Physica A, 228, 1996 p.189-211

The interacton equation

Benno Fuchssteiner

Automath Institute
University of Paderborn
Paderborn, Germany

Dedicated to Hans Capel on the occasion of his 60th birthday

Abstract

Given the Lie algebra $\mathcal{L}_{fp}$ of vector fields of a free particle then in a joint field of interacting particles an *interacton* is defined as a direct summand isomorphic to $\mathcal{L}_{fp}$. For several soliton equations it is shown how interactons can be obtained from various methods of symmetry analysis and how their dynamics can be expressed in terms of self-interaction alone; this procedure leads to new integrable systems. Furthermore, in case of nonlinear Schrödinger equations it is shown how the dynamics of interactons can be formulated in terms of self-consistent potentials with antlinear parts.

1 Introduction

First we demonstrate that there is still some need to clarify the notion of *particle* in an interacting field.

We consider a number of equal particles in $\mathbb{R}^3$ with mass 1, say r particles acting on each other by a force coming from a joint potential V. The potential is assumed to depend additively on the differences between the positions. We assume that asymptotically these r particles are free. That means that each particle, for time t going to infinity, moves along a line $\vec{\rho}(t) = \vec{q} + \vec{k} * t$ where $\vec{q}$ is a phase and $\vec{k}$ the asymptotic speed. The total energy of this system of r particles is $\mathcal{E} = \mathcal{E}_{kin} + \mathcal{E}_{pot}$ with kinetic and potential energy

$$\mathcal{E}_{kin} := \frac{1}{2}\sum_i \vec{v}_i(t)^2, \quad \vec{v}_i(t) := \vec{\rho}_i(t)_t$$

$$\mathcal{E}_{pot} := \frac{1}{2}\sum_{i,j} V(\vec{\rho}_i(t) - \vec{\rho}_j(t)) \ .$$

By the principle of least action the equation of motion in phase space is found to be

$$\frac{d}{dt}\begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \\ \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_r \end{pmatrix} = \begin{pmatrix} 0 & -I \\ +I & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \\ \nabla_{\vec{\rho}_1}\mathcal{E}_{pot} \\ \vdots \\ \nabla_{\vec{\rho}_r}\mathcal{E}_{pot} \end{pmatrix} \tag{1.1}$$

where $\nabla_{\vec{\rho}_i}\mathcal{E}_{pot}$ denotes the gradient with respect to $\vec{\rho}_i$ and where I is the unit matrix in $\mathbb{R}^{3^r}$.

Usually the mass concentrated at the point $\rho_i(t)$ is considered to be the i-th particle, and $\rho_i(t)$ is considered to be its orbit. Since we are going to shed some doubt on this notion of a *particle* we call it for the moment the *phenomenological particle.* Sometimes however, there is no doubt that we really have to do with particles. For example, in the following system where no interaction takes place

$$\frac{d}{dt}\begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \\ \vec{\rho}_1 \\ \vdots \\ \vec{\rho}_r \end{pmatrix} = \begin{pmatrix} 0 & -I \\ +I & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.2}$$

We agree on the principle:
wether or not a particle is present in a flow should not depend on the parametrization on the manifold on which the flow takes place. This requirement of *differential geometric invariance* is a principle which we cherish in honour to Albert Einstein.

However, the notion of a particle as given above for the phenomenological particle is not independent of the parametrization of the manifold. To see this let us do the following gedankenexperiment: We want to reparametrize $\mathbb{R}^{2m}, m := r * 3$. Consider therefore $2r$ points $\vec{x_1}, \cdots, \vec{x_r}, \vec{y_1}, \cdots, \vec{y_r}$ in $\mathbb{R}^3$ as elements of $2m$-dimensional space. In order to find a new parametrization connected to our dynamical system (1.1) we consider this point in $\mathbb{R}^{2m}$ to be the set of initial positions and speeds of the r-particle system at time $t = 0$. In our gedankenexperiment we let this system evolve according to the dynamic (1.1). Then we measure its asymptotical phases and speeds $\vec{q_1}, \cdots, \vec{q_r}, \vec{k_1}, \cdots, \vec{k_r}$. The set of these $2r$ vectors we consider as the new coordinate of the original point $\vec{x_1}, \cdots, \vec{x_r}, \vec{y_1}, \cdots, \vec{y_r}$. Thus we have found a new parametrization of $\mathbb{R}^{2m}$.

Independent of the fact that it may be difficult to compute the diffimorphism necessary for effectively obtaining our new parametrization of the manifold, we can easily express our r-particle flow in these new parameters. To do so, we consider the flow with asymptotic speeds $\vec{k_1}, \cdots, \vec{k_r}$ and phases $\vec{q_1}, \cdots, \vec{q_r}$ at time $t = 0$. Then obviously the initial state expressed in the new parameters has the values given by these speeds and phases. Now consider the positions and speeds of the same system at time $t = t_0$. For finding the new coordinates we have to take the positions and speeds at that time as a new initial state for the evolution of the system according to the same dynamic. Obviously the speeds we obtain by this gedankenexperiment are the same as before and the phases are just shifted by $\vec{k}_i t_0$. So the orbit which our r-particle system runs through in time with respect to the new parameters is given by

$$\vec{q_i}(t) = \vec{q_i}(0) + \vec{k}_i t \quad \text{and} \quad \vec{k}_i(t) = \vec{k}_i(0) \ .$$

Hence the flow (1.1) now has the following simple form:

$$\frac{d}{dt}\begin{pmatrix}\vec{k}_1\\ \vdots\\ \vec{k}_r\\ \vec{q}_1\\ \vdots\\ \vec{q}_r\end{pmatrix} = \begin{pmatrix}0 & -I\\ +I & 0\end{pmatrix}\begin{pmatrix}\vec{k}_1\\ \vdots\\ \vec{k}_r\\ 0\\ \vdots\\ 0\end{pmatrix} \tag{1.3}$$

However, up to a renaming of variables, this is the same flow as in (1.2). Hence, if there we were right in identifying the particles, and if the notion is a differential geometric invariant notion[1], then here also the particle must be given by the $\vec{k}_i$ and the $\vec{q}_i$. Then in original parameters the particles are obtained by going back with the diffeomorphism (given by the change in parameters) to the original manifold. And surprisingly, this is not the same as the quantity which we have termed phenomenological particle.

The problem remains to determine what we really mean by *particle*. Certainly, an important quantity like the notion of a particle in an interacting field has to be independent of time and asymptotical considerations and therefore should be found from the dynamics of the joint field by a group theoretical reduction. Indeed, the particles which we have identified in the example above by reparametrization do fulfill this requirement because there the i-th particle can be obtained in connection with the directional derivative of the flow with respect to the i-th phase and since the speed k_i is a conserved quantity this corresponds to a group theoretical reduction.

The formalization of this principle for continuous fields will be one of the aims of this paper. Furthermore, the resulting notions will be studied in context of soliton equations, which turn out to have the property that the dynamics of the resulting particles can be expressed in terms of self-interaction. As a consequence it turns out that with intergrable equations there are canonically associated other integrable equations which describe the particles in terms of self-interaction, these equations are called *interacton equations*, their solutions are the *interactons* of the corresponding field. They will be explicitely determined in case of recursion operators or auto-Bäcklund transformations. Furthermore, in the general case, the connection of interactons with self-consistent potentials is pointed out.

2 The free and the interacting particle

Surprisingly, not much attention has been directed to the question how to recognize individual particles in an interacting field, particles are mostly characterized as asymptotic phenomana. Of course, the reason for this is that for most parts of physics, this is the only relevant question.

We shall formalize the notion of particle in the situation of flows on general manifolds. To make things simple, in the following all tensors and manifolds are assumed to be C^∞, the dimension of a manifold is either assumed to be finite and even or to be infinite.

Let M be a symplectic manifold, denote its arbitrary point by u and let $G(u)$ be a hamiltonian vector field on M. We consider the flow

$$u_t = G(u) \tag{2.1}$$

[1]Since (1.3) obviously is an integrable flow it looks as if we had proven that all N-body problems are integrable. However, since the new parameters are not necessarily obtained from the old parameters by quadratures we are far away from having proven such an erroneous statement. Indeed our requirement of differential geometric invariance is far more than what we usually rquire, for example, in general relativity. Here, we not only required differential geometric invariance on the physical space-time manifold but also on the manifold of all solutions for the dynamics under consideration.

Its solutions we denote by $u(t)$. For simplicity, we assume that $G(u)$ is such that the map

$$u(0) \stackrel{R(t)}{\longrightarrow} u(t) \tag{2.2}$$

is a C^∞-semigroup of diffeomorphisms on M. Particles in the joint field u are assumed to have n degrees of freedom.

2.1 Free particles

Free particles, which shall be special solutions of (2.1), are assumed to be characterized by n action and n angle variables (see [2] for action-angle variable coordination) . The manifold of all possible states of a free particle therefore is a $2n$-dimensional invariant manifold on which the restriction of the flow is an integrable hamiltonian flow. To see this:
Consider the angle variables and denote by $\vec{k} = (k_1, \ldots, k_n)$ their time derivatives, which have to be constant by definition of the notion of angle variables.

By $T_1(u), \cdots T_n(u)$ we denote the vector fields given by taking on the special invariant manifold the derivative of u with respect to these angle variables. The n vector fields $T_i(u)$, on the special manifold under consideration, mutually commute, and they all commute with $G(u)$. Furthermore, the $T_1, \ldots, T_n$ are linear independent in each point on the manifold. We write

$$T_{\vec{k}}(u) = \sum k_i T_i(u) \ . \tag{2.3}$$

then $u_t = T_{\vec{k}}(u)$ and for each $\vec{k}$ the submanifold

$$M_{\vec{k}} = \{u \in M \mid G(u) = T_{\vec{k}}(u)\}.$$

is invariant with respect to (2.1). The union

$$M_{fp} = \bigcup \{M_{\vec{k}} | \vec{k} \in \mathbb{R}^n\}$$

we call the *one-particle manifold*. The $M_{\vec{k}}$ are the fibres of M_{fp} and the index $\vec{k}$ is said to be the *abstract momentum* of the particle and $1/2 \sum k_i^2$ is called its *abstract energy*. The maps

$$v(0) \stackrel{R_{\vec{k}}(t)}{\rightarrow} v(t) \tag{2.4}$$

associated with the solutions of

$$v_t = T_{\vec{k}}(v) \tag{2.5}$$

do form a C^∞-group of diffeomorphisms on M_{fp}. The map $R_{\vec{k}}(1)$ at $t = 1$ we call the *translation* associated to $\vec{k}$.

For convenience, we use a C^∞-section S_{fp} in M_{fp} which, for each $\vec{k} \in \mathbb{R}^n$ exactly has one point in common with $M_{\vec{k}}$. This point we denote by $S_{\vec{k}}$ and call it the *origin* of the fibre $M_{\vec{k}}$.

In order to see that our notions are well chosen, we consider the non-compact case, i.e. the case that each $M_{\vec{k}}$ is simply connected. Then M_{fp} has a canonical representation since each point on any fibre $M_{\vec{k}}$ can be reached from the origin by a unique translation because of

$$M_{fp} = \bigcup \{R_{\vec{q}}(1) S_{fp} \mid \vec{q} \in \mathbb{R}^n\}. \tag{2.6}$$

This is easy to see: The fibres are invariant under all translations and there are n independent translations on each n-dimensional fibre, hence the translations have to span the fibres. So each $s \in M_{fp}$ is characterized by its momentum $\vec{k}$ and its *phase* $\vec{q}$ defined by

$$R_{\vec{q}}(-1)s \in S_{fp}. \tag{2.7}$$

Observation 2: The restriction of $u_t = G(u)$ to M_{fp} is diffeomorph to the flow of a mass-1 single particle in $\mathbb{R}^n$ in the absence of potential energy. Momentum, energy and space-translation of this flow are exactly the abstract quantities defined above.
Proof: To see this we parametrize S_{fp} by the $\vec{k} \in \mathbb{R}^m$ by assigning to each point $s \in S_{fp}$ the $\vec{k}$ such that $s \in M_{\vec{k}}$. By this, and the translations we obtain a parametrization of M_{fp} in terms of the $\vec{k}$ and $\vec{q}$. Now representing the points u on M_{fp} by the $\vec{q}$ and $\vec{k}$ then $u_t = G(u)$ reads

$$\begin{pmatrix} \vec{k} \\ \vec{q} \end{pmatrix}_t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \operatorname{grad} \frac{1}{2} \sum k_i^2 \tag{2.8}$$

where I denotes the $n \times n$ unit matrix. ■
Observation 3: Observe that the system (2.8) has two different hamiltonian fromulations namely

$$\begin{pmatrix} \vec{k} \\ \vec{q} \end{pmatrix}_t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \operatorname{grad} \frac{1}{2} \sum k_i^2 = \begin{pmatrix} 0 & -diag(\vec{k}) \\ diag(\vec{k}) & 0 \end{pmatrix} \operatorname{grad} \sum k_i \tag{2.9}$$

where $diag(\vec{k})$ is the diagonal matrix with the components of $\vec{k}$ as entries. Of course, only one of these hamiltonian formulations is in canonical form.
Observation 4: The physical energy

$$E_{fp} := \mathcal{E}(\vec{k}) \tag{2.10}$$

coming from the hamiltonian formulation of the system of the free paticle is not necessarily given by the abstract energy. Correspondingly, the original symplectic form of the hamiltonian system (2.1), considered on the manifold M_{fp}, may not coincide with the symplectic form given by the abstract hamiltonian formulation (2.8). All we know [2, p. 221, 280] is that for the symplectic form of the original system there are symplectic coordinates $\{I_1, \ldots, I_n, q_1, \ldots, q_n\}$ on M_{fp} such that for the *free particle energy*, expressed as a function in the $(I_1, \ldots, I_n)$

$$E_{fp} = E(I_1, \ldots, I_n) \text{ we have } \frac{\partial E_{fp}}{\partial I_i} = k_i \ .$$

Hence the symplectic coordinates are obtained by a transformation in the parameter space of the momenta alone.

The Lie algebra of the vector fields of the manifold M_{fp} has a canonical structure. Since the vector field Lie algebra does not depend on the special symplectic form under consideration we may use the hamiltonian system (2.8) and the correspondence between its Poisson brackets and the hamiltonian vector fields to investigate that Lie algebra. We consider the vector fields

$$M_i := \frac{\partial u}{\partial k_i}, T_i := \frac{\partial u}{\partial q_i} \tag{2.11}$$

which are given by differentiating the manifold points with respect to the components of abstract momentum and phase, respectively. Obviously, the latter are the T_i. The M_i, T_i which we call *angle fields* and *action fields*, respectively, they are the vector fields corresponding, via the symplectic form of (2.8), to the vector fields q_i and k_i, respectively. The infinitesimal generator of time translation is given by the linear combination over the action fields multiplied with the corresponding components of abstract momentum:

$$u_t = G(u) = T_{\vec{k}}(u).$$

Since, with respect to (2.8), the Poisson brackets $\{\ ,\ \}_{fp}$ of the $(k_1, \ldots, k_n, q_1, \ldots, q_n)$ fulfill

$$\{k_i, k_j\}_{fp} = 0, \ \{q_i, q_j\}_{fp} = 0, \ \{q_i, k_j\}_{fp} = \delta_{ij}. \tag{2.12}$$

we obtain as a consequence that the action fields as well as the angle fields all commute with each other and for the commutator with the generator of time translation we see

$$[T_i, G(u)] = 0,\ [M_i, G(u)] = T_i. \tag{2.13}$$

The fact that the last bracket does not vanish is due to the multiplication of the T_i by the k_i. Furthermore, since the transformation to the symplectic variables only acts upon the actions and not on the angles, the action fields are hamiltonian because they are hamiltonian with respect to (2.8).

Definition 5: Given a hamiltonian dynamic $u_t = G(u)$. An abelian Lie algebra of dimension $2n$ consisting of vector fields $T_i, M_i,\ i = 1, \ldots, n$ with hamiltonian T_i and such that (2.13) holds, is called an n-dimensional *free particle Lie algebra*.

At the end we remark that if we drop the assumption that the fibres are non-compact, then the only difference is that the phases are not unique but periodic.

2.2 Interacting particles

We are now in the position to define a particle in a hamiltonian joint field of interacting particles. Heuristically our definition reads:
A particle with n degrees of freedom is characterized by a set of dynamical variables, called eigen-variables *of the particle, such that for the remaining field the effect of small changes in of any of the eigen-variables is negligible compared to the change itself. In addition it is required that the eigen-variables can be rescaled such that their corresponding vector fields form a free particle Lie algebra.*

As a consequence we obtain that the derivative of the whole field with respect to any of the eigen-variables only changes the eigen variables itself. Using the fact that we know the Lie algebra structure of the corresponding vector fields we find that the derivatives with respect to the action-eigen-variables yield functions in the angle-eigen-variabels and the derivatives with respect to the angle-eigen-variables yield functions in the action eigen-variabels which, in particular are invariants of the dynamics. Transferring this intuitive definition to the vector field Lie algebra we see that the vector fields split into those coming from the particle Lie algebra and those which are not effected by these. Hence transferring this intuitive picture of a particle to the vector field Lie algebra we obtain:

Definition 6: Consider a submanifold invariant against the flow under consideration and let $\mathcal{L}$ be the Lie algebra of all vector fields on that subamnifold.

i) A free particle Lie algebra $\mathcal{L}_p$ which is a direct summand in $\mathcal{L}$ i.e.

$$\mathcal{L} = \mathcal{L}_p \oplus \mathcal{L}_p^{\perp}$$

is said to be an *interacting particle Lie algebra*. When $\vec{k} = (k_1, \ldots, k_n)$ are the action fields of that Lie algbra then the quantity given by (2.10) is said to be the energy of $\mathcal{L}_p$ and is denoted by $\mathcal{E}(\mathcal{L}_p)$.

ii) The Lie algebra $\mathcal{L}$ is said to be a *multi-particle Lie algebra* if it is the direct sum of free particle Lie algebras $\mathcal{L}_{p_i}$

$$\mathcal{L} = \bigoplus_{i=1..N} \mathcal{L}_{p_i}$$

such that the total energy of the system

$$E = \sum_{j=1}^{j=N} \mathcal{E}(\mathcal{L}_{p_i})$$

is given by the sum of the free particle energies.

iii) Whenever the action-angle fields $\{M_j, T_j \mid j = 1 \ldots n\}$ in $\mathcal{L}_{p_i}$ are given as in (2.11) then

$$u_{t_i} := T_{\vec{k}_i} := \sum_{j=1}^{n} k_j T_j \tag{2.14}$$

is called the *eigentime derivative* of u with respect to the i-th particle $\mathcal{L}_{p_i}$. The quantity U_i defined by

$$\frac{\partial U_i}{\partial t} := u_{t_i}$$

is called an *interacton* (abbreviation for interacting soliton).

Obviously the eigen-variables of a particle, as defined heuristically, give rise to a direct summand on the smallest invariant submanifold generated by all possible changes of these eigen-variables. On the other hand, one can construct the eigenvariables once the direct summand $\mathcal{L}_p$ is given. To see this, we first fix the potentials $\{k_1, ..k_n\}$ of the hamiltonian action fields in $\mathcal{L}_p$. We further claim that for the angle fields in $\mathcal{L}_p$ there are integrating factors, only depending on the k_i, which make these hamiltonian. Thus the corresponding potentials are providing the remaining eigen-variables. To find these factors, for a fixed u in the submanifold one takes the $2n$-dimensional sub-submanifold spread by the vector fields in $\mathcal{L}_p$. This is a symplectic manifold with respect to the given symplectic form, and by the $\{k_1, ..k_n\}$ on that $2n$-dimensional manifold we have n scalar quantities which are in involution. Hence, by the classical results on integrable systems, for the remaining vector fields given by the angle fields of $\mathcal{L}_p$, there are integrating factors, depending on the $\{k_1, ..k_n\}$.

Observation 7: Since the eigentime derivatives are symmetry group generators their dynamics is given by the linearization of (2.1), i.e.

$$(u_{t_i})_t = G'(u)\,[u_{t_i}] \tag{2.15}$$

where $G'(u)[A]$ denotes the variational derivative of G at u in the direction of the vector field A. For a multi-particle solution u we furthermore have that

$$u_t = \sum_{i=1}^{N} u_{t_i} \tag{2.16}$$

the time derivative of the field variable u is the sum of eigentime derivatives. This is an immediate consequence of (2.13). In terms of the interactons the dynamic is given by

$$(U_i)_{tt} = G'(u)\,[(U_i)_t] \tag{2.17}$$

and the multi-particle decomposes (if suitable constants of integration are chosen)

$$u = \sum_{i=1}^{N} U_i \ . \tag{2.18}$$

Definition 8: In case we are able to express the dynamics (2.17) only in terms of the interacton, i.e. when everything can be expressed in terms of self-interaction, then the resulting equation is called the *interacton equation.*

2.3 Asymptotic free particles

We consider the case when the flow

$$u_t = G(u) \tag{2.1}$$

is a flow on a suitable manifold of C^∞-functions in the independent variable $x \in \mathbb{R}^n$. The functions are assumed to vanish with all their derivatives at $x = \infty$. For simplicity, we consider the case when $x \in \mathbb{R}^1$ since not much changes for the general case of several dimensions (apart from having fewer relevant examples). Consider those solutions which asymptotically decompose

$$u(x,t) \simeq \sum_{i=1}^{N} s_i(x + k_i t + q_i) \quad \text{for } t \to \infty \tag{2.19}$$

in such a way that all the energy is carried by the asymptotically free waves $s_i(x + k_i t + q_i)$ which also must be solutions of (2.1) (this is seen from the assumption that the solutions of (2.1) have to vanish at ∞). The k_i, q_i we call *asymptotic data*, more specifically the k_i *asymptotic speeds* and the q_i *asymptotic phases*. In order to analyse this kind of solution we perform the same gedankenexperiment is in section 1 and reparametrize the manifold of all such solutions by the asymptotic data. In order to find a new parametrization on that manifold connected to the dynamical system (2.1), we consider an arbitrary point u as initial condition $u(0) := u$ for the dynamical system (2.1) and then measure its asymptotic data. Then because of

$$k_i(u(t)) = k_i(u(0))$$

$$q_i(u(t)) = q_i(u(0)) + k_i t$$

we obtain the dynamic as in (2.8). Using this form of the dynamic one easily sees that the derivatives with respect to the asymptotic data are for this case the action-angle fields, and the special solutions having the prescribed asymptotic decomposition, are multi-particles in the sense of definition 6. The asymptotic waves in (2.19) are just the asymptotic states of the interactons.

3 Interacton equations

The aim of this section is to show that most integrable systems or soliton equations admit interactons and yield corresponding interacton equations by symmetry analysis. The idea how to obtain for the interactons the dynamical laws where only self-interaction occurs is quite simple. Consider the manifold of all multi-particles of

$$u_t = G(u) \ . \tag{2.1}$$

Then the flow on this manifold has two hamiltonian formulations (as we have seen in observation 4). By combination of these two hamiltonian formulations one obtains a co-contravariant tensor which is invariant with respect to the flow under consideration (see [12]). The action-angle fields are eigenvectors for this operator, which are either invariant (action fields) or having an invariant time derivative (angle fields).

With respect to the special parametrization this tensor (considered as a mapping from tanget space to tangent space) looks like

$$\Psi = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

where

$$\Lambda = \begin{pmatrix} k_1 & 0 & .. & 0 \\ 0 & k_2 & .. & 0 \\ : & : & : & : \\ 0 & .. & 0 & k_r \end{pmatrix}$$

Its special properties of an invariant diagonal matrix can be described in a more structural way by the observation that it can only depend on the action variables (because it is invariant) and by the property that its variational derivative fulfills

$$\Psi\Psi'[B] = \Psi'[\Psi B] \ .$$

for any vector field B. However, since we use here the variational derivative (which is not given by a differential geometric invariant operation) this is not a differential geometric invariant characterization. However, a differential geometric invariant characterization is obtained by reformulating all derivatives in terms of Lie derivatives (see [12]). Doing this results in

$$\Psi L_K(\Psi) = L_{\Psi K}(\Psi) \tag{3.1}$$

for all vector fields K. An operator with such a property is said to be a hereditary operator (see [7], [14], [6]). These operators have very strong properties with respect of the group ananlysis of the underlying flows:

Theorem 9: *Let Φ be some hereditary operator which is invariant with repect to the vector field G, i.e. for the Lie derivative L_G with respect to G we have*

$$L_G\Phi = 0 \ .$$

Then the vector fields

$$G, \Phi G, \Phi^2 G, \Phi^3 G, ...$$

are forming the basis of an abelian Lie algebra, i.e. a family of commuting vector fields.

The problem however is that obviously for multi-particle systems we only know such operators on special manifolds, namely those given by the multi-particle solutions. Nevertheless, it seems worthwile to look for extensions of such operators on the manifold of all solutions for (2.1). Indeed, in most 1+1-dimensional cases such operators have been found, for example

$$\Phi_{KdV} = D^2 + 2DuD^{-1} + 2u \tag{3.2}$$

for the KdV (see [21] for its first appearance). So, in case an invariant hereditary operator is given for (2.1) the dynamical laws for the eigenvectors are good candidates for interactons and those solutions which decompose exactly into eigenvectors are candidates for multi-particles. Indeed, one easily sees that these yield multi-particles because under vanishing boundary conditions at ∞ these solutions decompose asymptotically like (2.19) for which case we have already seen the multi-particle structure.

3.1 Interactons for KdV and the like

In order to get some idea for deriving a method for obtaining interacton equations in the absence of hereditary operators we briefly point out how to find these equations in case of the presence of a hereditary operator in 1+1-dimension(see [10] or [11]).

Let $\Phi(u)$ be a hereditary recursion operator for (2.1), i.e. apart from being hereditary, $\Phi(u)$ must be invariant under the flow. Furthermore we assume that (2.1) is translation invariant and of the form

$$u_t = \Phi(u)u_x \ . \tag{3.3}$$

Then for any asymptotically emerging travelling wave

$$s_k := s(x + kt + q)$$

it holds

$$(s_k)_t = \Phi(s_k)\ s_{kx} = ks_{kx} \tag{3.4}$$

hence asymptotically s_{kx} is an eigenvector of Φ. Observe that the eigenvectors of Φ

$$\Phi(u)\sigma_k = k\sigma_k \tag{3.5}$$

have the same dynamics as the generators of one-parameter symmetry groups (see [7], [8]):

$$\sigma_t = G'(u)[\sigma] := \frac{\partial}{\partial\epsilon}_{|\epsilon=0} G(u+\epsilon\sigma)\ . \tag{3.6}$$

From here follows that any multi-particle solution which consists of asymptotically free particles must be of the form

$$u_x = \sum_{j=1}^{n} \sigma_{k_j} \tag{3.7}$$

because this relation, which holds for $t \to \infty$, defines an invariant manifold. Now put

$$(S_j)_x := \sigma_{k_j}.$$

Then, under appropriate boundary conditions at infinity, we obtain for multi-particles

$$u = \sum_{j=1}^{n} S_{k_j} \tag{3.8}$$

By use of the considerations of the last section we see that in this case the eigentime derivative of the j-th particle is

$$(U_j)_t := u_{t_j} := k_j \frac{\partial S_j}{\partial q_j}$$

where q_j is the asymptotic phase. Hence the particles can be obtained by taking derivatives with respect to asymptotic data. Since asymptotically for the symmetry generators $(U_j)_t$ and $(S_j)_x$ we have $(U_j)_t = k_j(S_j)_x$, which again defines an invariant submanifold, we obtain independently of any asymptotics

$$(U_j)_t = k_j(S_j)_x = k_j\sigma_{k_j}\ , \tag{3.9}$$

which defines the interacting particle in terms of the eigenvector of the recursion operator.

The problem remains how the dynamics can be uncoupled such that only self-interaction is present. In order to do that we consider (3.5) as a Lie-Bäcklund-transformation between u and σ. This transformation usually is unique modulo an additional symmetry (given by rescaling the amplitude of the eigenvector). For suitable boundary conditions we solve this relation for u when σ is given and obtain $u = F(\sigma) = F(S_x),\ S_x := \sigma$. We insert this in (3.6) in order to obtain the interacton equation. The resulting equation is again integrable because, modulo uniqueness up to a symmetry for the given system, it is related to the original integrable equation by a transformation between dependent variables, hence we have to do with a manifold transformation which does not destroy integrability. Indeed by this manifold transformation all the invariants can be carried over from the original equation to the corresponding interacton equation (see [10]).

Example 10: For the mKdV (modified Korteweg de Vries equation)

$$u_t = u_{xxx} + 6u_x u^2 =: G(u)$$

the eigenvector equation given by the recursion operator is

$$k_j S_x = S_{xxx} + 4DuD^{-1}uS_x\ .$$

This procedure yields

$$u_x u^{-3}(S_{xx} - k_j S) - u^{-2}(S_{xxx} - k_j S_x) = 4S_x \ .$$

Solving the homogeneous equation and applying the method of variation of constants gives

$$u = \frac{k_j S - S_{xx}}{2(\sqrt{k_j S^2 - S_x^2 + C})}$$

where $C = 0$ because of the boundary condition at infinity. Inserting this into

$$S_{xt} = G(u)'[S_x]$$

we obtain the interacting soliton equation for this case

$$S_t = S_{xxx} + \frac{3(k_j S - S_{xx})^2}{2(k_j S^2 - S_x^2)} S_x.$$

Inserting here $(U)_t = k(S)_x$ we find the associated interacton equation. □

Example 11: By the same procedure we obtain for the sine-Gordon equation

$$S_{xt} = \frac{S}{2} \cos \left(\int_{-\infty}^{x} \frac{k_j S(\xi) - S_{\xi\xi}(\xi)}{2\sqrt{k_j S(\xi)^2 - S_\xi(\xi)^2}} d\xi \right)$$

and the Korteweg de Vries:

$$S^2 S_t = S^2 S_{xxx} - 3 S S_x S_{xx} + \frac{3}{2} S_x^3 + \frac{3}{2} k_j S^2 S_x$$

In both cases the interacton equation is obtained by the substitution $(U)_t = k(S)_x$.

In case of the nonlinear Schrödinger equation we get

$$|\psi|^2 \psi_t \ = \ -i\psi_{xx} \mid \psi \mid^2 \ + \ i\psi \mid m\psi \ + \ \frac{i}{2}\psi_x \mid^2 \ - \ i(m\psi \ + \ i\psi_x)^2 \overline{\psi}.$$

Where ψ is connected to the interacton U by

$$(U)_{tx} = k(\psi) \ .$$

See [10] for the details of the computation. Indeed, in all these cases the complete action-angle representation can be carried out in terms of the spectrum of the recursion operator, see [15] and [22]. □

3.2 Interactons in the absence of hereditary operators

In many cases the eigenvector problem for the recursion operator is equivalent to a nonlinear eigenvector problem given by the auto-Bäcklund transformation (ABT) of the system ([9], [13]). We briefly review this:
Consider for

$$u_t = G(u) \tag{2.1}$$

an auto-Bäcklund transformation

$$B(u, \bar{u}, \lambda) = 0 \ . \tag{3.10}$$

Hence, for any value of λ, we have that when u is a solution of (2.1) then $\bar{u}$ also has to be a solution. We consider the following *nonlinear spectral problem*:

Given a solution u of (2.1), consider the partial variational derivative $B_u(u, \bar{u}, \lambda)$ of $B(u, \bar{u}, \lambda)$ (with respect to u) and find those λ such that there is some non-zero vector field ω and some $\bar{u}$ on the manifold under consideration such that

$$B_u(u, \bar{u}, \lambda)[\omega] = 0 \quad \text{and} \quad B(u, \bar{u}, \lambda) = 0 \ . \tag{3.11}$$

That means we try to find those values of λ where the *implicit function theorem* is not valid. Then in most cases where a recursion operator exists, this nonlinear spectral problem is equivalent to the linear one given by the recursion operator. In these cases the nonlinear problem is easily linearized in a purely algorithmic way.

Example 12: For the KdV there is the well known auto-Bäcklund transformation

$$B(u, \bar{u}, \lambda) = u + \bar{u} + \lambda D^{-1}(u - \bar{u}) + \frac{1}{2}\{D^{-1}(u - \bar{u})\}^2 = 0 \ . \tag{3.12}$$

Variational derivative of (3.12) with respect to u provides the operator:

$$B_u = I + (D^{-1}(u - \bar{u}))D^{-1} + \lambda D^{-1} \ . \tag{3.13}$$

And the corresponding nonlinear spectral problem (3.11) is

$$0 = \omega + (D^{-1}(u - \bar{u}))D^{-1}\omega + \lambda D^{-1}\omega \ . \tag{3.14}$$

This only is formally linear since $\bar{u}$ and ω are not independent. Abbreviation $D^{-1}\omega = v$ allows to write

$$D^{-1}(u - \bar{u}) = -(\frac{v_x}{v} + \lambda) \ . \tag{3.15}$$

Writing $u + \bar{u}$ as $2u - (u - \bar{u})$ and replacing all terms $u - \bar{u}$ in (3.12) by (3.15) we obtain

$$2u + (\frac{v_x}{v} + \lambda)_x + \frac{1}{2}(\frac{v_x}{v} + \lambda)^2 - \lambda(\frac{v_x}{v} + \lambda) = 0 \tag{3.16}$$

which certainly is a *nonlinear* eigenvalue equation. By multiplication with v^2 we get

$$2uv^2 + v_{xx}v - \frac{1}{2}v_x v_x = \frac{1}{2}\lambda^2 v^2 \ . \tag{3.17}$$

If this problem can be linearized there must be operators $A(v)$ and $\Psi(u)$ such that $A(v)v = Cv^2$ and $A(v)\Psi(u)v$ is equal to the left hand side of (3.17). Comparison of suitable terms yields in an algorithmic way:

$$D^{-1}vD\{v_{xx} + 2uv + 2D^{-1}(uv_x)\} = \lambda^2 D^{-1}vDv \ . \tag{3.18}$$

Hence $A(v) = D^{-1}vD$ and $\Psi(u) = D^2 + 2u + 2D^{-1}uD$. Going back to $\omega = v_x$ we see that ω is a solution of (3.14) if and only if ω is an eigenvector of

$$\Phi(u) = D\Psi(u)D^{-1} = D^2 + 2u + 2DuD^{-1},$$

the recursion operator of the KdV.□

So in principle the results which can be obtained from the recursion operator can be derived from the auto-Bäcklund transformation. However this approach is more general because there are cases of integrable systems where recursion operators do not exist whereas auto-Bäcklund transformations are easily found. Therefore, also for these equations with only an ABT, the interacton equation is hidden in that relation. However, we have to modify our approach slightly. For simplicity we do this for the 1+1-dimensional case. We consider an equation

$$u_t = G(u) \tag{2.1}$$

such that there is an ABT

$$B(u, \bar{u}, \lambda) = 0. \tag{3.19}$$

For a given solution u with emerging soliton $s_i(x + kt + q)$, having, at $t \to \infty$, asymptotic speed k and phase q we consider the spectral parameter $\lambda_0 = \lambda(k)$ such that in the solution $\bar{u}$ this soliton is annihilated by the ABT. These asymptotic data we consider als action-angles of our particle. Now, taking the gradient of (3.19), for $\lambda = \lambda_0$ with respect to q, we obtain

$$\nabla_q B(u, \bar{u}, \lambda_0) = B_u(u, \bar{u}, \lambda_0)[\nabla_q u] = 0 \ . \tag{3.20}$$

Or, after defining the time derivative of the interacton as before

$$U_t = k \frac{\partial u}{\partial q} \tag{3.21}$$

we get

$$\nabla_q B(u, \bar{u}, \lambda_0) = B_u(u, \bar{u}, \lambda_0)[U_t] = 0 \ . \tag{3.22}$$

Now we use this equation together with (3.19) in order to express u and $\bar{u}$ in terms of U_t or U. These representations we use to replace u in the dynamic

$$U_{tt} = G(u)'[U_t] \tag{3.23}$$

to obtain the interacton equation only in terms of self-interaction. This equation is integrable and many of its solutions can be obtained by differentiation of solutions of the original equation with respect to the asymptotic phases.

3.2.1 The interacton for the BO

Consider the Benjamin-Ono equation ([4] and [23])

$$u_t = H u_{xx} + 2 u u_x \ . \tag{3.24}$$

where H denotes the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} \int_{+\infty}^{-\infty} \frac{f(\xi) d\xi}{\xi - x} \quad \text{(principal value integration)} \ .$$

We want to derive the interacton equation (see [17] for more details) for this case where no recursion operator exists.

As usual we define $H1 = 0$. Equation (3.24) has ([1, p.204])

$$s_k(t, x) = \frac{ik}{k(x - x_0) + k^2 t + i} - \frac{ik}{k(x - x_0) + k^2 t - i} \tag{3.25}$$

as one-soliton solutions, i.e. solutions of $k u_x = H u_{xx} + 2 u u_x$. A typical two soliton looks like

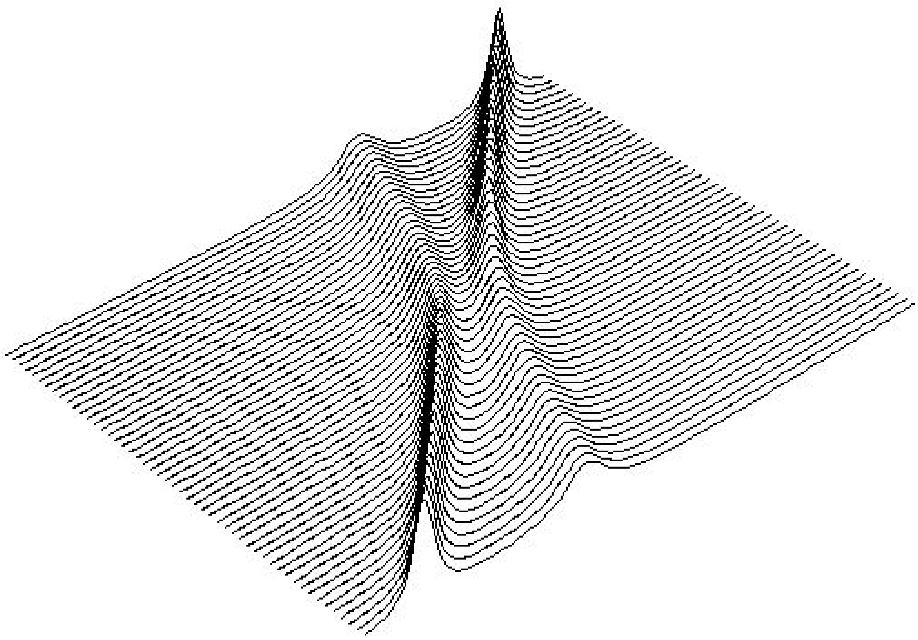

Fig. 1: A two soliton for the BO[2]

For the BO we have the well known ABT [20], [9]

$$\exp(iD^{-1}(u-\bar{u}))-1-\frac{1}{k}\{iH(u-\bar{u})+(u+\bar{u})\}=0 \tag{3.26}$$

which annihilates those solitons in u with asymptotic speed k. Equation (3.20) now has the form

$$is\exp(iD^{-1}(u-\bar{u}))-\frac{1}{k}(iHs_x+s_x)=0\ . \tag{3.27}$$

Introducing projection operators

$$P_\pm:=\frac{1}{2}(I\pm iH)$$

we write (3.26) and (3.27) as

$$\exp(iD^{-1}(u-\bar{u}))-1+\frac{2}{k}P_-(u-\bar{u})-\frac{2}{k}u=0 \tag{3.28}$$

and

$$D^{-1}(u-\bar{u})=-i\ln\left(-\frac{2i}{ks}P_+s_x\right)\ . \tag{3.29}$$

Inserting this last result into (3.28) we get

$$-\frac{2i}{ks}P_+(s_x)-1-\frac{2i}{k}P_-\left(\ln\left(\frac{2}{iks}P_+s_x\right)\right)_x=\frac{2u}{k}\ . \tag{3.30}$$

[2]Here the line in front is parallel to the x-axis, the chosen asymptotic speeds are $k_1:=0.7$ and $k_1:=0.25$ and the t-slices have been scaled by 3/2.

Observe that the decomposition given by the projection $P_\pm$ is the usual decomposition into functions being analytic in the upper and lower half of the complex plane, respectively. Hence terms like

$$P_-\left(\frac{P_+ s_{xx}}{P_+ s_x}\right) = 0$$

vanish. Using this we rewrite (3.30) as

$$u = -\frac{k}{2} - \frac{i}{s} P_+(s_x) + i P_-\left(\frac{s_x}{s}\right) \ . \tag{3.31}$$

The dynamics (3.23) in case of the BO is

$$s_{xt} = K(u)'[s_x] = H s_{xxx} + 2 s_x u_x + 2 u s_{xx}$$

or

$$s_t = H s_{xx} + 2 u s_x. \tag{3.32}$$

Eliminating u by use of (3.31) we find for the equation

$$\begin{aligned} s_t &= H s_{xx} - k s_x + 2 i s_x P_- s^{-1} s_x - 2 i s^{-1} s_x P_+ s_x \\ &= H s_{xx} - k s_x + s_x H(s^{-1} s_x) + s^{-1} s_x H(s_x) \end{aligned} \tag{3.33}$$

As before, the interacton equation is obtained by introducing

$$U_t = k s_x \ . \tag{3.34}$$

This equation is integrable, its invariants can easily be obtained from those of the BO by using (3.31) as a manifold transformation where the invariants of the BO are easily computed by the usual mastersymmetry approach [5]:
To be concrete, replacing in any conserved scalar field of the BO the variable u by (3.31) leads to a conserved quantity for (3.33). Replacing in a symmetry generator of the BO the quantity u, then mapping the result by the inverse of the variational derivative of the right hand side of (3.30) yields the symmetry generators for (3.33). For finding the corresponding angle-variables see [17].

Solutions are easily found for equation (3.33): Since Fig. 1 represents a two soliton for the BO with asymptotic speeds $k_1 := 0.7$ and $k_1 := 0.25$ the corresponding derivatives with respect to the phases lead, after one integration, to the interacting solitons for the equations with these k-values. Plots show the expected asymptotic behaviour, and the effects due to the nonlinearity:

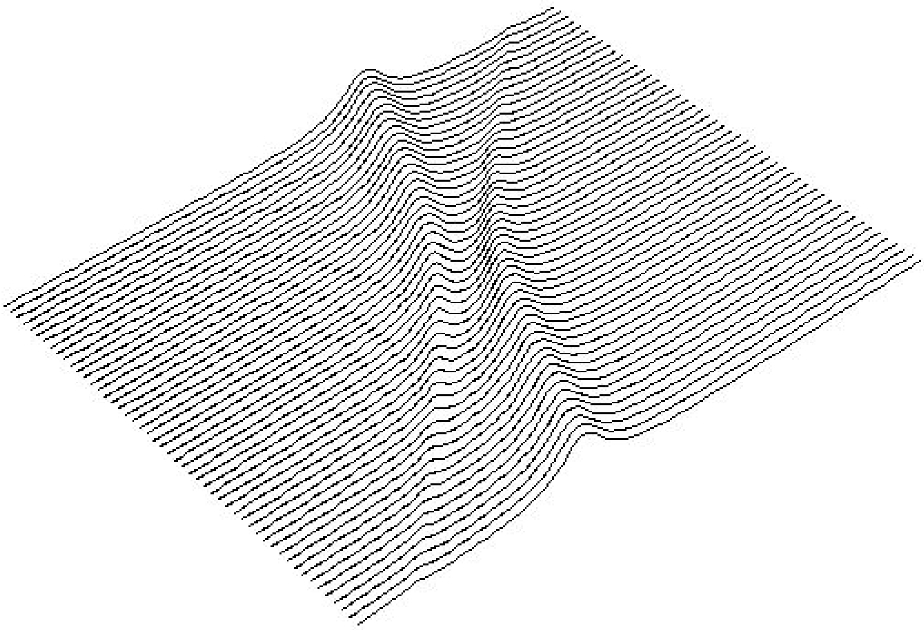

Fig. 2: The slower interacton for the BO, $k_1 := 0.25$

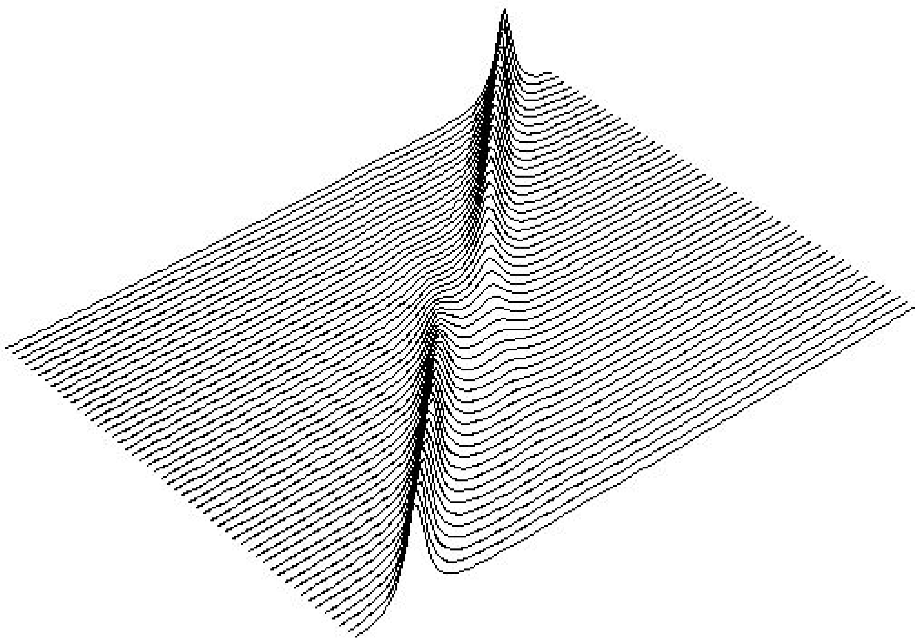

Fig. 3: The faster interacton for the BO, $k_1 := 0.6$

It should be observed that the foregoing considerations do not depend of the dimension of the space of independent variables. Therefore similar arguments can be carried out in order to give the interacton equations for integrable systems in 2+1 dimensions like the KP (see [16]).

3.3 Self-consistent potentials

Nonlinear Schrödinger equations often (see [24]) play a role in the search of valid descriptions of interacting particles. Take for example Yukawas model for nucleon meson interaction, or, on another level, the nonlinear phenomena in nuclear models (see [18] [19]). Therefore we put special emphasis in this section on the investigation of self-consistent potentials for interactons.

For a field ψ we consider the Schrödinger equation where we assume that the potential energy is of higher than second order in ψ. This obviously results in a nonlinear Schrödinger equation. Furthermore, we assume that asymptotically the field decomposes in N different particles $\varphi_1, \ldots, \varphi_N$ which carry all the energy of the field, i.e. that the solution is a multi-particle in the sense of section 2.3. The *joint wave* function is

$$\psi = \sum_{l=1}^{N} \varphi_l \tag{3.35}$$

where φ_l denotes the field function of the interacton. The energy of the N-particle system shall be

$$\mathcal{E}(\psi) = \mathcal{E}_{kin}(\psi) + \mathcal{E}_{pot}(\psi) = \frac{\hbar}{2m} \langle \nabla\psi, \nabla\psi \rangle + \int_{\mathbb{R}^3} V(\psi, \bar{\psi}) dx. \tag{3.36}$$

We assume that the potential energy $V(\psi, \bar{\psi})$ is gauge invariant, were a quantity $\mathcal{F}(\psi)$ is said to be *gauge invariant* if ψ can be multiplied with any number of modulus 1 (and $\bar{\psi}$ with its complex conjugate) without changing the value of $\mathcal{F}$.

We measure time in units of $\hbar$ and we scale physical units and space variables such that $\hbar = 2m$. Hence the usual Schrödinger equation has the form

$$i \frac{\partial}{\partial t} \Phi = -\Delta \Phi + U \Phi, \tag{3.37}$$

with potential energy U. We want to know if the dynamics of a single particle, i.e. the dynamic for some φ_l can be written as Schrödinger equation with respect to a joint potential (self-consistent potential). A positive answer to this problem is not at all evident since the dynamics of the φ_l is fixed by definition of energy. To see this observe that the dynamics for ψ is

$$\frac{\partial}{\partial t} \psi = -i \ \mathrm{grad} \mathcal{E}(\psi) \tag{3.38}$$

where the gradient has to be taken with respect to ψ and $\bar{\psi}$. Here we used the fact that for Schrödinger theorie the symplectic form appearing in the hamiltonian formulation is given by multiplication with i. To be precise, we look for a potential such that the dynamics of the φ_l can be written as

$$i \frac{\partial}{\partial t} (i\varphi_l) = -\Delta(i\varphi_l) + U(i\varphi_l) \ . \tag{3.39}$$

At this point taking instead of φ_l the infinitesimal generator $i\varphi_l$ of its gauge group seems completely irrelevant, but later on we see its importance.

Now observe that as a consequence of the conservation laws

$$\langle \varphi_l, \varphi_l \rangle = 1$$

the generator $i\varphi_l$ of the gauge group of each particle must be a symmetry generator of the dynamics of the joint field ψ. Hence the dynamics for $i\varphi_l$ is given by the linearization of (3.38):

$$\frac{\partial}{\partial t} (i\varphi_l) = -i \ \mathcal{E}'' \ i\varphi_l \tag{3.40}$$

where $\mathcal{E}''$ denotes the sesquilinear form given by the second variational derivative

$$\mathcal{E}''[\phi_a,\phi_b] := \frac{\partial}{\partial\lambda_2}\frac{\partial}{\partial\lambda_1}\mathcal{E}(\phi+\lambda_1\phi_a+\lambda_2\phi_b)_{|\lambda_1=\lambda_2=0}\ . \tag{3.41}$$

To determine the self-consistent potential energy operator we observe that we know the expectation of U

$$\mathcal{E}_{pot}(\psi) = \langle\vec{\varphi}, U\vec{\varphi}\rangle$$

where $\vec{\varphi} = (\varphi_1,\varphi_2,\ldots,\varphi_N)$. In the linear case operator representations for observables $\mathcal{E}$ are recovered from their expectation $\mathcal{E}(\phi)$. For example if it is known as a function for general wave functions ϕ, then we find

$$2\big\langle\phi_b,\mathcal{E}\ \phi_a\big\rangle = \mathcal{E}''[i\phi_a,i\phi_b] - i\mathcal{E}''[i\phi_a,\phi_b] \tag{3.42}$$

where again $\mathcal{E}''$ is the second variational derivative. This formula was obtained only for the situation where the operator representation for the observable does not depend on the vector ϕ which represents the state of the system (*linear case*), however we consider it for the general case, even when scalar quantities are not bilinear in the field function.

Definition 13: *Given a gauge invariant scalar quantity $\mathcal{F}(\phi)$, with $\mathcal{F}=0$ when $\phi=0$, and given two vectors ϕ_a,ϕ_b then the application to ϕ_a,ϕ_b of the self-consistent operator F assigned to $\mathcal{F}(\phi)$ is defined to be*

$$\big\langle\phi_b, F\ \phi_a\big\rangle := \frac{1}{2}\left\{\mathcal{F}(\phi)''[i\phi_a,i\phi_b] - i\mathcal{F}(\phi)''[i\phi_a,\phi_b]\right\} \tag{3.43}$$

Observe that $F=F(\phi)$ may depend on ϕ, therefore its expectation is defined to be

$$\text{expct}(F) := 2\int_0^1 \lambda\big\langle\phi, F(\lambda\phi)\phi\big\rangle d\lambda$$

Of course this definition only makes sense if $\text{expct}(F)=\mathcal{F}(\phi)$. To see this we observe that gauge invariance implies for the variational derivative in direction of $i\phi$ that $\mathcal{F}'(\phi)[i\phi]=0$. Second derivatives in direction of ϕ and $i\phi$ then yield

$$\mathcal{F}''(\phi)[\phi,i\phi] = 0 \tag{3.44}$$

$$\mathcal{F}''(\phi)[i\phi,i\phi] = \mathcal{F}'(\phi)[\phi] \tag{3.45}$$

By (3.44) and (3.43) we obtain

$$\begin{aligned} 2\int_0^1 \lambda\big\langle\phi, F(\lambda\phi)\phi\big\rangle &= \int_0^1 \lambda\mathcal{F}(\lambda\phi)''[i\phi,i\phi]d\lambda \\ &= \int_0^1 \mathcal{F}(\lambda\phi)'[\phi]d\lambda \\ &= \mathcal{F}(\phi) \end{aligned}$$

In this computation the second-last line followed from (3.45) and the last line from the fact that in the second-last line the integrand was a complete derivative with respect to λ.

Observation 14: If U is taken to be the self-consistent operator of the potential energy then equations (3.39) and (3.40) coincide. Hence the dynamics of the interacton in a nonlinear Schrödinger field is given by the Schrödinger equation with self-consistent potential. Since the linearization equation (3.40) represents a flow on the tangent bundle, the self-consistent potential is a two times covariant tensor, therefore a quantity defined in a differential geometric invariant way.

Observe that the self-consistent operator coming from a scalar quantity is by definition an operator which acts on tangent vectors, therefore it made sense to consider the asymptotic Schrödinger equation (3.39) as an equation acting on the generator of the gauge group instead of the wave function itself. That this also leads to different computational results will be seen from the following example.

We compute the self-consistent potential U for the potential energy

$$\mathcal{E}_{pot}(\psi) = -\frac{1}{2}\int_{\mathbb{R}^3} |\psi|^4 dx.$$

We obtain

$$\langle \phi_b, U\phi_a \rangle = -\langle \phi_b, \psi^2 \bar{\phi_a} + 2|\psi|^2 \phi_a \rangle$$

which indeed has the prescribed expectation

$$\operatorname{expct}(U) = -\frac{1}{2}\int_{\mathbb{R}^3} |\psi|^4 dx.$$

Observe that $U : \phi_a \to \psi^2 \bar{\phi_a} + 2|\psi|^2 \phi_a$ has an *antilinear* part, therefore it makes a difference wether this operator is applied to the generator of the gauge group instead to the wave function itself. Observe that by the same computation the self-consistent operator assigned to the energy is $-\Delta + U$.

Writing down equation (3.39) for the single particle we obtain

$$i\frac{\partial}{\partial t}(i\varphi_l) = -\Delta(i\varphi_l) - \left(\psi^2\ \overline{(i\varphi_l)} + 2|\psi|^2(i\varphi_l)\right). \tag{3.46}$$

This yields for the wave functions

$$i\frac{\partial}{\partial t}\varphi_l = -\Delta\varphi_l + \psi^2\bar{\varphi}_l - 2|\psi|^2\varphi_l.$$

Going over to the joint wave we obtain by summation

$$i\frac{\partial}{\partial t}\psi = -\Delta\psi - |\psi|^2\psi, \tag{3.47}$$

which is the the well known nonlinear Schrödinger equation (NLS), which is completely integrable in (1+1) dimension and for which the interacton equation was given in Example 3.3.

Thus for one space dimension the N-particle state is a special multisoliton solution of the integrable cubic Schrödinger equation. Furthermore, one should observe that in case of a single particle and due to the fact that the Schrödinger operator defines an isospectral formulation for the Korteweg de Vries equation (KdV) $u = |\psi|^2$ must be a single soliton of the KdV. Hence in case of a single particle the linear part of the self-consistent potential must be a Bargmann potential, i.e. a reflectionless potential (see [3], [25]).

References

[1] M. J. Ablowitz and H. Segur. Asymptotic solutions of the Korteweg-de Vries equation. *Studies in Appl.Math.*, 57:13–44, 1977.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60. Springer Verlag, Berlin-Heidelberg-New York, 1978.

[3] V. Bargmann. On the connection between phase shift and scattering potentials. *Rev. Mod. Phys.*, 21:488–493, 1949.

[4] T. B. Benjamin. Internal waves of finite amplitude and permanent form. *J.Fluid Mech.*, 29:559–592, 1967.

[5] A. S. Fokas and B. Fuchssteiner. The hierarchy of the Benjamin-Ono equation. *Phys. Lett.*, 86 A:341–345, 1981.

[6] A.S. Fokas. Symmetries and integrability. *Stud. Appl. Math.*, 77:253–299, 1987.

[7] B. Fuchssteiner. Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Analysis TMA*, 3:849–862, 1979.

[8] B. Fuchssteiner. The lie algebra structure of nonlinear evolution equations admitting infinite dimensional abelian symmetry groups. *Progr. Theor. Phys.*, 65:861–876, 1981.

[9] B. Fuchssteiner. From single solitons to auto/Bäcklund transformations and hereditary symmetries. *in: Topics in Soliton Theory and Exactly solvable Nonlinear equations (eds: M. Ablowitz, B. Fuchssteiner, M. Kruskal) World Scientific Publ.*, pages 230–254, 1987.

[10] B. Fuchssteiner. Solitons in interaction. *Progress of Theoretical Physics*, 78:1022–1050, 1987.

[11] B. Fuchssteiner. The dynamical behavior of interacting solitons. *in: Nonlinear Evolutions (J.J. P. Jerome ed.), Singapore, World Scientific Publishers*, pages 13–32, 1988.

[12] B. Fuchssteiner. Hamiltonian structure and integrability. *in: Nonlinear Systems in the Applied Sciences, Mathematics in Science and Engineering Vol. 185 Academic Press, C. Rogers and W. F. Ames eds.*, pages 211–256, 1991.

[13] B. Fuchssteiner. Linear aspects in the theory of solitons and nonlinear integrable equations. *Journal of the Phys. Soc. Japan*, 60:1473–1496, 1991.

[14] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their, Bäcklund transformations and hereditary symmetries. *Physica*, 4 D:47–66, 1981.

[15] B. Fuchssteiner and G. Oevel. Geometry and action-angle variables of multisoliton systems. *Reviews in Mathematical Physics*, 1:415–479, 1990.

[16] B. Fuchssteiner and T. Schulze. The interacton equation for the KP. *preprint*, 1995.

[17] B. Fuchssteiner and T. Schulze. A new integrable system: The interacting soliton of the BO. *Phys. Lett.*, A 204:336–342, 1995.

[18] E.F. Hefter and G.Papini. Nuclear matter as a nonlinear medium. *Il Nuovo Cimento*, 91 A:193–204, 1986.

[19] E.F. Hefter and G.Papini. Self-energy of nuclei and real and imaginary optical potentials. *Phys. Rev.*, 34 C:L3020–L3023, 1986.

[20] Y. Kodama, M. J. Ablowitz, and J. Satsuma. Direct and inverse scattering problems of the nonlinear intermediate long wave equation. *J.Math.Phys.*, 23:564–576, 1982.

[21] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[22] G. Oevel and B. Fuchssteiner. Unified approach to action-angle representation of real and complex multisolitons. *Physica*, A 181:364–384, 1992.

[23] J.H. Ono. Algebraic solitary waves in stratified fluids. *J.Phys.Soc. Japan*, 39:1082–1091, 1975.

[24] T. Ozawa and Y. Tsutsumi. Asymptotic behaviour of solutions for the coupled Klein-Gordon-Schrödinger equations. *Adv. Stud. Pure Math., Spectral and scattering theory and applications, 23, Math. Soc. Japan*, pages 295–305, 1994.

[25] P. Sabatier. *Inverse Problems in Quantum Scattering Theory.* Springer Verlag, Berlin-Heidelberg-New York, 1989.