Appl. Math. Lett. Vol. 10, No. 2, pp. 57–61, 1997

0893-9659/97 $7.00 + 0.00

Flows in Infinite Networks Represented by Vector Lattices

B. FUCHSSTEINER AND K. MORISSE
University of Paderborn, Paderborn, Germany

(Received May 1996; accepted August 1996)

Abstract—We are concerned with minimal cost flows in infinite networks. As an application of a Hahn-Banach type monotone extension theorem for convex cones, an abstract theorem characterizing minimal cost flows by local price systems is obtained for a general vector lattice situation. This result extends the usual finite network result, or rather its extension to some $L^1 - L^\infty$ situation, which states that the minimal transportation cost is the supremum (taken over all local price systems) of the difference between consumption cost and transportation profit.

Keywords—Extension theorem, Minimal cost flow problem, Infinite networks.

1. INTRODUCTION

We focus on the classical minimal cost flow problem (MCFP), which is to determine a minimum cost shipment of a commodity through a network such that it satisfies demand at certain nodes by use of available supplies at other nodes. In a previous paper [1], this problem is treated for infinite networks in a measure theoretic framework where flows are L^∞-functions and minimal cost flows are characterized by local price systems.

In the present paper, we obtain a considerably generalization concerning the existence of flows in infinite networks. In order to prove such a general minimal cost flow theorem in the situation of abstract vector lattices, we have to make use, as in [2–4], of a Hahn-Banach type extension theorem for cones endowed with a collection of order relations. We treat networks in a rather abstract setting, so that we can omit all measure theoretic arguments, thus making the essential mathematical structure of the problem more transparent. It turns out that tools like desintegration theorems or the Radon-Nikodym theorem can be completely avoided by this approach.

The organization of this paper is as follows. To make the paper self-contained, we first collect some principal tools from the theory of convex cones in Section 2. In Section 3, we describe the scenario for our main theorem and state the theorem itself with its proof. Finally, in Section 4, we deduce the L^1-L^∞ formulation of the flow problem from [1] as an application.

2. TOOLS

We list here the necessary technical results. For most of these, the proof follows from the general theory of convex cones (for a reference, see [5]).

Let F be a preordered convex cone, i.e., an additive semigroup provided with a multiplication with nonnegative scalars and a compatible preorder. All functionals on F are assumed to map to the reals extended by $-\infty$. We define for a linear ω on a subcone G of F and a monotone and sublinear π on F with $\omega \leq_G \pi$, (i.e., $\omega \leq \pi$ on G) the set

$$\mathrm{Lin}_{G<F}(\omega, \pi) := \{\nu \mid \nu \text{ monotone and linear with } \omega \leq_G \nu \text{ and } \nu \leq_F \pi\}.$$

Typeset by $\mathcal{AMS}$-TEX

THEOREM 2.1. (DOMINATED EXTENSION THEOREM).

(1) *Let G be a subcone of F, let ω be linear on G and π monotone and sublinear on F with $\omega \leq \pi$ on G. Then there is a monotone linear μ on F with $\omega \leq_G \mu$ and $\mu \leq_F \pi$, i.e., we have* $\mathrm{Lin}_{G<F}(\omega, \pi) \neq \emptyset$.

(2) *For an arbitrary fixed $f \in \mathcal{F}$, the functional μ can be chosen such that $\mu(f)$ is minimal, i.e.,*

$$\mu(f) = \inf \{\nu(f) \mid \nu \in \mathrm{Lin}_{G<F}(\omega, \pi)\}. \tag{1}$$

The first part of the theorem is a generalization of the classical Hahn-Banach theorem. The second part is obtained by constructing first from $\mathrm{Lin}_{G<F}(\omega, \pi)$ a suitable sublinear functional $\pi^{\vee} \geq_G \omega$, having on f as value the minimum which f attains on $\mathrm{Lin}_{G<F}(\omega, \pi)$. Thereafter, we apply the first part of the theorem to ω and $\pi^{\vee}$.

For the characterization of a minimal cost flow we need a technical lemma, which determines the minimal value of a linear functional for some fixed element.

LEMMA 2.1. *Let G be a subcone of the preordered cone F, let μ be linear on G and π monotone and sublinear on F with $\mu \leq_G \pi$ and consider the following sublinear functional $\omega(f) := \sup\{\mu(g) - \pi(h) \mid g \prec f + h,\ g \in G,\ h \in F\}$. For those $f \in F$ with $\omega(f) \neq -\infty$, we then have*

$$\omega(f) = \inf \{\nu(f) \mid \nu \in \mathrm{Lin}_{G<F}(\mu, \pi)\}. \tag{2}$$

PROOF. (See [6] for details). From the assumptions, it is easily seen that $\omega(f)$ is less than or equal to the right-hand side. Now we fix some $f \in F$ and define a monotone sublinear functional $\leq \pi$ by $\rho(h) := \inf\{\lambda\omega(f) + \pi(g) \mid h \prec \lambda f + g,\ g \in F,\ \lambda \geq 0\}$. Because of $\mu \leq \rho$ on G, we use now the above theorem (in particular, 2) to obtain the other inequality.

3. THE MAIN THEOREM

Let Ω be a nonempty set and E a real vector lattice of functions on $\Omega \times \Omega$. For functions f on Ω, we denote by $\otimes$ the usual tensor product $(f \otimes g)(\omega_1, \omega_2) := f(\omega_1)g(\omega_2)$. On E, a linear functional $\mathcal{J}$ is given which is monotone with respect to the pointwise order. Furthermore, we have a space E_Ω of functions on Ω with the following properties.

C1. If $f \in E_\Omega$, then $f \otimes 1$ and $1 \otimes f$ are elements of E.

C2. For $g \in E$, the diagonal element g_D, given by $g_D(\omega) := g(\omega, \omega)$, is an element of E_Ω.

C3. For $g \in E$ and $\tilde{\omega} \in \Omega$, the function $\omega \in \Omega \mapsto \hat{g}_{\tilde{\omega}}(\omega) := g(\tilde{\omega}, \omega)$ is an element of E_Ω.

We denote by E_Ω^+ and E^+ the positive cones of E_Ω and E. We call $\Omega \times \Omega$ a *network* with *capacity* $\mathcal{J}$. A linear μ on E_Ω is called a *demand*. A *flow* is a monotone linear ν on E. We say that a flow is *admissible*, if $\nu \leq \mathcal{J}$ on E^+. Given a demand μ, then a flow is said to satisfy this demand, if $\mu(f) \leq \nu(1 \otimes f - f \otimes 1)$ holds for all $f \in E_\Omega^+$. As the main theorem, we obtain the following.

THEOREM 3.1. ABSTRACT MINIMAL COST FLOW THEOREM. *Let γ be an element of E with $\gamma_D = 0$, and μ a linear functional on E_Ω.*

(1) *There is an admissible flow ν which satisfies the demand μ if and only if for all $f \in E_\Omega^+$, we have*

$$\mu(f) \leq \mathcal{J}(\max(0, 1 \otimes f - f \otimes 1)). \tag{3}$$

(2) *If there is an admissible flow satisfying the demand μ, then the flow can be taken such that*

$$\nu(\gamma) = \sup \left\{\mu(f) - \mathcal{J}(\max(0, 1 \otimes f - f \otimes 1 - \gamma)) \mid f \in E_\Omega^+\right\}. \tag{4}$$

3.1. Technical Tools

Before we start to prove the theorem, we have to define some mappings, orders, and spaces. In E_Ω, we define a collection $\{\prec_{(\omega_1,\omega_2)} \mid \omega_1, \omega_2 \in \Omega\}$ of order relations by $f \prec_{(\omega_1,\omega_2)} g$ iff $f(\omega_2) \leq g(\omega_2)$ and $f(\omega_1) \geq g(\omega_1)$. For arbitrary $g \in E$, we define a sublinear operator $\mathcal{E}_g : E_\Omega \to E$ by

$$\mathcal{E}_g(f) := \max(0, 1 \otimes f - f \otimes 1 - g). \tag{5}$$

$\mathcal{E}_0$ denotes $\mathcal{E}_g$ for $g = 0$. If $\varphi : \Omega \times \Omega \to E_\Omega$, then a function $P(\varphi) : \Omega \times \Omega \to \mathbb{R}$ is defined by $P(\varphi)(\omega_1, \omega_2) := \mathcal{E}_0(\varphi(\omega_1, \omega_2))(\omega_1, \omega_2)$, $\forall\, \omega_1, \omega_2 \in \Omega$. Let Φ be those mappings $\varphi : \Omega \times \Omega \to E_\Omega$, such that the functions $(\omega_1, \omega_2) \in \Omega \times \Omega \mapsto \varphi_L(\omega_1, \omega_2) := (\varphi(\omega_1, \omega_2))(\omega_1)$ and $(\omega_1, \omega_2) \in \Omega \times \Omega \mapsto \varphi_R(\omega_1, \omega_2) := (\varphi(\omega_1, \omega_2))(\omega_2)$ are elements of E. Since E is a vector lattice, we have $P(\Phi) \subseteq E$. On Φ, we define $\pi(\varphi) := \mathcal{J}(P(\varphi))$ for all $\varphi \in \Phi$. Moreover, we endow Φ with a preorder $\prec_\Phi$, defined by $\varphi_1 \prec_\Phi \varphi_2 \iff \varphi_1(\omega_1, \omega_2) \prec_{(\omega_1,\omega_2)} \varphi_2(\omega_1, \omega_2)$ for all $\omega_1, \omega_2 \in \Omega$. Finally, we consider a mapping $L : E \to (E_\Omega)^{\Omega \times \Omega}$ given by $L(g)(\omega_1, \omega_2)(\omega) := g(\omega_1, \omega)$ for all $g \in E$, $(\omega_1, \omega_2) \in \Omega \times \Omega$, $\omega \in \Omega$. Because of C1 and C2, we have $L(E) \subseteq \Phi$.

LEMMA 3.1. *Φ is a vector space. P and $\pi : \Phi \to E$ are sublinear and monotone with respect to $\prec_\Phi$.*

PROOF. One has to observe that $P(\varphi) = \max(0, \varphi_R - \varphi_L)$. Since $\varphi \to \varphi_R$, $\varphi \to \varphi_L$, and $\varphi \to 0$ are linear, P must be sublinear. One easily sees that $\varphi \prec_\Phi \tilde{\varphi}$ induce $\varphi_R \leq \tilde{\varphi}_R$ and $\varphi_L \geq \tilde{\varphi}_L$. Using this in $P(\varphi) = \max(0, \varphi_R - \varphi_L)$, one has the monotony of P.

That π is monotone is an immediate consequence of the monotony of P and that of $\mathcal{J}$. The sublinearity of π follows from the linearity of $\mathcal{J}$ and the obvious sublinearity of P. ∎

For the following, we consider the elements of E_Ω as constant functions $\Omega \times \Omega \to E_\Omega$, hence, E_Ω is regarded as a subspace of Φ.

LEMMA 3.2. *For $f \in E_\Omega$ and $g \in E$ with $g(\omega, \omega) = 0$ for all $\omega \in \Omega$, we have $P(f - L(g)) = \mathcal{E}_g(f)$.*

PROOF. Since $g(\omega_1, \omega_1) = 0$, we get $P(f - L(g))(\omega_1, \omega_2) = \max(0, f(\omega_2) - f(\omega_1) - g(\omega_1, \omega_2) + g(\omega_1, \omega_1)) = \max(0, f(\omega_2) - f(\omega_1) - g(\omega_1, \omega_2)) = \mathcal{E}_g(f)(\omega_1, \omega_2)$. ∎

3.2. Proof of the Main Theorem

By the definitions and by monotony, it follows that if there is some admissible flow which satisfies demand μ, then (3) must hold. Therefore, we only have to show that the existence of a flow for which (2) holds can be deduced from (3).

As before, we regard E_Ω^+ as a subcone of Φ. Condition (3) then can be rewritten as $\mu \leq \pi$ on E_Ω^+, because $\pi(f) = \mathcal{J}(P(f)) = \mathcal{J}(\mathcal{E}_0(f)(\omega_1, \omega_2)) = \mathcal{J}(\max(0, 1 \otimes f - f \otimes 1))$ holds for $f \in E_\Omega^+$. By the extension Theorem 2.1, together with Lemma 2.1, there is a linear monotone $\hat{\nu}$ on Φ with $\mu \leq \hat{\nu}$ on E_Ω^+, and $\hat{\nu} \leq \pi$ on Φ, such that

$$\hat{\nu}(L(\gamma)) = \sup\left\{\mu(f) - \pi(h) \mid f \prec_\Phi L(\gamma) + h,\ f \in E_\Omega^+,\ h \in \Phi\right\}. \tag{6}$$

We define a functional ν on E $\nu := \hat{\nu} \circ L$ and show that this yields a flow which is admissible and satisfies the demand.

We consider $g \in E^+$, then using $\nu := \hat{\nu} \circ L$, $\hat{\nu} \leq \pi$ on Φ, and the definition of π, we obtain

$$\nu(g) = \hat{\nu}(L(g)) \leq \pi(L(g)) = \mathcal{J}(P(L(g))), \tag{7}$$

and because of the positivity of g, we have

$$\begin{aligned} P(L(g))(\omega_1, \omega_2) &= \mathcal{E}_0(L(g)(\omega_1, \omega_2))(\omega_1, \omega_2) \\ &= \max(0, (L(g)(\omega_1, \omega_2))(\omega_2) - (L(g)(\omega_1, \omega_2))(\omega_1)) \\ &= \max(0, g(\omega_1, \omega_2) - g(\omega_1, \omega_1)) \leq \max(0, g(\omega_1, \omega_2)) = g(\omega_1, \omega_2). \end{aligned} \tag{8}$$

Hence, from (7) and (8), we get $\nu(g) \leq \mathcal{J}(g)$ for all $g \in E^+$. Therefore, ν is admissible.

Consider $f \in E_\Omega^+$. Since $P(L(f\otimes 1))(\omega_1,\omega_2) = \mathcal{E}_0(L(f\otimes 1)(\omega_1,\omega_2))(\omega_1,\omega_2) = \max(0,(L(f\otimes 1)(\omega_1,\omega_2))(\omega_2) - (L(f\otimes 1)(\omega_1,\omega_2))(\omega_1)) = \max(0,(f\otimes 1)(\omega_1,\omega_2) - (f\otimes 1)(\omega_1,\omega_1)) = \max(0, f(\omega_1) - f(\omega_1)) = 0$, we can deduce $\nu(f\otimes 1) = \hat{\nu}(L(f\otimes 1)) \le \pi(L(f\otimes 1)) \le \mathcal{J}(P(L(f\otimes 1))) \le 0$. Together with $\mu \le \hat{\nu}$ on E_Ω^+ and the fact that $L(1\otimes f) = f$, we have $\nu(1\otimes f - f\otimes 1) \ge \nu(1\otimes f) = \hat{\nu}(L(1\otimes f)) = \hat{\nu}(f) \ge \mu(f)$. Therefore, the flow satisfies the demand μ.

For the completion of the proof, we have to prove (4). The left-hand side of (6) is equal to $\nu(\gamma)$. Since $h := f - L(\gamma)$ is the minimal element of those $\tilde{h}$ satisfying $f \prec_\Phi L(\gamma) + \tilde{h}$, the right-hand side of equation (6) is equal to $r := \sup\{\mu(f) - \pi(f - L(\gamma)) \mid f \in E_\Omega^+\}$. Now, using Lemma 2.1 and (5) gives the desired estimate $\nu(\gamma) = r = \sup\{\mu(f) - \mathcal{J}(\mathcal{E}_\gamma(f)) \mid f \in E_\Omega^+\}$. ∎

4. AN APPLICATION

In [1], as a direct generalization of the minimal cost flow problem for finite networks, the minimal cost flow problem for infinite networks was considered in a measure theoretic framework. Let us briefly recall the situation.

Consider a measurable space (Ω, Σ) with a nonempty set Ω of consumers and a suitable σ-algebra Σ on Ω. The set $\Omega \times \Omega$ is endowed with the product σ-algebra $\Sigma \otimes \Sigma$. Furthermore, the following measures are given: a signed measure μ on Σ measuring the demand, a σ-finite measure τ on $\Sigma\otimes\Sigma$ representing the capacity, and the cost given by a $\mathcal{L}^1(\tau)$-function γ with $\gamma(\omega,\omega) = 0$ almost everywhere on Ω. A flow ν is defined to be a $\mathcal{L}^\infty(\tau)$-function with $\nu \le 1$, $\int_{A\times B} \nu\, d\tau \ge 0$ for disjoint measurable sets A, B, such that for all $A \in \Sigma$, we have $\mu(A) \le \int_{(\Omega\setminus A)\times A} \nu\, d\tau - \int_{A\times(\Omega\setminus A)} \nu\, d\tau$. The cost of a flow is $\Gamma(\gamma,\nu) := \int \gamma\nu\, d\tau$. Here we used $\mathcal{L}^\infty(\tau), \mathcal{L}^1(\tau)$ for the actual function spaces and $L^\infty(\tau), L^1(\tau)$ for their *almost-everywhere* equivalence classes.

For a suitable characterization of flows with minimal cost one introduces *local price systems*. These are positive measurable functions on Ω which are integrable with respect to both marginal measures of τ. Introducing the notion of *consumption cost* under a local price system $C(f) := \int f\, d\mu$ and the *optimal transport profit* as $\rho(f,\gamma) := \int \max(0, f(\omega_2) - f(\omega_1) - \gamma(\omega_1,\omega_2))\, d\tau$, one obtains the following result (see [1]).

THEOREM ON L^∞ MINIMAL COST FLOW UNDER L^1 COST.

(1) *There is a flow iff* $\mu(A) \le \tau((\Omega\setminus A)\times A)$ *for all* $A \in \Sigma$.

(2) *If a flow exists, then there is a flow* ν *such that the transport cost are equal to the supremum (over all local price systems) of the difference between consumption cost and optimal transport profit, i.e.,*

$$\Gamma(\gamma,\nu) = \sup\{C(f) - \rho(f,\gamma) \mid f \textit{ local price system}\}.$$

Necessarily ν *is a flow with minimal cost.*

We can deduce this theorem easily from our main theorem: let E be the space of absolutely τ-integrable functions, i.e., $\mathcal{L}^1(\tau)$, and let E_Ω be the space $\mathcal{L}^1(\tau_1) \cap \mathcal{L}^1(\tau_2)$, where τ_1, τ_2 are the marginal measures of τ. Finally, define $\mathcal{J}(h) := \int h\, d\tau$ for all $h \in E$.

Obviously, E is a vector lattice and $\mathcal{J}$ has the required properties of linearity and monotony. One easily verifies that E_Ω has the desired features. Therefore, our main theorem in this scenario now reads as the theorem above.

REFERENCES

1. B. Fuchssteiner and K. Morisse, Infinite networks: Minimal cost flows, *Appl. Math. Lett.* **9** (1), 89–93 (1996).
2. B. Fuchssteiner, An abstract disintegration theorem, *Pacific Journal of Mathematics* **94** (2), 303–309 (1981).
3. H. König and M. Neumann, *Mathematische Wirtschaftstheorie (mit einer Einführung in die konvexe Analysis)*, Volume 100 of *Mathematical Systems in Economics*, Verlag Anton Hain, Königstein/Ts., (1986).

4. M. Neumann, The theorem of Gale for infinite networks and applications, In *Infinite Programming, Proceedings of an International Symposium on Infinite Dimensional Linear Programming*, September 1984, (Edited by E.J. Anderson and A.B. Philpott), Volume 259 of *Lecture Notes in Economics and Mathematical Systems*, pp. 154–171, Springer-Verlag, Berlin.
5. B. Fuchssteiner and W. Lusky, Convex Cones, In *Mathematics Studies*, Volume 56, Elsevier Science Publishers B.V. (North-Holland), Amsterdam, (1981).
6. K. Morisse, Kostenminimale Flüsse in endlichen und unendlichen Netzwerken, Dissertation, Universität Paderborn, (1995).
7. B. Fuchssteiner, Disintegration methods in mathematical economics, In *Game Theory and Mathematical Economics*, (Edited by O. Moeschlin and D. Pallaschke), pp. 193–204, Elsevier Science Publishers B.V. (North-Holland), Amsterdam, (1981).