degree of its nonlinearity mustn't be too large, compared with its differential-difference degree. Usually an integrable equation in soliton theory is referred as to an equation possessing infinitely many symmetries [3] [4]. Moreover these symmetries form beautiful algebraic structures [3] [4].

The appearance of master symmetries [5] gives a common character for integrable differential equations both in $1+1$ dimensions and in $1+2$ dimensions, for example, the KdV equation and the KP equation. The resulting symmetries are sometimes called τ-symmetries [6] and constitute centreless Virasoro algebras together with time-independent symmetries [7]. Moreover this kind of τ-symmetries may be generated by use of Lax equations [8] or zero curvature equations [9]. In the case of lattice equations, there also exist some similar results. For instance, a lot of lattice equations have τ-symmetries

[1]Email: benno@uni-paderborn.de
[2]Email: wenxiuma@uni-paderborn.de

and centreless Virasoro symmetry algebras [10]. So far, however, there has not been a systematic theory to construct this kind of τ-symmetries for lattice equations.

The purpose of this paper is to provide a procedure to generate those master symmetries for a given lattice hierarchy. The discrete zero curvature equation is our basic tool to give such a procedure. A Volterra-type lattice hierarchy and the Toda lattice hierarchy are chosen and analyzed as two illustrative examples, which have one dependent variable and two dependent variables, respectively.

The paper begins by discussing discrete zero curvature equations. It will then go on to establish an approach to master symmetries by using discrete zero curvature equations. The fourth section will give rise to applications of our approach to two concrete examples of lattice hierarchies. Finally, the fifth section provides some concluding remarks.

2 Discrete zero curvature equations

Let $u(n,t)$ be a function defined over $Z\times \mathbb{R}$, E be a shift operator: $(Eu)(n) = u(n+1)$, and $K^{(m)} = E^m K,\ m \in Z,\ K$ being a vector function. Consider the discrete spectral problem

$$\begin{cases} E\phi = U\phi = U(u,\lambda)\phi, \\ \phi_t = V\phi = V(u,\lambda)\phi, \end{cases} \tag{2.1}$$

where U, V are the same order square matrix operators and λ is a spectral parameter. Its integrability condition is the following discrete zero curvature equation

$$\begin{aligned} U_t &= (EV)U - UV = ((E-1)V)U - UV + VU \\ &= ((E-1)V)U - [U,V] = (\Delta_+ V)U - [U,V]. \end{aligned} \tag{2.2}$$

Recall that the continuous zero curvature equation reads as

$$U_t = V_x - [U,V].$$

Therefore we see that there is a slight difference between two zero curvature equations. The Gateaux derivative of $X(u)$ at a direction $S(u)$ is defined by

$$X'[S] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} X(u+\varepsilon S). \tag{2.3}$$

Throughout the paper, we assume that the spectral operator U has the injective property, that is, if $U'[K] = 0$, then $K = 0$. Therefore the Gateaux derivative U' is an injective linear map. For example, in the case of Toda lattices, the spectral operator U reads as [11]

$$U = \begin{pmatrix} 0 & 1 \\ -v & \lambda - p \end{pmatrix}, \ u = \begin{pmatrix} p \\ v \end{pmatrix}, \tag{2.4}$$

and thus we have

$$U'[K] = \begin{pmatrix} 0 & 0 \\ -K_2 & -K_1 \end{pmatrix} = 0 \Rightarrow K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0,$$

which means that the spectral operator of the Toda lattice hierarchy has the injective property. We also need another property of the spectral operator

$$\text{if } (EV)U - UV = U'[K] \text{ and } V|_{u=0} = 0, \text{ then } V = 0, \tag{2.5}$$

which is called the uniqueness property. Further we obtain $K = 0$ from (2.5) by the injective property. It may be shown that (2.4) share such a property, when V is Laurent-polynomial dependent on λ.

Note that $U_t = U'[u_t] + \lambda_t U_\lambda = U'[u_t] + f(\lambda)U_\lambda$ when $\lambda_t = f(\lambda)$, where $U_\lambda = \frac{\partial U}{\partial \lambda}$. If (V, K, f) satisfies a so-called key discrete zero curvature equation

$$(EV)U - UV = U'[K] + fU_\lambda, \tag{2.6}$$

then we can say that when $\lambda_t = f(\lambda)$,

$$\underset{\text{a discrete evolution equation}}{u_t = K(u)} \qquad \Longleftrightarrow \qquad \underset{\text{a discrete zero curvature equation}}{U_t = (EV)U - UV.}$$

This result may be proved as follows.

Proof: $(\Rightarrow)$

$$U_t = U'[u_t] + \lambda_t U_\lambda = U'[K] + fU_\lambda = (EV)U - UV.$$

$(\Leftarrow)$

$$\begin{aligned} U'[K] + fU_\lambda &= (EV)U - UV = U_t = U'[u_t] + \lambda_t U_\lambda \\ &= U'[u_t] + fU_\lambda \Rightarrow U'[u_t - K] = 0 \Rightarrow u_t = K, \end{aligned}$$

where the linearity of U' and the injective property of U are used in the last two steps. ▌

Definition 2.1 *A matrix operator V is called a Lax operator corresponding to the spectral operator U with a spectral evolution law $\lambda_t = f(\lambda)$ if a key discrete zero curvature equation (2.6) holds. Moreover V is called an isospectral Lax operator if $f = 0$ or a nonisospectral Lax operator if $f \neq 0$.*

The equation (2.6) exposes an essential relation between a discrete equation and its discrete zero curvature representation. It will play an important role in the context of our construction of master symmetries.

3 An approach to master symmetries

What are master symmetries? For a given evolution equation $u_t = K(u)$, where $K(u)$ does not depend explicitly on t, the definition of master symmetries is the following [5].

Definition 3.1 *A vector field $\rho(u)$ is called a master symmetry of $u_t = K(u)$ if $[K, [K, \rho]] = 0$, where the commutator of two vector fields is defined by*

$$[K, \rho] = K'[\rho] - \rho'[K]. \tag{3.1}$$

If $\rho(u)$ is a master symmetry of $u_t = K(u)$, then the vector field $\tau = t[K, \rho] + \rho$, depending explicitly on the time t, is a symmetry of $u_t = K(u)$, namely, to satisfy the linearized equation of $u_t = K(u)$:

$$\frac{d\tau}{dt} = K'[\tau], \ i.e. \ \frac{\partial \tau}{\partial t} = [K, \rho]. \tag{3.2}$$

Main Observation:

Nonisospectral flows with $\lambda_t = \lambda^{n+1} \Rightarrow$ master symmetries.

So for (K, V, f) and (S, W, g), we introduce a new product

$$[\![V, W]\!] = V'[S] - W'[K] + [V, W] + gV_\lambda - fW_\lambda, \tag{3.3}$$

in order to discuss master symmetries. An algebraic structure for the key discrete zero curvature equation is shown in the following theorem.

Theorem 3.1 *If two discrete zero curvature equations*

$$(EV)U - UV = U'[K] + fU_\lambda, \tag{3.4}$$
$$(EW)U - UW = U'[S] + gU_\lambda, \tag{3.5}$$

hold, then we have the third discrete zero curvature equation

$$(E[\![V, W]\!])U - U[\![V, W]\!] = U'[T] + [\![f, g]\!]U_\lambda, \tag{3.6}$$

where $T = [K, S]$ and $[\![f, g]\!]$ is defined by

$$[\![f, g]\!](\lambda) = f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda). \tag{3.7}$$

Proof: The proof is an application of equalities (3.4), (3.5) and

$$(U'[K])'[S] - (U'[S])'[K] = U'[T], \ T = [K, S],$$

which is a similar result to one in the continuous case in [12] and may be immediately checked. We first observe that

$$(\text{Equ. (3.4)})'[S] - (\text{Equ. (3.5)})'[K] + g(\text{Equ. (3.4)})_\lambda - f(\text{Equ. (3.5)})_\lambda,$$

and then a direct calculation may give the equality (3.6). ▮

This theorem shows that a product equation $u_t = [K, S]$ has a discrete zero curvature representation

$$U_t = (E[\![V, W]\!])U - U[\![V, W]\!] \quad \text{with} \quad \lambda_t = [\![f, g]\!],$$

when two evolution equations $u_t = K(u)$ and $u_t = S(u)$ have the discrete zero curvature representations:

$$U_t = (EV)U - UV \quad \text{with} \quad \lambda_t = f(\lambda), \; U_t = (EW)U - UW \quad \text{with} \quad \lambda_t = g(\lambda), \tag{3.8}$$

receptively. According to the above theorem, we can also easily find that if an equation $u_t = K(u)$ is isospectral ($f = 0$), then the product equation $u_t = [K, S]$ for any vector field $S(u)$ is still isospectral because we have $[\![f, g]\!] = [\![0, g]\!] = 0$, where g is the evolution law corresponding $u_t = S(u)$ (see [13] for the continuous case).

Let us now assume that we already have a hierarchy of isospectral equations of the form

$$u_t = K_k = \Phi^k K_0, \; k \geq 0, \tag{3.9}$$

associated with a discrete spectral problem

$$E\phi = U\phi, \; \phi = (\phi_1, \cdots, \phi_r)^T. \tag{3.10}$$

Usually a discrete spectral problem $E\phi = U\phi$ yields a hereditary operator Φ (see [14] for instance), i.e. a square matrix operator to satisfy

$$\Phi^2[K, S] + \Phi[\Phi K, \Phi S] - \{\Phi[K, \Phi S] + \Phi[\Phi K, S]\} = 0$$

for any vector fields K, S.

In order to generate nonisospectral flows, we further introduce an operator equation of $\Omega(X)$:

$$(E\Omega(X))U - U\Omega(X) = U'[\Phi X] - \lambda U'[X]. \tag{3.11}$$

We call it a characteristic operator equation of $E\phi = U\phi$.

Theorem 3.2 *Let two matrices V_0, W_0 and two vector fields K_0, ρ_0 satisfy*

$$(EV_0)U - UV_0 = U'[K_0], \tag{3.12}$$

$$(EW_0)U - UW_0 = U'[\rho_0] + \lambda U_\lambda, \tag{3.13}$$

and $\Omega(X)$ be a solution to (3.11). If we define a hierarchy of new vector fields and two hierarchies of square matrix operators as follows

$$\rho_l = \Phi^l \rho_0, \ l \geq 1, \tag{3.14}$$

$$V_k = \lambda^k V_0 + \sum_{i=1}^{k} \lambda^{k-i}\Omega(K_{i-1}) \ , \ k \geq 1, \tag{3.15}$$

$$W_l = \lambda^l W_0 + \sum_{j=1}^{l} \lambda^{l-j}\Omega(\rho_{j-1}) \ , \ l \geq 1, \tag{3.16}$$

then the square matrix operators V_k, W_l, $k, l \geq 0$, satisfy

$$(EV_k)U - UV_k = U'[K_k], \ k \geq 0, \tag{3.17}$$

$$(EW_l)U - UW_l = U'[\rho_l] + \lambda^{l+1}U_\lambda, \ l \geq 0, \tag{3.18}$$

respectively. Therefore $u_t = K_k$ and $u_t = \rho_l$ possess the isospectral ($\lambda_t = 0$) and nonisospectral ($\lambda_t = \lambda^{l+1}$) discrete zero curvature representations

$$U_t = (EV_k)U - UV_k, \ U_t = (EW_l)U - UW_l,$$

respectively.

Proof: We prove two equalities (3.17) and (3.18). We can compute that

$$\begin{aligned}
& (EV_k)U - UV_k \\
= \ & \lambda^k[(EV_0)U - UV_0] + \sum_{i=1}^{k} \lambda^{k-i}[E\Omega(K_{i-1})U - U\Omega(K_{i-1})] \\
= \ & \lambda^k U'[K_0] + \sum_{i=1}^{k} \lambda^{k-i}\{U'[\Phi K_{i-1}] - \lambda U'[K_{i-1}]\} \\
= \ & \lambda^k U'[K_0] + \sum_{i=1}^{k} \lambda^{k-i}\{U'[K_i] - \lambda U'[K_{i-1}]\} \\
= \ & U'[K_k], \ k \geq 1; \\
& (EW_l)U - UW_l \\
= \ & \lambda^l[(EW_0)U - UW_0] + \sum_{j=1}^{l} \lambda^{l-j}[E\Omega(\rho_{j-1})U - U\Omega(\rho_{j-1})] \\
= \ & \lambda^l\{U'[\rho_0] + \lambda U_\lambda\} + \sum_{j=1}^{l} \lambda^{l-j}\{U'[\Phi\rho_{j-1}] - \lambda U'[\rho_{j-1}]\} \\
= \ & \lambda^l\{U'[\rho_0] + \lambda U_\lambda\} + \sum_{j=1}^{l} \lambda^{l-j}\{U'[\rho_j] - \lambda U'[\rho_{j-1}]\} \\
= \ & U'[\rho_l] + \lambda^{l+1}U_\lambda, \ l \geq 1.
\end{aligned}$$

Note that we have used the characteristic operator equation (3.11). The rest is obviously and the proof is therefore finished. ▌

This theorem gives rise to the structure of Lax operators associated with the isospectral and nonisospectral hierarchies. In fact, the theorem provides us with a method to construct an isospectral hierarchy and a nonisospectral hierarchy associated with a discrete spectral problem (3.10) by solving two initial key discrete zero curvature equations (3.12) and (3.13) and by solving a characteristic operator equation (3.11), if a hereditary operator is known. However, the hereditary operator Φ and the isospectral hierarchy (3.9) are often determined from the spectral problem (3.10) simultaneously. Therefore we obtain just a new nonisospectral hierarchy (3.14). A operator solution to (3.11) may be generated by changing the element K_k (or G_k) in the following equality

$$\Omega(K_k) = V_{k+1} - \lambda V_k, \tag{3.19}$$

which may be checked through (3.15). The Lax operator matrices V_{k+1} and V_k are known, when the isospectral hierarchy has already been given. Therefore the whole process of construction of nonisospectral hierarchies becomes an easy task: finding ρ_0, W_0 to satisfy (3.13) and computing $V_{k+1} - \lambda V_k$.

The nonisospectral hierarchy (3.14) is exactly the required master symmetries. The reasons are that the product equations of the isospectral hierarchy and the nonisospectral hierarchy are still isospectral by Theorem 3.1, and the isospectral equations often commute with each other. Therefore it is because there exists a nonisospectral hierarchy that there exist master symmetries for lattice equations derived from a given spectral problem. In the next section, we will in detail explain our construction process by two concrete examples and establish their corresponding centreless Virasoro symmetry algebras.

4 Application to lattice hierarchies

We explain by two lattice hierarchies how to apply the method in the last section to construct master symmetries.

Example 4.1. *A Volterra-type lattice hierarchy.* Let us first consider the discrete spectral problem [15]:

$$E\phi = U\phi,\ U = \begin{pmatrix} 1 & u \\ \lambda^{-1} & 0 \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{4.1}$$

The corresponding isospectral lattice hierarchy:

$$u_t = K_k = \Phi^k K_0 = u(a_{k+1}^{(1)} - a_{k+1}^{(-1)}),\ K_0 = u(u^{(-1)} - u^{(1)}),\ k \geq 0, \tag{4.2}$$

where the hereditary operator Φ is defined by

$$\Phi = u(1+E^{-1})(-u^{(1)}E^2+u)(E-1)^{-1}u^{-1}.$$

The associated Lax operators are as follows

$$V_k = (\lambda^{k+1}V)_{\geq 1} + \begin{pmatrix} a_{k+1} & 0 \\ c_{k+1} & a_{k+1}^{(-1)} \end{pmatrix}, \ k \geq 0, \tag{4.3}$$

where $(P)_{\geq i}$ denotes the selection of the terms with degrees of λ no less than i. The matrix $V = \sum_{i\geq 0} V_{(i)}\lambda^{-i} = \sum_{i\geq 0} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}$ solves the stationary ($U_t = 0$) zero curvature equation $(EV)U - UV = 0$, and we choose $a_0 = \frac{1}{2}, b_0 = u, c_0 = 0$, $a_1 = -u, b_1 = -(u^2+uu^{(-1)}), c_1 = 1$ and require that $a_{i+1}|_{u=0} = b_{i+1}|_{u=0} = c_{i+1}|_{u=0} = 0$, $i \geq 1$. Actually we have

$$\begin{cases} a_{i+1}^{(1)} - a_{i+1} = -u^{(1)}c_{i+1}^{(2)} + uc_{i+1}, \ i \geq 1, \\ b_{i+1} = uc_{i+2}^{(1)}, \ i \geq 1 \\ c_{i+1} = a_i + a_i^{(-1)}, \ i \geq 1. \end{cases} \tag{4.4}$$

In particular, we can obtain

$$V_0 = \begin{pmatrix} \frac{1}{2}\lambda - u & \lambda u \\ 1 & -\frac{1}{2}\lambda - u^{(-1)} \end{pmatrix}.$$

Nonisospectral Hierarchy:

Step 1: To solve the nonisospectral ($\lambda_t = \lambda$) initial key discrete zero curvature equation (3.13) yields a pair of solutions:

$$\rho_0 = u, \ W_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{4.5}$$

Step 2: We compute that

$$\begin{aligned} & V_{k+1} - \lambda V_k \\ &= (\lambda^{k+2}V)_{\geq 1} + \begin{pmatrix} a_{k+2} & 0 \\ c_{k+2} & a_{k+2}^{(-1)} \end{pmatrix} - \lambda(\lambda^{k+1}V)_{\geq 1} - \lambda \begin{pmatrix} a_{k+1} & 0 \\ c_{k+1} & a_{k+1}^{(-1)} \end{pmatrix} \\ &= \begin{pmatrix} a_{k+2} & \lambda b_{k+1} \\ c_{k+2} & -\lambda(a_{k+1} + a_{k+1}^{(-1)}) + a_{k+2}^{(-1)} \end{pmatrix}. \end{aligned}$$

On the other hand, by (4.2) and (4.4), we have

$$\begin{cases} a_{k+1} = (E - E^{-1})^{-1} u^{-1} X_k, \\ a_{k+2} = (E - E^{-1})^{-1} u^{-1} \Phi X_k, \\ c_{k+2} = a_{k+1} + a_{k+1}^{(-1)} = (1 + E^{-1}) a_{k+1}, \\ b_{k+1} = u c_{k+2}^{(1)} = u(E+1) a_{k+1}. \end{cases}$$

Now by interchanging the element X_k into X in the quantity $V_{k+1} - \lambda V_k$, we obtain a solution to the corresponding characteristic operator equation:

$$\Omega(X) = \begin{pmatrix} \Omega_{11}(X) & \Omega_{12}(X) \\ \Omega_{21}(X) & \Omega_{22}(X) \end{pmatrix}, \tag{4.6}$$

where $\Omega_{ij}(X)$, $i, j = 1, 2$, are given by

$$\begin{aligned} \Omega_{11}(X) &= (E-1)^{-1}(-u^{(1)}E^2 + u)(E-1)^{-1}u^{-1}X \\ \Omega_{12}(X) &= \lambda u E (E-1)^{-1} u^{-1} X \\ \Omega_{21}(X) &= (E-1)^{-1} u^{-1} X \\ \Omega_{22}(X) &= [-\lambda + E^{-1}(E-1)^{-1}(-u^{(1)}E^2 + u)](E-1)^{-1}u^{-1}X. \end{aligned}$$

Therefore we obtain a hierarchy of nonisospectral discrete equations $u_t = \rho_l = \Phi^l \rho_0,\ l \geq 0$, by Theorem 3.2.

Symmetry Algebra:

Step 3: We make the following computation at $u = 0$:

$$\begin{aligned} &K_k|_{u=0} = 0,\ \rho_l|_{u=0} = \Phi^l \rho_0|_{u=0} = 0,\ k, l \geq 0, \\ &V_k|_{u=0} = \lambda^k \begin{pmatrix} \frac{1}{2}\lambda & 0 \\ 1 & -\frac{1}{2}\lambda \end{pmatrix},\ k \geq 0, \\ &W_l|_{u=0} = \lambda^l \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + (1 - \delta_{l0})\lambda^{l-1} \begin{pmatrix} 0 & 0 \\ [n] & -\lambda[n] \end{pmatrix},\ l \geq 0, \end{aligned}$$

where δ_{l0} represents the Kronecker symbol, and $[n]$ denotes a multiplication operator $[n] : u \mapsto [n]u$, $([n]u)(n) = nu(n)$, involved in the construction of nonisospetral hierarchies. While computing $W_l|_{u=0}$, we need to note that $\Omega(\rho_0)|_{u=0} \neq 0$, but $\Omega(\rho_l)|_{u=0} = 0,\ l \geq 1$. Now we may find by the definition (3.3) of the product of two Lax operators that

$$\begin{cases} [\![V_k, V_l]\!]|_{u=0} = 0,\ k, l \geq 0, \\ [\![V_k, W_l]\!]|_{u=0} = (k+1)V_{k+l}|_{u=0},\ k, l \geq 0, \\ [\![W_k, W_l]\!]|_{u=0} = (k-l)W_{k+l}|_{u=0},\ k, l \geq 0. \end{cases} \tag{4.7}$$

Step 4: It is easy to prove that

$$[\![V_k, V_l]\!], [\![V_k, W_l]\!] - (k+1)V_{k+l}, [\![W_k, W_l]\!] - (k-l)W_{k+l},\ k, l \geq 0,$$

are all isospectral ($\lambda_t = 0$) Lax operators. For example, the spectral evolution laws of the Lax operators of the third kind are

$$[\![\lambda^{k+1}, \lambda^{l+1}]\!] - (k-l)\lambda^{k+l+1} = (k+1)\lambda^k\lambda^{l+1} - (l+1)\lambda^{k+1}\lambda^l - (k-l)\lambda^{k+l+1} = 0.$$

Then by the uniqueness property of the spectral problem (4.1), we obtain a Lax operator algebra

$$\begin{cases} [\![V_k, V_l]\!] = 0,\ k, l \geq 0, \\ [\![V_k, W_l]\!] = (k+1)V_{k+l},\ k, l \geq 0, \\ [\![W_k, W_l]\!] = (k-l)W_{k+l},\ k, l \geq 0. \end{cases} \tag{4.8}$$

Further, due to the injective property of U', we finally obtain a vector field algebra of the isospectral hierarchy and the nonisospectral hierarchy

$$\begin{cases} [K_k, K_l] = 0,\ k, l \geq 0, \\ [K_k, \rho_l] = (k+1)K_{k+l},\ k, l \geq 0, \\ [\rho_k, \rho_l] = (k-l)\rho_{k+l},\ k, l \geq 0. \end{cases} \tag{4.9}$$

This implies that $\rho_l,\ l \geq 0$, are all master symmetries of each equation $u_t = K_{k_0}$ of the isospectral hierarchy, and the symmetries

$$K_k,\ k \geq 0,\ \tau_l^{(k_0)} = t[K_{k_0}, \rho_l] + \rho_l = (k_0+1)tK_{k_0+l} + \rho_l,\ l \geq 0, \tag{4.10}$$

constitute a symmetry algebra of Virasoro type possessing the same commutator relations as (4.9).

Example 4.2: *The Toda lattice hierarchy.* Let us second consider the discrete spectral problem [11]:

$$E\phi = U\phi,\ U = \begin{pmatrix} 0 & 1 \\ -v & \lambda - p \end{pmatrix},\ u = \begin{pmatrix} p \\ v \end{pmatrix},\ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{4.11}$$

which is equivalent to $(E + vE^{-1} + p)\psi = \lambda\psi$. The corresponding isospectral ($\lambda_t = 0$) integrable Toda lattice hierarchy reads as

$$u_t = K_k = \Phi^k K_0,\ K_0 = \begin{pmatrix} v - v^{(1)} \\ v(p - p^{(-1)}) \end{pmatrix},\ k \geq 0, \tag{4.12}$$

where the hereditary operator Φ is given by

$$\Phi = \begin{pmatrix} p & (v^{(1)}E^2 - v)(E-1)^{-1}v^{-1} \\ v(E^{-1}+1) & v(pE - p^{(-1)})(E-1)^{-1}v^{-1} \end{pmatrix}. \tag{4.13}$$

The first nonlinear discrete equation is exactly the Toda lattice [16]

$$\begin{cases} p_t(n) = v(n) - v(n+1), \\ v_t(n) = v(n)(p(n) - p(n-1)), \end{cases} \tag{4.14}$$

up to a transform of dependent variables. The corresponding Lax operators read as

$$V_k = (\lambda^{k+1}V)_{\geq 0} + \begin{pmatrix} b_{k+2} & 0 \\ 0 & 0 \end{pmatrix}, \ k \geq 0, \tag{4.15}$$

Here $V = \sum_{i\geq 0} V_{(i)}\lambda^i = \sum_{i\geq 0} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}$ solves the stationary discrete zero curvature equation $(EV)U - UV = 0$, and we choose $a_0 = \frac{1}{2},\ b_0 = 0,\ c_0 = 0,\ a_1 = 0,\ b_1 = -1,\ c_1 = -v$ and require that $a_{i+1}|_{u=0} = b_{i+1}|_{u=0} = c_{i+1}|u = 0 = 0,\ i \geq 1$. More precisely, we have

$$\begin{cases} a_{i+1}^{(1)} - a_{i+1} = p(a_i^{(1)} - a_i) + (vb_i - v^{(1)}b_i^{(2)}),\ i \geq 1, \\ b_{i+1}^{(1)} = pb_i^{(1)} - (a_i^{(1)} + a_i),\ i \geq 1, \\ c_{i+1} = -vb_{i+1}^{(1)},\ i \geq 1. \end{cases} \tag{4.16}$$

Now it is easy to find an isospectral Lax operator

$$V_0 = \begin{pmatrix} \frac{1}{2}\lambda - p^{(-1)} & -1 \\ v & -\frac{1}{2}\lambda \end{pmatrix}.$$

Nonisospectral Hierarchy:

Step 1: To solve find the nonisospectral ($\lambda_t = \lambda$) initial key discrete zero curvature equation (3.2) leads to a pair of solutions

$$\rho_0 = \begin{pmatrix} p \\ 2v \end{pmatrix},\ W_0 = \begin{pmatrix} [n]-1 & 0 \\ 0 & [n] \end{pmatrix},$$

$[n]$ still being a multiplication operator $[n] : u \mapsto [n]u$, $([n]u)(n) = nu(n)$.

Step 2: To compute $V_{k+1} - \lambda V_k$ leads to a solution to the corresponding characteristic operator equation:

$$\Omega(X) = \begin{pmatrix} \Omega_{11}(X) & \Omega_{12}(X) \\ \Omega_{21}(X) & \Omega_{22}(X) \end{pmatrix},\ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

where $\Omega_{ij}(X)$, $i, j = 1, 2$, are given by

$$\begin{aligned} \Omega_{11} &= E^{-1}(E-1)^{-1}X_1 + (p^{(-1)} - \lambda)(E-1)^{-1}v^{-1}X_2, \\ \Omega_{12} &= (E-1)^{-1}v^{-1}X_2, \\ \Omega_{21} &= vE(E-1)^{-1}v^{-1}X_2, \\ \Omega_{22} &= (E-1)^{-1}X_1. \end{aligned}$$

At this stage, we obtain a hierarchy of nonisospectral discrete equations $u_t = \rho_l = \Phi^l \rho_0,\ l \geq 0$, by Theorem 3.2.

Symmetry Algebra:

Step 3: We make the following computation at $u = 0$:

$$K_k|_{u=0} = 0,\ \rho_l|_{u=0} = \Phi^l \rho_0|_{u=0} = 0,\ k, l \geq 0,$$
$$V_k|_{u=0} = \lambda^k \begin{pmatrix} \frac{1}{2}\lambda & -1 \\ 0 & -\frac{1}{2}\lambda \end{pmatrix},\ k \geq 0,$$
$$W_l|_{u=0} = \lambda^l \begin{pmatrix} [n]-1 & 0 \\ 0 & [n] \end{pmatrix} + (1-\delta_{l0}) \begin{pmatrix} -2\lambda[n] & 2[n] \\ 0 & 0 \end{pmatrix},\ l \geq 0.$$

Now we may similarly find by the product definition (3.3) of the product of two Lax operators that

$$\begin{cases} [\![V_k, V_l]\!]|_{u=0} = 0,\ k, l \geq 0, \\ [\![V_k, W_l]\!]|_{u=0} = (k+1)V_{k+l}|_{u=0},\ k, l \geq 0, \\ [\![W_k, W_l]\!]|_{u=0} = (k-l)W_{k+l}|_{u=0},\ k, l \geq 0. \end{cases}$$

Step 4: Because $[\![V_k, V_l]\!]$, $[\![V_k, W_l]\!]-(k+1)V_{k+l}$, $[\![W_k, W_l]\!]-(k-l)W_{k+l}$, $k, l \geq 0$, are still isospectral ($\lambda_t = 0$) Lax operators, a Lax operator algebra may be similarly obtained by using the uniqueness property:

$$\begin{cases} [\![V_k, V_l]\!] = 0,\ k, l \geq 0, \\ [\![V_k, W_l]\!] = (k+1)V_{k+l},\ k, l \geq 0, \\ [\![W_k, W_l]\!] = (k-l)W_{k+l},\ k, l \geq 0. \end{cases} \tag{4.17}$$

Further, through the injective property of U', a vector field algebra is yielded

$$\begin{cases} [K_k, K_l] = 0,\ k, l \geq 0, \\ [K_k, \rho_l] = (k+1)K_{k+l},\ k, l \geq 0, \\ [\rho_k, \rho_l] = (k-l)\rho_{k+l},\ k, l \geq 0. \end{cases} \tag{4.18}$$

This shows that the symmetries for $u_t = K_{k_0}$:

$$K_k,\ k \geq 0,\ \tau_l^{(k_0)} = t[K_{k_0}, \rho_l] + \rho_l = (k_0+1)tK_{k_0+l} + \rho_l,\ l \geq 0, \tag{4.19}$$

constitute the same centreless Virasoro algebra as (4.18).

5 Concluding remarks

We introduced a simple procedure to construct master symmetries for isospectral lattice hierarchies associated with discrete spectral problems. The crucial points are an algebraic structure related to discrete zero curvature equations and the structure of Lax operators of isospectral and nonisospectral lattice hierarchies. The procedure may be divided into four steps. The first two steps yields the required nonisospectral lattice hierarchy, by solving an initial key nonisospectral discrete zero curvature equation, and by computing a difference $V_{k+1} - \lambda V_k$ between two isospectral Lax operators V_{k+1} and V_k. The second two steps yields the required centreless Virasoro symmetry algebra, by using an algebraic relation between two discrete zero curvature equations, and the uniqueness property and the injective property of the corresponding spectral operators. Two lattice hierarchies are shown as illustrative examples.

There is a common Virasoro algebraic structure for symmetry algebras of a Volterra-type lattice hierarchy and the Toda lattice hierarchy. In general, we have

$$[K_k, \rho_l] = (k + \gamma)K_{k+l}, \ \gamma = const., \quad (semi - direct\ product), \tag{5.1}$$

but the other two equalities do not change:

$$\begin{cases} [K_k, K_l] = 0 & (Abelian\ algebra), \\ [\rho_k, \rho_l] = (k - l)\rho_{k+l} & (centreless\ Virasoro\ algebra). \end{cases} \tag{5.2}$$

This is also a common characteristic for continuous integrable hierarchies [17]. Interestingly, this kind of centreless Virasoro algebra itself may also be used to generate variable-coefficient integrable equations which possess higher-degree polynomial time-dependent symmetries [18].

We point out that there are some other methods to construct master symmetries by directly searching for some suitable vector fields ρ_0 [10] [19] [20]. However the approach above takes full advantage of discrete zero curvature equations, and therefore it is more systematic and may be easily applied to other lattice hierarchies.

Acknowledgments: One of the authors (W. X. Ma) would like to thank the Alexander von Humboldt Foundation for financial support. He is also grateful to Prof. R. K. Bullough and Dr. P. J. Caudrey for their warm hospitality at UMIST and to Prof. W. I. Fushchych for his kind and helpful conversations.

References

[1] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, in: *What is Integrability?*, ed. V. E. Zakhalov (Springer Verlag, Berlin, 1991) p115–184.

[2] D. Levi and R. I. Yamilov, “Classification of evolutionary equations on the lattice. I. The general theory”, solv-int/9511006.

[3] B. Fuchssteiner and A. S. Fokas, *Physica D*, **4**(1981), 47–66.

[4] A. S. Fokas, *Studies in Appl. Math.*, **77**(1987), 253–299.

[5] B. Fuchssteiner, *Prog. Theor. Phys.*, **70**(1983), 1508–1522.

[6] W. X. Ma, *J. Phys. A: Math. Gen.*, **23**(1990), 2707–2716.

[7] H. H. Chen and Y. C. Lee, in: *Advances in Nonlinear Waves* Vol. II, ed. L. Debnath, Research Notes in Mathematics 111 (Pitman, Boston, 1985) p233-239.

[8] W. X. Ma, *J. Phys. A: Math. Gen.*, **25**(1992), L719-L726.

[9] W. X. Ma, *Phys. Lett. A*, **179**(1993), 179–185; *Proceedings of the 21st International Conference on the Differential Geometry Methods in Theoretical Physics*, ed M. L. Ge (World Scientific, Singapore, 1993) p535-538.

[10] W. Oevel, B. Fuchssteiner and H. W. Zhang, *Prog. Theor. Phys.*, **81**(1989), 294–308.

[11] G. Z. Tu, *J. Phys. A: Math. Gen.*, **23**(1990), 3903–3922.

[12] W. X. Ma, *J. Phys. A: Math. Gen.*, **25**(1992), 5329–5343.

[13] W. X. Ma, *J. Phys. A: Math. Gen.*, **26**(1993), 2573–2582.

[14] A. S. Fokas and R. L. Anderson, *J. Math. Phys.*, **23**(1982), 1066–1073.

[15] H. W. Zhang, G. Z. Tu, W. Oevel and B. Fuchssteiner, *J. Math. Phys.*, **32**(1991), 1908–1918.

[16] M. Toda, *Theory of Nonlinear Lattice* (Springer-Verlag, New York, 1981).

[17] W. X. Ma, *J. Math. Phys.*, **33**(1992), 2464–2476.

[18] W. X. Ma, R. K. Bullough and P. J. Caudrey, preprint, 1996.

[19] I. Y. Cherdantsev and R. I. Yamilov, Physica D **87**, 140–144 (1995).

[20] B. Fuchssteiner, S. Ivanov and W. Wiwianka, Algorithmic determination of infinite dimensional symmetry groups for integrable systems in $1+1$ dimensions, preprint, 1995.