Computer algebra:
Does it offer new means for doing and teaching mathematics?

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Lecture at ICTMT 3, Koblenz 1997

Abstract:

After lengthy introductory remarks on the shortcomings of teaching mathematics, current trends in technical developments of Computer algebra systems and their necessity for educational systems are reviewed. The impact of such developments on creating future multimedia environments for doing mathematics is outlined and the possible effect on education in mathematics is evaluated. The possibilities of systems that learn on their own and thus can stay in pace with the broadened knowledge of their users are discussed.

Introduction

I am leader of a group developing the CAS (computer algebra system) MuPAD (Multi Processor Algebra Datatool).

If you look at its catch phrase, you certainly get the idea that we are developing a high level system, which consequently cannot have anything to do with everyday teaching of mathematics in our schools. This suggests one of the basic questions I will pose in my lecture:

If CAS do have an impact at all on mathematics education, do we need simpler tools or more sophisticated ones?

Another, even more fundamental question is whether or not CAS, and more generally modern technology tools, are of any use for the education in a science which in its rigor and beauty dates back more than 2000 years.

Opinions are divided as we see in the following slide:

- **Computer algebra systems are a genuine tool for teaching.** Meinolf Sellmann.
- **The themes that emerge from [the availability of technology in instruction] permeate all the [calculus] reform courses.** Alan H. Schoenfeld in UME Trends [UME, p.4]
- **The technology aspect of calculus reform courses is especially disturbing.** Gorge E. Andrews: in [UME, p.17]
- **NIL on modern technology.** The GDM (Gesellschaft für Didaktik der Mathematik) in a June 1997 report for the joint conference of German Education Ministers on the possibilities of bringing German mathematical education to its full potential.

This last contribution to the problem of improving instruction in mathematics is remarkable insofar as it is an impressive collection of abstract general nonsense, for example finding the reason for the lack of efficiency in German math education mainly in shortcomings of the society and the fact that collaboration between teachers and German didactic is not close enough. This level of awareness in the discussion of problems and chances given in mathematical education by modern technology, shown by such statements, may be a German problem only, and as such could be left aside. But such statements are pointing to the problem that even among the highly industrialized nations the awareness for such problems differs dramatically (for example in the US we have since 1986 a lively discussion of these matters, at least after the Tulane conference on UME (Undergraduate Mathematics Education) highlighted the problems).

In any case the general influence of tools like CAS to the development of modern life and science cannot be denied:

- **Mathematics is the basis of technological progress and technological progress is a key for international competitiveness. Automating an important part of the mathematical problem solving process is a key technology for a nation that wishes to control structure and accelerate technological progress. The automation of the solution of mathematical problems is a powerful lever with which human productivity and expertise can be amplified many times... Symbolic computation is a part of a key technology, namely scientific and engineering computation that is becoming increasingly important to science, technology and society** Report to the National Science Foundation: A. C. Hearn, Ann Boyle and B.F. Caviness, Future Directions for Research in Symbolic Computation, Siam Reports on Issues in the Mathematical Sciences, Philadelphia, 1990

- **Computers may be dumb, but they are not too dumb to take your job.** Robert Wright Viewpoint Time Magzin May 26. 1997

Since education also should reflect all important developments of society, the relation between technology and education [in mathematics] has to be discussed, this even more, as the use of CAS may be a decisive factor in what people outside of mathematics believe the content of mathematics to be. A general observation is:

More jobs than ever will have a mathematical component, and technology will play a key role in how this mathematics is done. Janet Ray in [UME, p. 9]

So even if these systems are of no help for a better understanding and learning of mathematics (both are not the same and in German schools we may put to much emphasis on the latter) these tools will gain their forceful entrance in our schools and universities, to a much higher extent than we imagine today.

In the following I will not discuss:

- whether or not the influence of modern technology on education of mathematics will be positive or negative.
- I just assume that this influence will happen, that changes will be brought about by these possibilities, and not only superficial changes.

However, since I believe that the quality of these changes more depends on us than on technology, I will make some remarks how we can influence these developments to the better.
Basically it is simple what to do:

- **Technology is not the solution to pedagogical problems [in mathematics] ... but rather an opportunity to think about and solve those problems in a new way.**

David A. Smith in [UME , p. 14]

One point is important to me:
According to almost all opinions CAS will have a major influence on

How we do mathematics.

Some say, that this influence will be far greater than that on

How we teach mathematics.

It also seems to be taken for granted, that CAS will influence the perception others have of mathematics. Such assertions demonstrate that there is a conceptual gap between

Doing mathematics and *Teaching mathematics*

and it is tacitly assumed that this gap will widen. A fact, if it comes true, I greatly deplore, and I strongly believe such a gap to be adverse to the development of mathematics and the development of society as well. So one of the relevant questions should be, if the means of modern technology cannot help to bridge that gap instead of widening it.

Also in education the statement: **Computers may be dumb, but they are not too dumb to take your job,** may be a relevant assertion, meaning that if you cannot go along with these changes you are left behind.

Present State of teaching

When we ask

What is wrong with our teaching of mathematics?

Answers come in abundance:

- **We concentrate to much on the formalities of mathematics, on syntax and formal tautologies**
- **We do not present enough applications and examples**
- **We neglect intuition, or at least we do not teach intuition**
- **There is a considerable gap between what we teach and what we use in every day life**
- **We fool ourselves about the success of our teaching**
- **There should be more cooperative learning**
- **We need to teach more understanding of concepts and ideas**
- **We impose to many subjects on our students instead of having them concentrate on the real understanding of a few**
- **Our students do not experience success often enough**
- **We believe to much in our own perception of the subject, this insofar as we tend to take the understanding we have reached after a thorough study as starting point for teaching others**

However:

All that is just a collection of good intentions and a statement of the obvious.

The relevant question is not whether we have these deficiencies but if we have viable alternatives, if we can do better at all. Because one is certain: We run into these difficulties not out of our bad intentions, and most often not out of incompetence.
Nevertheless, before we concentrate on finding out if modern technology can lead us out of this impasse, we like to show that these points are shared by others.

Items:

- **We concentrate to much on the formalities of mathematics, on syntax and formal tautologies**

Indeed this aspect of mathematics certainly does not yield stimulating effects for our students:
Mathematics depicted as symbol manipulation is dull boring and frustrating. .. If we drill on symbol manipulation we will be able to do more symbol manipulation, but not necessarily more mathematics.
Wade Ellis, Trends in [UME, p. 10].

With respect to syntax and the understanding of the formalities Richard von Mises, one of the outstanding mathematicians of this century goes even further:
.. in high school .. all of us had occasion to learn certain axioms .. they .. soon came to haunt our memories like nightmares. .. familiar examples [are] every quantity is equal to itself. The student normally does not feel any apprehension towards the assertion of obviousness. For, how, indeed could he imagine that a quantity is not equal to itself. [TWM, p. 1723]

Using syntax and symbol manipulation without deeper understanding of what lies behind is not at all motivating. In addition students have limited manipulative skills and they always will have. The German author Thomas Mann gives us a convincing description about what ordinary humans feel in face of mathematical syntax and symbols:

- *Was er sah, war sinnverwirrend. Ein phantastischer Hokuspokus, ein Hexensabbat verschränkter Runen bedeckte die Seiten. Griechische Schriftzeichen waren mit lateinischen und mit Ziffern in verschiedener Höhe verkoppelt, mit Kreuzen und Strichen durchsetzt, ober- und unterhalb waagerechter Linien bruchartig aufgereiht, durch andere Linien zeltartig überdacht, durch Doppelstrichelchen gleichgewertet, durch runde Klammern zu großen Formelmassen vereinigt. Einzelne Buchstaben, waren rechts und oberhalb der umklammerten Gruppen ausgesetzt. Kapitalistische Male, vollständig unverständlich dem Laiensinn, umfaßten mit ihren Armen Buchstaben und Zahlen, während Zahlenbrüche ihnen voranstanden und Zahlen und Buchstaben ihnen zu Häupten und zu Füßen schwebten. Sonderbare Silben, Abkürzungen geheimnisvoller Worte waren überall eingestreut, und zwischen den nekromantischen Kolonnen standen geschriebene Sätze und Bemerkungen in täglicher Sprache, deren Sinn gleichwohl so hoch über allen menschlichen Dingen war, daß man sie lesen konnte, ohne mehr davon zu verstehen, als von einem Zaubergemurmel.*

  Thomas Mann in “Königliche Hoheit”

- **We do not present enough applications and examples**

When in [UME, p. 6] a student is reported to say

then I can only add that this is true for me as well. Almost all mathematics was discovered or created by abstraction from examples. To show that certainly we do not give enough examples just one highlight:
When in a German test in 1978 several thousand students were asked if they had learned about groups in high school 54 % gave an affirmative answer, but only 21 % were able to present a single example.
So most of these students certainly did not learn by example, the knowledge about groups for half of them is just intellectual garbage.

However I should caution: Applications and examples alone will not help us to overcome the apparent difficulties, complex applications even less.

Adding new examples and technology will not help, more deep changes are necessary [UME, p. 5]

- **We neglect intuition, or at least we do not teach intuition**

I have to admit that intuition is not easy to teach. In addition, we often may be reluctant to put more emphasis on intuition because some of us have the feeling that it contradicts rigor.

Here I can offer an excellent excuse Emanuel Kant himself , this tower of rigor in philosophy is the one who assigned to intuition the crucial importance with respect to all our knowledge:

Thus, intuition and concepts constitute the elements of all our knowledge. [TWM, p. 1956].

Although mathematical progress in itself may have lead to a Crisis in Intuition (as Hans Hahn the German positivist and mathematician claims, ibid.) I believe that no reform in the education of mathematics can be successful which does not focus on how we can strengthen intuition.

- **There is a considerable gap between what we teach and what we use in every day life**

A sad fact is, you can do very well in modern life without knowing any mathematics at all. I should add, a sad fact for mathematicians, a lucky fact for most. The general public believes that even elementary mathematics is something for the specialist. By the general public, no science is less understood in its aims virtues and chances, as well as in its importance for society, than mathematics. When a few years ago the German mathematician Heimann published his findings about what really is used in daily life of the mathematics the average student learns, then these findings were utterly misunderstood by the public as if the author were advocating for a reduction of mathematics education to 7 years in school. Such a misunderstanding usually indicates a resonance between the faulty conclusion and the secret desire of the public, so it is no surprise that the findings of the author were generally applauded in public (and equally detested among mathematicians).

- **We fool ourselves about the success of our teaching**

This is a natural behavior.
Humans could not live in society if they did not fool each other (and themselves one might add). [LRF, p. 31].

That we are missing serious assessment of the success of our teaching is general knowledge:

Students completing a course with decent grades didn't really understand basic concepts. [UME, p. 9]

We must start where students are rather than we wish they were. UME page 10.
We need a more complex approach to student feedback than we have ever thought about in the past. [UME, p. 6]

I recently did some tests with several hundred freshman about which skills from 7th to 9th grade they still master. The results were devastating. However I should add, that I am convinced that if we do similar tests about the success of University education in mathematics, then the results would be even more devastating. This lack of success happily is without consequences because we have formed our society in such a way that one easily can do without any mathematics in future life.

- **There should be more cooperative learning**

This I only included in order to show that I have absorbed some of the usual catch phrases.

- **We need more understanding of concepts and ideas**

This is a serious point, on which nobody will take offense.

G. H. Hardy certainly is right when he states that *a mathematician is a maker of patterns* [TWM, p. 2027], but that does not mean that we need to put to much emphasis on patterns in the education of mathematics, because Hardy continues that these patterns *are made of ideas*, so obviously the ideas have to come first.

Mathematics may be an *axiomatized deductive system* (as Carl G. Hempel states it in [TWM, p. 1622]) but this insight may not necessarily help in education. We eventually will discover that in all our teaching of mathematics there is to much of this understanding. Certainly, the notion of *axiomatized deductive system* does not fully explain the applicability of math. We may be willing to believe that *The book of nature is written in the language of mathematics* as Galilei claims in his Discorsi, but most of us refuse to believe that the Universe is written in the language of an *axiomatized deductive system.*

The value of all our education in mathematics certainly is something else:

Capability, not scholarly knowledge is attained by doing science. The value of exercising a rigorous science lies not primarily in the results obtained, because these can only be a drop in the ocean of those items worth to be known. [SCH, Vol. . 5 p. 239].

- **We impose to many subjects in our students instead of having them concentrate on the real understanding of a few**

Here I also rely on the authority of Schopenhauer:

Observing all this scholarship I ask myself how seldom this person must have had thoughts in order to have the time to learn all this. . [SCH, Vol. . 5 p. 507].

- **Our students do experience success too seldom**

This has a disastrous effect insofar as without experiencing success a person does not gain confidence in its intellectual power. That this is the case you can see from the fact that the more mathematics education someone may have had, the more prudent he becomes in applying any of this.

Why is apprehension, the essential element for the researcher and philosopher, a lustful experience? Because you become aware of your power. [SCH, Vol. . 5 p. 236].

- **We believe to much in our own perception of the subject, this insofar as we tend to take the understanding we have reached after a thorough study of a subject as starting point when we teach others**

To often in our enthusiasm about science we overlook one basic fact:

Science gives much to those who do it, but few to those who have to learn it. [SCH, Vol. . 5 p. 235].

The fiber of mathematics was created by melting all relevant examples into sublime abstractions, this process has taken more than 2000 years, as a result we now have rather rigorous definitions of objects like real numbers and other basic notions. But believing that one can learn the content of math by absorbing these sublime abstractions without having experienced a considerable number of examples, is ignorant at best, arrogant at worst.

Can Computer algebra systems cure that - or why can't they?

The general opinion I encounter is:

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Computer algebra sytems represent important technological progress but are not suitable for schools.

A major argument for that opinion seems to be that CAS offer far more mathematics, than everyday life in schools requires. In order to see if this opinion is well founded:

Imagine: You are traveling back in time to 1910 and you offer the people you meet a modern day car, say a fancy Japanese car. Certainly, you would be told, this car is not suitable for our roads. However you know, their roads are not fit for this car.

So, in analogy, my first thesis is:

Schools are not suitable for computer algebra sytems

However, we can exploit this example a little bit further.

Imagine: You insist that those people you meet, drive a modern car, what kind of car do they need, a simple one or a more sophisticated one?

The answer is obvious: The most sophisticated and adaptable one you can get, a four wheel drive at least. Bumpy roads do not allow for cars who indulge in luxury instead of providing a high technical level. To stay in that picture:

What is wrong with the roads in mathematics.

Imagine: You have to teach poetry or literature to a group of people suffering under dyslexia (Legasthenie in German).

What will you discover? First of all that you have a difficult job, since these people cannot read. And obviously all the beauty of a poem is lost when , instead of reading it, you start spelling its words. Convince yourself, start spelling*: Ich weiß nicht was soll es beuten, daß ich so traurig bin, ein Märchen aus uralten Zeiten das geht mir nicht aus dem Sinn.* When you are French or English start spelling *Les sanglots longs des violons de l'automne*, or *friends Romans and countrymen.*

Then after teaching a little bit longer you will discover that in spite of the disability of your students some of them may very well understand the ideas of poems, some have a deep feeling for poems, some even may be able to write beautiful poems on their own. You only have to teach them poetry by avoiding that they read them on their own, because that destroys all beauty the poet has put into his work. You either read the poems to them, or, even that is better than allowing them to read them on their own, you let them scan the poems in a computer, and have them read by a synthesizer.

Do not misunderstand me: I do not advocate, that learning to read is of no value. I only claim that when teaching poetry you are allowed to avoid reading if that circumvents the obstacles for understanding poems.

The analogy is obvious, I claim that, with respect to mathematical formulas and the rigor of mathematical syntax, we are mostly teaching to dyslexics. I even go further by claiming that we all , more or less, suffer under this disease.

Just take the definition of dyslexia and do some obvious replacements:

- **Dysmathia (similar to dyslexia)**
- Dysmathia is a developmental disorder marked by difficulty in learning to manipulate mathematical formulas despite adequate intelligence, conventional instruction, and sociocultural opportunity. Research suggests that it results from various causes and that a number of subtypes of dysmathia probably exist with different origins and associated symptoms. Contrary to earlier theories, very few individuals with dysmathia appear to have perceptual problems--that is, the problem does not lie in perceiving mathematics correctly. Recent research indicates that dyslexia is instead usually related to some kind of symbolic impairment and is often associated with visual memory problems.

Let me give some proofs that dysmathia really exists, and that a large part of the population suffers under it.

When, after introducing Taylor series, and presenting a lot of examples, I ask my freshmen students to identify the series or the function given by:

$$1+x+x^2+x^3+x^4+x^5\ldots$$

usually none of them has any idea, even if I give them the hint that they have encountered this example in school. However, when after that I give them

$$1+q+q^2+q^3+q^4+q^5+\ldots$$

instead, then about 40 % of them start shouting that they recognized the geometric series. Incidentally, among those students were some who had heard either in school or elsewhere, about Gödels theorem, some of them were even able to give me some reasonable explanation what Gödels had found out.

Another example:
In a recent test less than 30 % of my freshman (in economics) were able to determine the slope of the line

$$\frac{x}{2}+3$$

When I presented this frustrating result to one of my colleagues, a colleague who is working in didactic and obviously has to know about such things, he pointed out that the result would not have been frustrating at all if I had asked them instead to determine the slope of

$$\frac{1}{2}x + 3$$

That is dysmathia pure!

Indeed, later on it was shown that these „stupid“ students (as one of my colleagues in pure math called them) where a rather intelligent bunch of young men and women, who even learned some math after all (but none of them knew about Gödels theorem).

So:

Dysmathia exists!

And I believe that by support of **adequate** electronic tools we may be able to overcome part of it and will be able to teach the ideas and the beauty of math even to those. I should however point out that I emphasized the word *adequate.*

Why is present day school not suitable for computer algebra?

Because we have teachers who - to stay in that picture - are used during all their life to teach poetry to dyslexics, and out of a sudden they are given means which bear the potential of circumventing the necessity for reading.

Do we have alternatives ?

Yes, I believe so, and furthermore I believe that modern technology can make a difference. That CAS embedded in Multimedia working environments will improve the conceptual understanding of mathematics as well as the ability to incorporate mathematical methods in daily work dramatically. However: only

in the long run .

I admit that proponents and proselytes of multimedia are coming along so often with so many ridiculous arguments and such a naive understanding of human intellectuality that speaking nevertheless in favor of multimedia needs courage.

In the general opinion and in the media multimedia is understood as a colorful Internet site with sound, pictures and films.
So as one of my collaborators stated it last week
„Multimedia is loud and noisy, has buttons and links, and contains elements which move“.
Remarkably, in most public statements, especially by politicians, never the possible content of multimedia is discussed. If multimedia is what is presently understand by it, then, indeed it offers no hope for a better understanding of mathematics - it only will perpetuate superficiality.
However, we overlook that CAS are some of the few examples where Multimedia may not only mean better access to more information, may not only mean use of a new world of noisy, wild and colorful data, but means also an opportunity to gain and create new knowledge, genuinely new knowledge, not just old knowledge and perception combined anew an newly painted and then bottled in new containers.
CA already nowadays, generates - among the working scientists from many fields - new insight, new understanding and push the frontiers of human perception forward. CAS already today allow in many areas an exploration of the unknown.
If, what even many pure, formal and rigorous mathematicians claim, one good example is worth more than a couple of theorems, then this is obvious, since we now have the opportunity to handle, to create and to understand a new universe of examples, examples of a quality which was out of reach up to now.
This however does not automatically mean that we are offered new ways of teaching by that. Of course , it would be a disaster if we did not find new ways of teaching and new ways of incorporating these new tools, which will win over mathematical usage outside of schools rapidly. Because this would mean a widening gap between what we teach and what we use - a gap which certainly would make scholarly education less relevant than it already seems nowadays.

So what are the obstacles and chances?

First of all, we should not forget change is something people naturally resist! So, if you are not only speaking about new ways in teaching but insist on really doing it, then you will experience that fact every day, sometimes you might even encounter outright hostility. In the eyes of your colleagues you too easily become a traitor of the cherished cultural values humanity has piled up through the ages.

On the other hand, rapid change is not always an advantage but sometimes dangerous, we had - at least in Germany - too many changes in mathematical teaching in too few time, and always we were told that the most recent change will cure all our needs. So, I believe it good advice to be careful if somebody tells us that he found what we call in Germany the Nuremberg Trichter for mathematics.

An apparently easy to implement course structure has, at some places, brought the pedagogical house down. [UME, p. 4].

Secondly, rapid change always really is bad - and not possible anyway most of the time. Take for example the situation already considered when we offered fancy cars to the person in 1910. Even if they were willing for change and adoption of modern ways of transportation, it just would not have been possible.

Only including modern techniques into education may eventually bring more harm than good. Would you really be understanding the subtleties of Bach’s music by playing it in a disco?

So having now done my good share in warning, I may now step forward to suggest modest change:
Playing Bach in a Disco may not be useful, but putting wonderful music on discs or CD’s, so that unseen masses may be enjoying it, certainly offers a new dimension for the understanding of music, which was not there before. And of course, by not relying on teaching music only to those who can produce it on there own offers new possibilities in education.
The same is true for mathematics and the new technical possibilities for applying it without going to through hardship only to discover that one suffers incurably under dysmathia.

And here CAS will make a difference, a considerable one! CAS will help us, or rather those who are suffering under dysmathia, to get away from many of the technicalities overshadowing the understanding, they will help to have a new and creative encounter with math. CAS, properly embedded in suitable environments will give an example for a new dimension in multimedia, by not only giving access to data and prefabricated knowledge, but by allowing creation of knowledge the electronic environment did not have beforehand.

CAS can contribute to the creation of knowledge instead of the storage of knowledge!

What is needed, what of the current trends will prevail for systems which **also** aim at teaching? Here for me the ‘also’ is important insofar as I do not believe in tools for education to be different from tools in serious applications, only their interfaces and their appearance may differ from that of their professional versions.

What Modern Tools in education need to be, can be compared to driving lessons, there you do not exercise with toy cars, the value of the education comes from the fact that you use the same cars as in real life, only they may be equipped differently.

Technical details of what is needed:

Above all: **Object orientation !**

Object-oriented Concept:

- Domains = Freely definable datastructures (e.g. *Matrices*, *Polynomials*, *ODE's*, ...)
- Categories = Classes of domains with common properties (e.g. *Rings*, *Fields*, *Groups*, ...)
- Axioms = Properties of categories
- Overloading of Operators = One operator (e.g. . +) for several operations
- Polymorphism = Generic algorithms

Why?

Mathematics, in its 2500 years history, has mainly created objects, objects representing intuition, objects representing our idea of space, objects representing our ideas for formulating the laws of the universe, i.e.

user defined objects.

Domains and categories reflect mathematical structure! These data structures allow you to structure the order of your structures, and mathematics, in particular where it is successful, has to do a lot with order, in particular with the order of our thoughts**.**
Therefore any system representing mathematics has to be object oriented, has to allow for the creation of your own abject, has to allow for structuring your objects into classes, into categories, to allow for a mechanism that one object inherits its properties from another. No system will survive which disregards these fundamental facts, and which allows for a comfortable way for implementing such objects.
Another point is that a system should be able to scope with the paradigm of the user not to force its own paradigm onto the user. Thus, abuse of language should be possible, or to formulate it better, overloading of functionality has to be possible

Overloading of operators:

Define a number and a matrix ...

```
>> a := 7;        A := Matrix( [[1,2],[3,4]] );
```

Both are multiplied with the same operator:

```
>> x := a * a;    X := A * A;
```

Of course, this also holds for mixed operations:

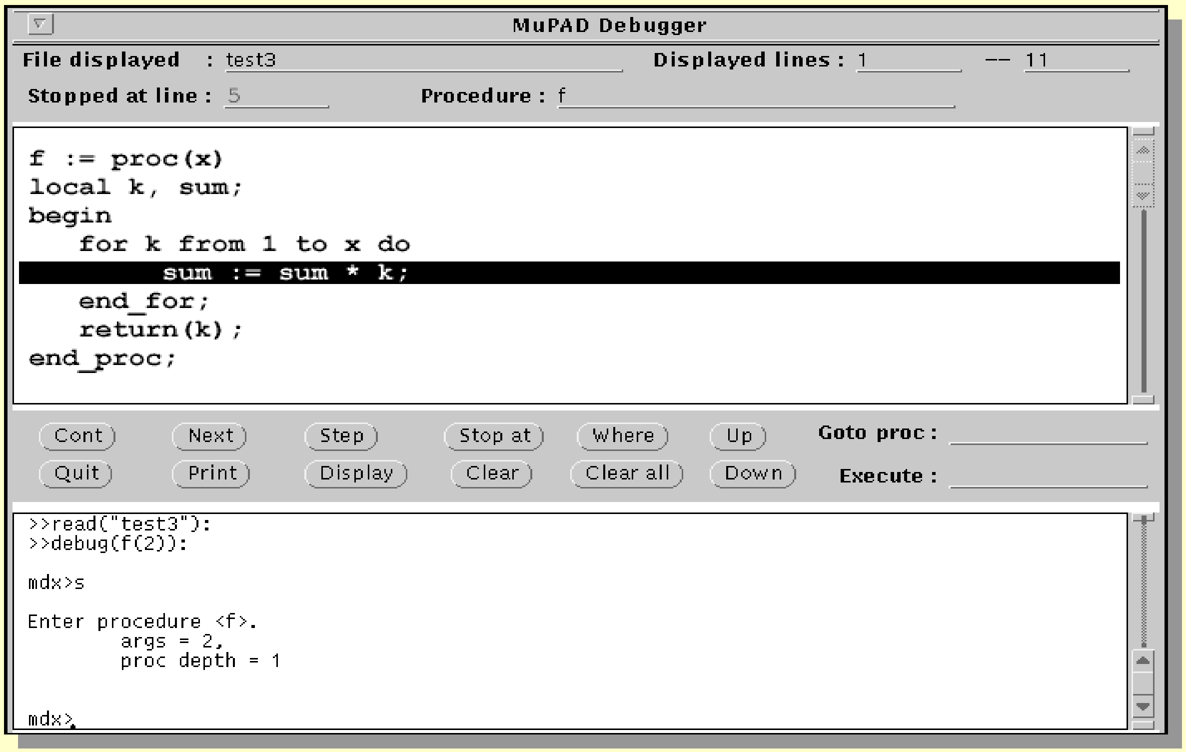
```
>> Z := a * A;
```

Systems which force you to write a new symbol for adding for example two matrices or remainder classes, are violating this simple and obvious requirement.

Then you need more information about the system - if you like. That is a matter of integrity. You must - in principle - be able to look inside, you need certain levels of user information, you need to be able to go stepwise through procedures you or others have written,. One way providing a bit of all this is for example an interactive debugger (but you need much more of all that).

MuPAD - **Multi Processing Algebra Datatool** **(www.mupad.de)**

Source-Level Debugger:



- Breakpoints
- Stepwise execution
- Monitoring of identifiers
- Exhibition of stack

You - at least if you are a teacher - need tools for the analysis of your problems, the efficiency of your solutions and so on. Remember, from now on we want to use in our schools these tools for math which are also needed for doing real life problems.
If a system are to be used for schools, they must allow for a reasonable downsizing, downsizing of mathematical sophistication, not downsizing of efficiency. This is a problem we have underestimated in its complexity. To see what we mean try to implement the following on the basis of any of the leading CAS.

- **Downsizing: 'Equations' for Sekundarstufe I**
  During the session with an arbitrary example the student has the choices:
- He commands an action and inserts the result

$$2x+8=\frac{1}{2}+3 \qquad | \ -\ 8$$

$$2x+8-8=\frac{1}{2}-5$$

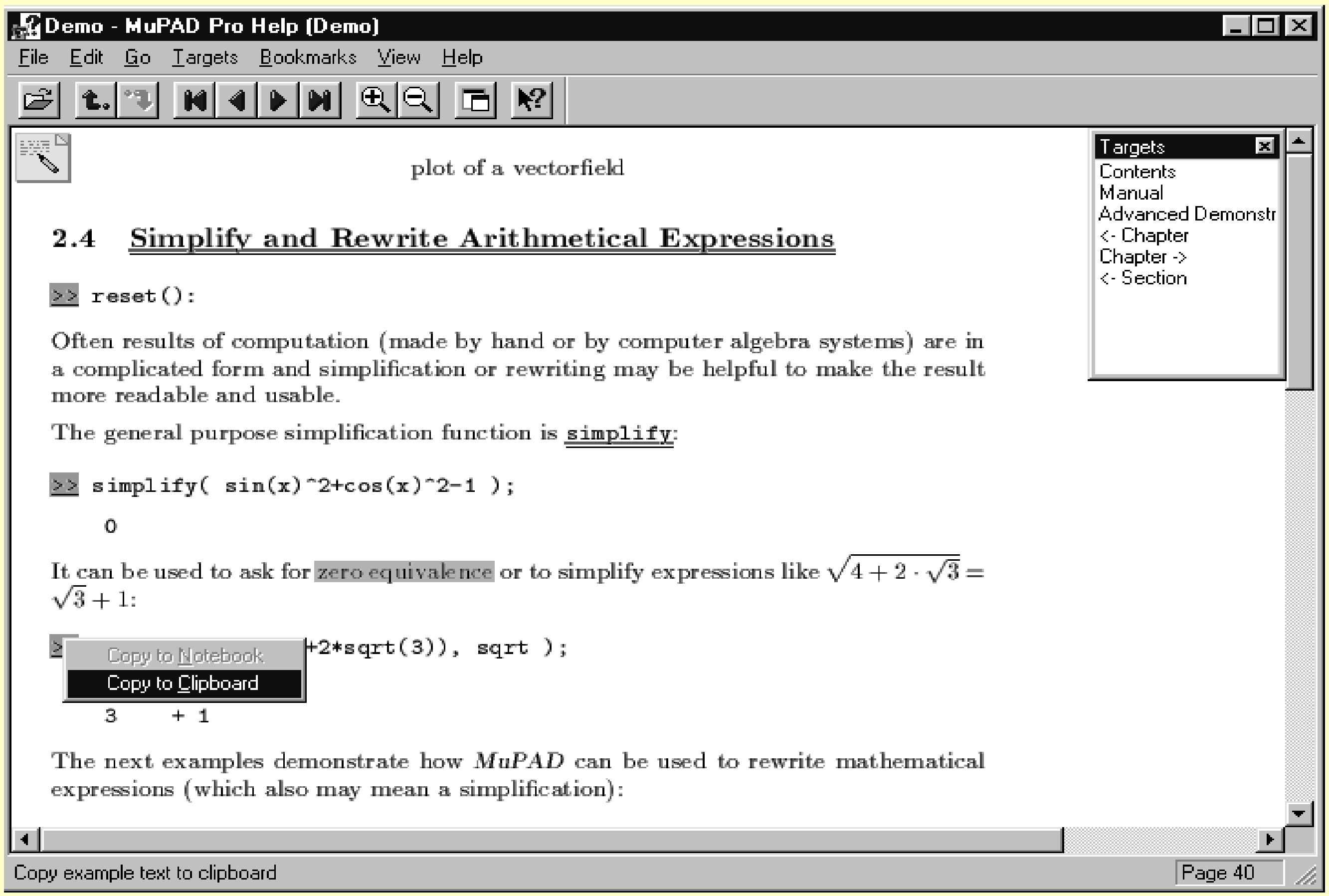
The system checks the adequacy of the command and the correctness of the result

- The student only gives a command for an action and
  the system checks the adequacy of the command and delivers the result.
- The system suggests an adequate command and the
  student has the choice between computing the result by himself or letting the system do it

This is part of a project of the MuPAD Group together with Cornelsen Soft, a project Frank Postel spoke about in more details in his presentation (see on this disc). Indeed it turned out, that the problems given with such functionality should not be underestimated. But that is no surprise: if a system allows a little bit for the freedom the student has it in classroom, then the system must be a complex one, side steps must be allowed, and so on, a very complex user model is to be implemented.
Furthermore, the design of such functionality is to be discussed. A consequence is that much didactical research is necessary about such issues - a statement for which a lot of my dear colleagues from pure math will try to kill me. But remember we have to develop something for the future, however also it must be something what teachers can use **now.** It has to have the essential features for the tomorrow’s use but it should be also well equipped for the bumpy roads now.
All these are interface issues, but the list is not complete. We need to embed that in multimedial environments, which to my opinion have to be structured along the lines of the paradigm of the classical book. Here I admit I am moving on thin ice, since even in my group this requirement is not completely adopted. But I firmly believe multimedia should not kill books (they may be more resistant than we might expect), but rather enlarge their functionality. Many believe that the linear order of knowledge and presentation should be given up in favor to almost no order at all represented by linking and clicking and trees. To my opinion all this introduces to much chaos and confusion. All that certainly is needed in multimedia embedding, but so often you will be lost, therefore you need some safe haven to take refuge, and this will be the extension of classical books to a multimedia environment. I give you a short glimpse on my favorite solution:

Sci Face Scientific Interfaces

MuPAD Pro 1.4



Here all the necessary multimedia features may be found, however it still is organized like a book. There is one important feature which none of the systems offers now:

object oriented output!

Take for example a system of linear differential equations. I want to see that as differential equations, however for the electrical engineer these just represent a bunch of circuits, magnets resistancies and so on. When he deals with differential equations he THINKS along these lines. So he should be able to program his favorite system in its user library such that when inputting differential equations he sees those circuits as output. I. e. when creating the objects of his intellectual desire he also has to be able to declare how they look on screen.

Apart from better, much better, interfaces we also need an increase of efficiency. In CAS a speedup of a factor 350 in the next 10 years seems possible to me (5 technical progress, 2, better data structures, 2 better algorithms, 2.5 multiprocessor machines, 7 Byte code compiler). This improvement of efficiency we need for building up on the system interface upon interface, library upon library and so on.
Such a system should allow efficient linkage to other components, systems and electronic universes, it has to be a real universal shell for doing math and natural sciences.

This universal shell then allows for a new use of math!

What really would be bad:

1. *Systems which shrink your freedom instead of expanding it*
2. *A system where you run linearly through prefabricated examples, this trains your students more in the sense of a Pawlow reflex*
3. *A system where you cannot explore areas of knowledge neither known to you nor to the system*
4. A system which forces its stupid paradigm on you, for example when you need an array it forces you to call that a vector, a system which does not allow you to add matrices by a plus sign, a system which when you invent something new it forbids you to use old notation.

We need exploration and creation of the unknown instead of tourism in the superficiality of the Internet world. To say the horrible, we do not need databases, but AI instead!

What are the possible effects of all that, of when we get what I ask for?

- **a.** The gap between the mathematics taught and the mathematics used will close.
- **b.** Technicalities will less overshadow the view on concepts.
- **c.** Free exploration of new territories will be possible
- **d.** We are given back the fun of doing math instead of memorizing it.
- **e.** We all will be doing more math than acquiring its formal knowledge and we will be getting rid of any axiomatic methods.

This last consequence is something even pure mathematicians ask for:

The formulations of axioms found in high school textbooks, being based on uncertain and imprecise customs of language and therefore unsuited for drawing unambiguous conclusions, is a failure. [Richard von Mises in TWM , p.1724]

Of course, by relying more on electronic systems than on our own mind, we loose something. We also loose skills which up to now we considered essential.

Is it really so bad what we loose?

Here one example: When I teach math to engineers or even more simple to economists, in examinations I usually let them demonstrate basic skills at simple objects, for example the Gauss algorithm on 2 by 2 matrices, or 3 by 3 matrices. Some years ago, one of my colleagues complained that by such ridiculous examinations the student are not able anymore to do the real problems, he instead required of them to apply the Gauss algorithm for 5 by 5 matrices. I am sure many of you agree with the critique from my colleague, even I was in doubt Then I asked in a large group of mathematicians: How many of you have in the last year inverted a 5 by 5, or higher order, matrix by use of the gauss algorithm by hand?
Not a single one had done that! Isn’t it, that we torture our students with skills even we do not apply?

What brings the future?

First a question: Why is math different from other subjects?

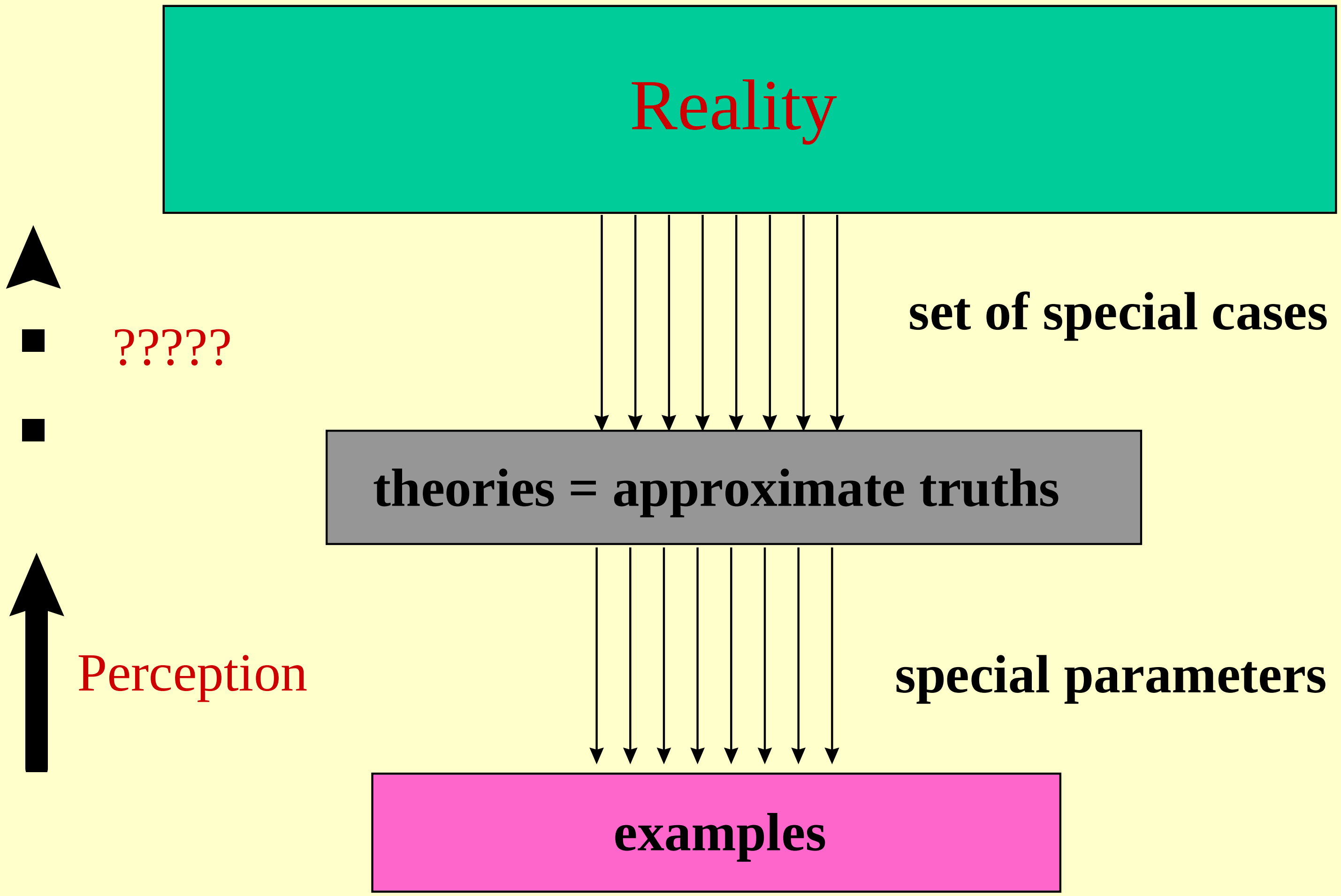
In other subjects you often have to do with *approximate truth*, this allows for a more intuitive approach and leads to a semi correct manipulation by using the wrong pictures and ideas. Nevertheless you may arrive by such doubtful procedure to the right conclusion. For this you may find many examples in, for example physics. The great work done by Newton's explanation of gravity in a rigorous sense of truth is obsolete since we know about general relativity. And general relativity may prove to be the wrong picture once we are convinced that a point n space-time is in reality a high-dimensional torus in a complicated universe, complicated beyond imagination compared to Einstein's general relativity. Sometimes you get even better results by the wrong explanations. So the results from the Ptolemean universe were still used when generally one already agreed on the model of Copernicus (which already was known to the Pythagorean).

This applies to all other subjects, Physics, philosophy, poetry and so on. Not knowing the truth, knowing that we never will know it, we nevertheless arrive at meaningful predictions.

So called 'nature', or at least what a physicists mean by that, turns out to be a man-designed apparatus, placed between reality and men. [Ortega y Gasset quoted according to [Bense vol. 2, p. 85].

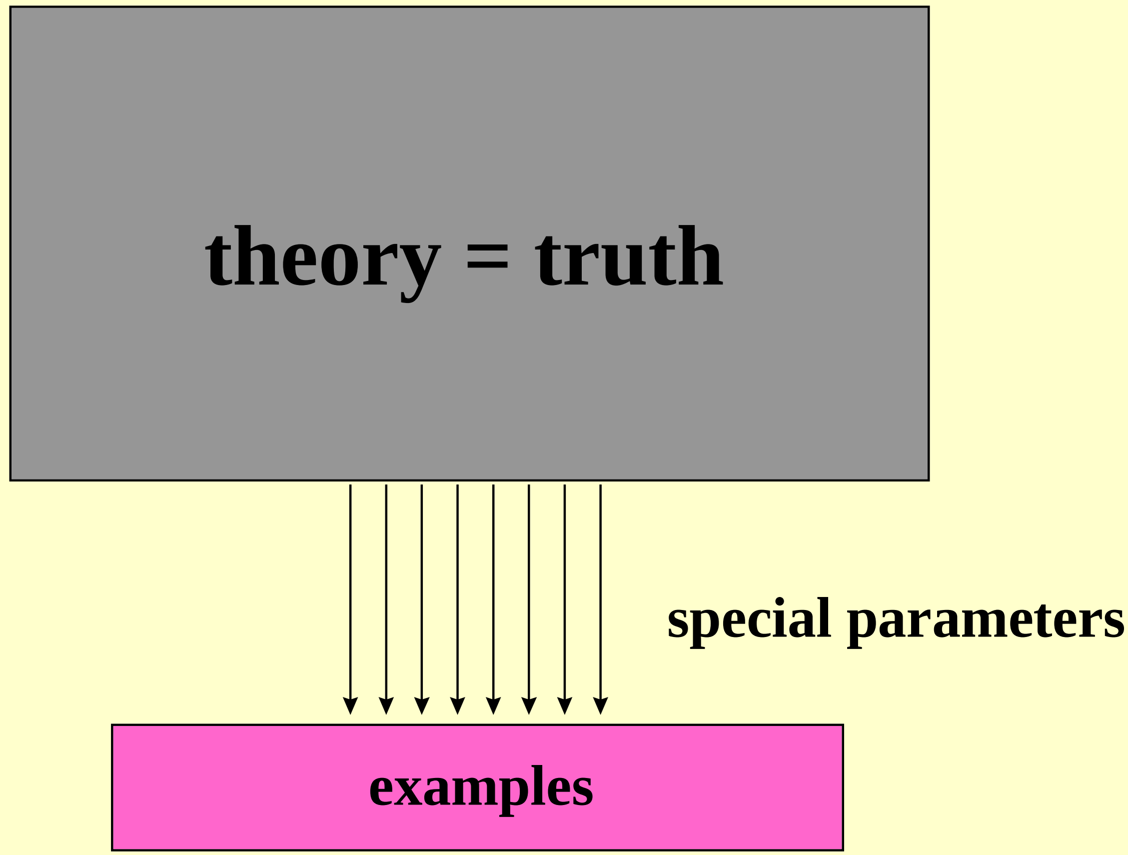
Modern physics does not deal at all with real things but rather with mathematical relations between certain pictures, which are the remainder of evaporated things. [Maybe Schopenhauer].

This impossibility of finding the truth however has one basic consequence, namely that since you have to provide your own picture anyway, there is only one way of finding it, namely by learning from example. This apparent shortcoming thus may be a methodical advantage.


Reality
set of special cases
?????
theories = approximate truths
Perception
special parameters
examples

Mathematics is different, the corresponding picture for mathematics would be:

Mathematics



- ?????
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This makes mathematics an *axiomatized deductive system* (Carl G. Hempel), which - principle - does not rely on learning by example.

Learning by example however is also done in math, but rather among them who invent math, instead of among those who apply math. Therefore we have a wider gap in math between these two groups, a gap wider than in all other subjects. Most of those who invent mathematics, and formal mathematics as well, may not even know the basic notions of their science, nevertheless they invent real deep mathematics. And a lot of what we teach, may not be known by all those who invent beautiful math.

In all sciences other than mathematics we solely learn from examples, we use the intuition developed by several examples to go back to the model, that is, we create the model as a substrata of the known examples, this very process is the essence of science. In all sciences other than mathematics a theory is a reduction of truth, the truth is hidden forever.

The basis of all perception and science is the unexplainable, to which all explanation leads. [SCH vol. 5, p. 9].

All other science, apart from logic which is mathematics, is empirical (even philosophy). Rigorously stated:

The theorems of logic and mathematics are tautological, but neither synthetic nor a prior. They are not synthetic, because they nothing say about reality, and not a priori because they do not come from a superempirical source but are the results of arbitrary definitions introduced by us. **[**Richard von Mises in TWM, p. 1734].

In math, when we learn from example, we want to go back all the way to the truth, which is only possible because we can create the ‘real life of math’ on our own, we create the truth. Our mind, when creating mathematics, does a map from examples to truth by creating the truth.

Opinions about the empirical character of math however are not unique:

*Mathematics in itself is an empirical science, the most general one***,** John Stuart Mill (1806 - 1873)

So the question is:

Can we do that by computer?

Mill may not be right but we may see more of that understanding in future applications. For example, in MuPAD we may write a function, which,

i. if you give her one suitable example of a matrix multiplication then it automatically creates the general function of matrix multiplication
ii. if you give her one suitable example for a Taylor series, then it automatically writes the general procedure for Taylor series
iii. if you give her one suitable example for multiplication in remainder classes, then it creates that as a general function, and so on.

How is it done? Mill explained the mechanism clearly, by stating that the only admissible method in science is by deducing the general from the special. Technically that means we have to map special values onto general variables and then we replace the special operations done with the special values by what we consider meaningful (or simple) general operations. Speaking in computers, by that we construct a map from some pairs (input, output) to the program which may have generated the output, i.e. we reverse the map given by a method or program. Since that map is not invertible, at least not in practical terms when not all pairs are given, we only can arrive at an approximate truth, which in an essential way depends on what we consider to be meaningful. If we can create such functionality on a more sophisticated level, then our computers can create math, then our computers may provide the user with methods to replace their intuition. However serious problems have to solved or least to be considered:

> **How, if there is no obvious choice, do we relate the pieces of output to the input variables (that is where 'simplicity' comes in)?**

We have,

> **to find hierarchies for mathematical simplicity, we have to struggle with randomness**

Not an unusual thought, unusual only if applied to the creation of math. Norbert Wiener for example advocated the advantage of randomness in learning. In learning of math we are not used to it, because we falsely believe in the unity of relevant mathematics for the explanation of algorithmic computations.

I believe problems related to such questions will be of unexpected urgency in the future of mathematics and its teaching. We shall arrive sooner than we believe at a point where computers create math. This is not so surprising because math, in all its apparent complexity is so much simpler than the real things of live., and this simplicity will help us to transfer some burden of the constant necessity of creating mathematical structures to computer (we should not be afraid of that, more mathematics than we ever can do will remain for the mind of those who love that science).

All this will have deep and everlasting effects on the teaching of mathematics. However in all our research about teaching and the role of modern technology related to it we should not forget

- As it is said that a good cook can even transform old shoes into delicacies, a good teacher can lead his students to the most pleasant perception and experience even on the issues dry-as-dust.

Free translation from Schopenhauer

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