

The Bi-Hamiltonian Structure of the Perturbation Equations of KdV Hierarchy

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Abstract

The bi-Hamiltonian structure is established for the perturbation equations of KdV hierarchy and thus the perturbation equations themselves provide also examples among typical soliton equations. Besides, a more general bi-Hamiltonian integrable hierarchy is proposed and a remark is given for a generalization of the resulting perturbation equations to $1 + 2$ dimensions.

Perturbation theory is the study of the effects of small disturbances. Its basic idea is to find approximate solutions to a concrete problem by exploiting the presence of a small dimensionless parameter. There have been a lot of works to investigate the perturbed soliton equations (see for example [1] [2] and references therein). Tamizhmani and Lakshmanan have considered a perturbation effect of the unperturbed KdV equation and they have given rise to infinitely many Lie-Bäcklund symmetries and a Hamiltonian structure for the resulting equations in Ref. [3]. However they haven't obtained the corresponding bi-Hamiltonian formulation.

In this letter, we would like to discuss the perturbation equations of the unperturbed whole KdV hierarchy, i.e. the effect of the disturbance around solutions of the following original KdV hierarchy

$$u_{t_n} = \Phi^n(u)u_x, \quad \Phi(u) = \partial^2 + 2u_x\partial^{-1} + 4u, \quad \partial = \frac{d}{dx}, \quad n \geq 1. \quad (1)$$

The first equation is exactly the usual KdV equation

$$u_{t_1} = u_{xxx} + 6uu_x, \quad (2)$$

which describes the unidirectional propagation of long waves of small amplitude and has a broad of applications in a number of physical contexts such as hydromagnetic waves, stratified internal waves, ion-acoustic waves, plasma physics and lattice dynamics (for details see, for example, [4] [5]). The operator $\Phi(u)$ is a common hereditary recursion operator for the whole KdV hierarchy. We shall show that all of the resulting perturbation equations possess a bi-Hamiltonian structure and thus they constitute a typical integrable

soliton hierarchy. We shall also point out a mistake on the Hamiltonian formulation in Ref. [3]. Finally, a more general bi-Hamiltonian integrable soliton hierarchy is established and some further discussion is presented.

We first consider the case of KdV equation (2) and then consider the case of higher order KdV equations. Let us make a perturbation expansion

$$\hat{u} = \sum_{i=0}^N \eta_i \varepsilon^i, \quad \eta_i = \eta_i(x, t_1, t_2, \dots), \quad N \geq 1, \quad (3)$$

and call the equation

$$\hat{u}_{t_1} = \Phi(\hat{u})\hat{u}_x + o(\varepsilon^N) = \hat{u}_{xxx} + 6\hat{u}\hat{u}_x + o(\varepsilon^N), \quad (4)$$

the N -th order perturbation equation of KdV equation (2). It is easy to find that

$$\Phi(\hat{u}) = \sum_{i=0}^N \Phi(\eta_i)\varepsilon^i, \quad \Phi(\eta_i) = \delta_{i0}\partial^2 + 2\eta_{ix}\partial^{-1} + 4\eta_i, \quad 0 \leq i \leq N,$$

where δ_{i0} is the Kronecker's symbol. Therefore the N -th order perturbation equation (4) may be rewritten as

$$\hat{u}_{t_1} \equiv \left(\sum_{i=0}^N \Phi(\eta_i)\varepsilon^i \right) \left(\sum_{i=0}^N \eta_{ix}\varepsilon^i \right) \pmod{\varepsilon^{N+1}}.$$

A balance of the coefficients of like powers of ε leads to the following equivalent equation

$$\eta_{t_1} = \hat{\Phi}(\eta)\eta_x, \quad \eta = (\eta_0, \eta_1, \dots, \eta_N)^T, \quad (5)$$

where the operator $\hat{\Phi}(\eta)$ reads as

$$\hat{\Phi} = \hat{\Phi}(\eta) = \begin{bmatrix} \Phi(\eta_0) & & & 0 \\ \Phi(\eta_1) & \ddots & & \\ \vdots & \ddots & \ddots & \\ \Phi(\eta_N) & \cdots & \Phi(\eta_1) & \Phi(\eta_0) \end{bmatrix}. \quad (6)$$

By its first component of (5), we see that the perturbation expansion (3) is an expansion around an exact solution η_0 of KdV equation (2). Moreover \hat{u} is an approximate solution of KdV equation (2) to a precision $o(\varepsilon^N)$ when η satisfies (5). However our aim here is not to get some approximate solutions. What we want to discuss is some algebraic structures that the perturbation equations possess.

Let us introduce two differential operators

$$\hat{M} = \begin{bmatrix} 0 & & & J(\eta_0) \\ & & \vdots & J(\eta_1) \\ & & \vdots & \vdots \\ J(\eta_0) & J(\eta_1) & \cdots & J(\eta_N) \end{bmatrix}, \quad \hat{J} = \begin{bmatrix} 0 & & \partial \\ & \partial & \\ \partial & & 0 \end{bmatrix} \quad (7)$$

with $J(\eta_i) = \delta_{i0}\partial^3 + 2\eta_{ix} + 4\eta_i\partial$, $0 \leq i \leq N$. Note that the operator $\hat{M} + \alpha\hat{J}$ is a Hamiltonian operator for any constant α (see for example Ref. [6]) and that

$$\hat{\Phi}(\eta) = \hat{M}\hat{J}^{-1}, \quad \hat{J}\hat{\Psi} = \hat{\Phi}\hat{J}, \quad \hat{\Psi} = \hat{\Phi}^*,$$

where the asterisk appended to $\hat{\Phi}$ denotes the conjugate operation. Therefore \hat{J}, \hat{M} constitute a Hamiltonian pair and further the operator $\hat{\Phi}(\eta)$ is hereditary [7],[8].

We are now in the position to construct a bi-Hamiltonian structure [9] [10] for the perturbation equation (5). We first have the first Hamiltonian structure

$$\eta_{t_1} = \hat{\Phi}(\eta)\eta_x = \hat{\Phi}(\eta)\hat{J}\frac{\delta H_0}{\delta\eta} = \hat{M}\frac{\delta H_0}{\delta\eta}, \quad H_0 = H_0(\eta) = \frac{1}{2}\sum_{i=0}^N \eta_i\eta_{N-i}. \quad (8)$$

Second noting that

$$\Psi(\eta_i) := \Phi^*(\eta_i) = \delta_{i0}\partial^2 + 2\eta_i + 2\partial^{-1}\eta_i\partial, \quad 0 \leq i \leq N,$$

we can compute

$$\begin{aligned} f_1(\eta) &:= \hat{\Psi}(\eta)\frac{\delta H_0}{\delta\eta} = \begin{bmatrix} \Psi(\eta_0) & \Psi(\eta_1) & \cdots & \Psi(\eta_N) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Psi(\eta_1) \\ 0 & & & \Psi(\eta_0) \end{bmatrix} \begin{bmatrix} \eta_N \\ \eta_{N-1} \\ \vdots \\ \eta_1 \\ \eta_0 \end{bmatrix} \\ &= \begin{bmatrix} \eta_{Nxx} + 3\sum_{\substack{i+j=N \\ i,j \geq 0}} \eta_i\eta_j \\ \eta_{N-1,xx} + 3\sum_{\substack{i+j=N-1 \\ i,j \geq 0}} \eta_i\eta_j \\ \vdots \\ \eta_{0xx} + 3\eta_0^2 \end{bmatrix} = \frac{\delta H_1}{\delta\eta}, \end{aligned}$$

where the Hamiltonian function H_1 is defined by

$$H_1 = \int_0^1 \langle f_1(\lambda\eta), \eta \rangle d\lambda = \frac{1}{2} \sum_{\substack{i+j=N \\ i,j \geq 0}} \eta_i\eta_j\eta_{xx} + \sum_{\substack{i+j+k=N \\ i,j,k \geq 0}} \eta_i\eta_j\eta_k.$$

Here and hereafter $\langle \cdot, \cdot \rangle$ stands for the standard inner on R^{N+1} . The above analysis allows us to conclude that the perturbation equation (5) possesses a bi-Hamiltonian structure

$$\eta_{t_1} = \hat{\Phi}(\eta)\eta_x = \hat{J}\frac{\delta H_1}{\delta\eta} = \hat{M}\frac{\delta H_0}{\delta\eta}. \quad (9)$$

When $N = 1$, $N = 2$, the perturbation equation (5) becomes

$$\begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix}_{t_1} = \begin{bmatrix} \eta_{0xxx} + 6\eta_0\eta_{0x} \\ \eta_{1xxx} + 6(\eta_0\eta_1)_x \end{bmatrix},$$

$$\begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix}_{t_1} = \begin{bmatrix} \eta_{0xxx} + 6\eta_0\eta_{0x} \\ \eta_{1xxx} + 6(\eta_0\eta_1)_x \\ \eta_{2xxx} + 6\eta_1\eta_{1x} + 6(\eta_0\eta_2)_x \end{bmatrix},$$

respectively. These two equations are all integrable coupling ones with KdV equation. Integrable coupling is an interesting subject in soliton theory [11]. Note that the second equation may be reduced to the first one if we take $\eta_1 = 0$.

For the case of higher order KdV equations, by induction we can engender the following equivalent equation

$$\eta_{t_n} = \hat{\Phi}(\eta)\eta_{t_{n-1}} = \hat{\Phi}^n(\eta)\eta_x \quad (n > 1) \quad (10)$$

to the N -th order perturbation equation of the n -th KdV equation

$$\hat{u}_{t_n} = \Phi^n(\hat{u})\hat{u}_x + o(\varepsilon^N), \quad (11)$$

namely,

$$\hat{u}_{t_n} \equiv \Phi^n(\hat{u})\hat{u}_x \equiv \left(\sum_{i=0}^N \Phi(\eta_i)\varepsilon^i \right) \left(\sum_{i=0}^N \eta_{it_{n-1}}\varepsilon^i \right) \pmod{\varepsilon^{N+1}}.$$

According to the bi-Hamiltonian theory (for a detailed description see [12], [13] and [14]), we know the vectors $\hat{\Psi}^n(\eta)\frac{\delta H_0}{\delta \eta}$, $n \geq 1$, are all gradient vectors because $\hat{\Psi}(\eta)\frac{\delta H_0}{\delta \eta}$ is a gradient vector and $\hat{\Psi}^* = \hat{\Phi}$ is hereditary. Therefore there exist functions H_n so that

$$f_n(\eta) := \hat{\Psi}^n(\eta)\frac{\delta H_0}{\delta \eta} = \frac{\delta H_n}{\delta \eta}, \quad n \geq 1.$$

Moreover the Hamiltonian functions can be computed by

$$H_n = H_n(\eta) = \int_0^1 \langle f_n(\lambda\eta), \eta \rangle d\lambda.$$

Therefore we see that the perturbation equation (10) of the n -th KdV equation in the hierarchy (1) possesses a bi-Hamiltonian structure

$$\eta_{t_n} = \hat{\Phi}^n(\eta)\eta_x = \hat{J}\frac{\delta H_n}{\delta \eta} = \hat{M}\frac{\delta H_{n-1}}{\delta \eta}. \quad (12)$$

In fact, the perturbation equation (10) has a multi-Hamiltonian structure

$$\eta_{t_n} = \hat{J} \frac{\delta H_n}{\delta \eta} = \hat{J} \hat{\Psi} \frac{\delta H_{n-1}}{\delta \eta} = \dots = \hat{J} \hat{\Psi}^n \frac{\delta H_0}{\delta \eta},$$

where $\hat{J} \hat{\Psi}^i$, $0 \leq i \leq n$, are all Hamiltonian operators and constitute Hamiltonian pairs with each other. It is also interesting to note the hereditary operator $\hat{\Phi}^n(\eta)$ has the following concrete form

$$\hat{\Phi}^n(\eta) = \begin{bmatrix} A_0 & & & 0 \\ A_1 & \cdots & & \\ \vdots & \cdots & \cdots & \\ A_N & \cdots & A_1 & A_0 \end{bmatrix},$$

where the operator A_j , $0 \leq j \leq N$, are determined by

$$A_j = \sum_{\substack{i_1 + \dots + i_n = j \\ i_1, \dots, i_n \geq 0}} \Phi(\eta_{i_1}) \cdots \Phi(\eta_{i_n}), \quad 0 \leq j \leq N.$$

Unlike the hereditary operator for the Kepler system [15], the above hereditary operator can generate new independent constants of motion H_n from a starting one H_0 .

By now, we have proposed a bi-Hamiltonian structure for the perturbation equations of the whole KdV hierarchy. This bi-Hamiltonian structure yields infinitely many symmetries and infinitely many corresponding constants of motion in involution for every perturbation equation and thus the perturbation equations of KdV hierarchy provide a typical hierarchy of soliton equations. It is worth noting that the algebraic structures of the first order perturbation equations may also be well described by the perturbation bundle [11] and cotangent bundle [16].

We remark that some cases, especially $N = 1$, for KdV equation (2) has in detail been considered in Ref.[3]. But they only described a Hamiltonian structure. Moreover that Hamiltonian structure is not suitable for the perturbation equations of KdV equation, which leads to some confusion of the symbols. For example, in the case of $N = 1$, we cannot write

$$\eta_{t_1} = \hat{\Phi} \eta_x = \begin{bmatrix} \partial & 0 \\ 0 & \partial \end{bmatrix} \frac{\delta H'_0}{\delta \eta}.$$

Actually, there doesn't exist this kind of Hamiltonian function H'_0 , because $(\eta_{0xx} + 3\eta^2, \eta_{1xx} + 6\eta_0\eta_1)^T$ is not a gradient vector. However this is a small problem, which is easily changed to the correct version mentioned above.

There also exists a more general result than the bi-Hamiltonian structure (12). Let a_s , $1 \leq s \leq p$, b_s , c_s , $1 \leq s \leq q$, d_s , $1 \leq s \leq r$, are arbitrary real constants, and k_s , $1 \leq s \leq p$, l_s , $1 \leq s \leq q$, m_s , $1 \leq s \leq r$, are distinct non-negative integers not greater than N . We further choose that $\tilde{J} = \hat{J}$ and

$$\tilde{M} = \begin{bmatrix} 0 & & & \tilde{J}(\eta_0) \\ & & & \vdots \\ & & & \tilde{J}(\eta_1) \\ & & \vdots & \\ & & \tilde{J}(\eta_0) & \tilde{J}(\eta_1) & \cdots & \tilde{J}(\eta_N) \end{bmatrix}, \quad (13)$$

where the operator $\tilde{J}(\eta_i)$, $0 \leq i \leq N$, are defined by

$$\begin{aligned} \tilde{J}(\eta_i) = & \sum_{s=1}^p a_s \delta_{ik_s} \partial^3 + \sum_{s=1}^q (b_s \delta_{il_s} \partial^3 + c_s \delta_{il_s} \partial) + \sum_{s=1}^r d_s \delta_{im_s} \partial \\ & + 2\eta_{ix} + 4\eta_i \partial, \quad 0 \leq i \leq N. \end{aligned} \quad (14)$$

Note that (14) means we choose

$$\begin{aligned} \tilde{J}(\eta_{k_s}) &= a_s \partial^3 + 2\eta_{k_s, x} + 4\eta_{k_s} \partial, \quad 1 \leq s \leq p, \\ \tilde{J}(\eta_{l_s}) &= b_s \partial^3 + c_s \partial + 2\eta_{l_s, x} + 4\eta_{l_s} \partial, \quad 1 \leq s \leq q, \\ \tilde{J}(\eta_{m_s}) &= d_s \partial + 2\eta_{m_s, x} + 4\eta_{m_s} \partial, \quad 1 \leq s \leq r, \\ \tilde{J}(\eta_i) &= 2\eta_{ix} + 4\eta_i \partial, \quad i \notin \{k_s, 1 \leq s \leq p; l_s, 1 \leq s \leq q; m_s, 1 \leq s \leq r\}. \end{aligned}$$

The differential operators \tilde{J} and \tilde{M} are still a Hamiltonian pair [6] and hence the operator $\tilde{\Phi}(\eta) = \tilde{M} \tilde{J}^{-1}$ is a hereditary symmetry operator. This leads to the following hierarchy of integrable equations which possesses a more general bi-Hamiltonian structure

$$\eta_{t_n} = \tilde{\Phi}^n \eta_x = \tilde{J} \frac{\delta \tilde{H}_n}{\delta \eta} = \tilde{M} \frac{\delta \tilde{H}_{n-1}}{\delta \eta}, \quad n \geq 1 \quad (15)$$

with the Hamiltonian functions defined by

$$\tilde{H}_n = \tilde{H}_n(\eta) = \int_0^1 \langle (\tilde{\Phi}^n \eta_x)(\lambda \eta), \eta \rangle d\lambda, \quad n \geq 0,$$

in which \tilde{H}_0 and \tilde{H}_1 read as

$$\begin{aligned} \tilde{H}_0 &= H_0 = \frac{1}{2} \sum_{i=0}^N \eta_i \eta_{N-i}, \\ \tilde{H}_1 &= \frac{1}{2} \sum_{s=1}^p a_s \sum_{\substack{i+j=N-k_s \\ i, j \geq 0}} \eta_i \eta_{jxx} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{s=1}^q \left(b_s \sum_{\substack{i+j=N-l_s \\ i,j \geq 0}} \eta_i \eta_j x x + c_s \sum_{\substack{i+j=N-l_s \\ i,j \geq 0}} \eta_i \eta_j \right) \\
& + \frac{1}{2} \sum_{s=1}^r d_s \sum_{\substack{i+j=N-m_s \\ i,j \geq 0}} \eta_i \eta_j + \sum_{\substack{i+j+k=N \\ i,j,k \geq 0}} \eta_i \eta_j \eta_k.
\end{aligned}$$

Evidently, (12) is a special case of the hierarchy (15), one reduction with $k_1 = 0$ and $a_1 = 1$, $a_s = 0$, $s \neq 1$, $b_s = c_s = d_s = 0$, $\forall s$.

We point out that our method can also be applied to discussing the forced integrable systems [17]. It is sufficient to further make the perturbation of value boundary conditions and then we can get new forced integrable systems. The resulting forced systems can be solved by the same approach as the one in the original forced integrable systems [17]. However in the present letter, we haven't considered this kind of interesting problems due to the limited space.

Finally, we suggest that it may be more fruitful to think of the perturbation expansion \hat{u} as depending on two independent space variables, x and $y = \varepsilon x$, i.e.

$$\hat{u} = \sum_{i=0}^N \eta_i(x, y, t_1, t_2, \dots) \varepsilon^i, \quad y = \varepsilon x. \quad (16)$$

This kind of expansions comes from the perturbation method of multiple scales [18] and the variable y is called a slow variable. The perturbation method of multiple scales for time variable has been applied to the study of perturbed soliton equations (see for instance [2]) and the study of solutions to soliton equations (see for instance [19]). Under the expansion (16), we have

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial y}.$$

Therefore from the N -th order perturbation equation of KdV equation (2), we can similarly obtain the following equation in 1 + 2 dimensions

$$\begin{cases}
\eta_{0t_1} = \eta_{0xxx} + 6\eta_0\eta_{0x}, \\
\eta_{1t_1} = \eta_{1xxx} + 3\eta_{0xxy} + 6(\eta_0\eta_1)_x + 6\eta_0\eta_{0y}, \\
\eta_{2t_1} = \eta_{2xxx} + 3\eta_{1xxy} + 3\eta_{0xyy} + 6(\eta_0\eta_2)_x + 6\eta_1\eta_{1x} + 6(\eta_0\eta_1)_y, \\
\eta_{jt_1} = \eta_{jxxx} + 3\eta_{j-1,xxy} + 3\eta_{j-2,xyy} + \eta_{j-3,yyy} \\
\quad + 6\left(\sum_{i=0}^j \eta_i\eta_{j-i,x} + \sum_{i=0}^{j-1} \eta_i\eta_{j-i-1,y}\right), \quad 3 \leq j \leq N.
\end{cases}$$

This is a generalization of the perturbation equations of KdV hierarchy. However, we don't know the complete answer on the integrability of the above equation, even in the case $N = 1$ where the above equation reads as

$$\begin{cases}
\eta_{0t_1} = \eta_{0xxx} + 6\eta_0\eta_{0x}, \\
\eta_{1t_1} = \eta_{1xxx} + 3\eta_{0xxy} + 6(\eta_0\eta_1)_x + 6\eta_0\eta_{0y}.
\end{cases}$$

This equation is also a coupling one with KdV equation. If integrable, it provides us an another example which gives an integrable coupling equation with KdV equation.

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