

## BINARY NONLINEARIZATION OF LAX PAIRS

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A kind of Bargmann symmetry constraints involved in Lax pairs and adjoint Lax pairs is proposed for soliton hierarchy. The Lax pairs and adjoint Lax pairs are nonlinearized into a hierarchy of commutative finite dimensional integrable Hamiltonian systems and explicit integrals of motion may also be generated. The corresponding binary nonlinearization procedure leads to a sort of involutive solutions to every system in soliton hierarchy which are all of finite gap. An illustrative example is given in the case of AKNS soliton hierarchy.

### 1 Introduction

Symmetry constraints become prominent because of the important role they play in the soliton theory <sup>1 10 15 16</sup>. A kind of very successful symmetry constraint method for soliton equations is proposed through the nonlinearization technique called mono-nonlinearization <sup>2 9</sup>. However, mono-nonlinearization involves only the Lax pairs of soliton equations. We would like to elucidate that the mono-nonlinearization technique can successfully be extended to the Lax pairs and the adjoint Lax pairs associated with soliton hierarchy. The corresponding symmetry constraint procedure is called a binary nonlinearization technique <sup>12 13 8</sup> because it involves the Lax pairs and the adjoint Lax pairs and puts the linear Lax pairs into the nonlinearized Lax systems. A kind of useful symmetries in our symmetry constraints is exactly the specific symmetries expressed through the variational derivatives of the potentials. The resulting theory provides a method of separation of variables for solving nonlinear soliton equations and exhibits integrability by quadratures for soliton equations. It also narrows the gap between finite dimensional integrable Hamiltonian systems and infinite dimensional integrable soliton equations. An illustrative example is carried out in the case of the three-by-three matrix spectral problem for AKNS soliton hierarchy.

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## 2 Basic idea of binary nonlinearization

This section reveals how to manipulate a binary nonlinearization procedure for a given soliton hierarchy along with a basic idea for the proof of the main result. Let  $\mathcal{B}$  denote the differential algebra of differential vector functions  $u = u(x, t)$ , and write for  $k \geq 0$

$$\mathcal{V}_{(k)}^s = \{(P^{ij} \partial^k)_{s \times s} \mid P^{ij} \in \mathcal{B}\}, \quad \tilde{\mathcal{V}}_{(k)}^s = \mathcal{V}_{(k)}^s \otimes C[\lambda, \lambda^{-1}], \quad \partial = \frac{d}{dx}.$$

For  $U = U(u, \lambda) \in \tilde{\mathcal{V}}_{(0)}^s$ , we choose a solution to the adjoint representation equation  $V_x = [U, V]$ :

$$V = V(u, \lambda) = \sum_{i \geq 0} V_i \lambda^{-i}, \quad V_i \in \mathcal{V}_{(0)}^s.$$

Suppose that the isospectral ( $\lambda_{t_n} = 0$ ) compatibility conditions  $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ ,  $n \geq 0$ , of the Lax pairs

$$\begin{cases} \phi_x = U\phi = U(u, \lambda)\phi, & U \in \tilde{\mathcal{V}}_{(0)}^s \\ \phi_{t_n} = V^{(n)}\phi = V^{(n)}(u, \lambda)\phi, & V^{(n)} = (\lambda^n V)_+ + \Delta_n, \quad \Delta_n \in \tilde{\mathcal{V}}_{(0)}^s \end{cases}$$

determine a soliton hierarchy

$$u_{t_n} = K_n = JG_n = J \frac{\delta H_n}{\delta u}, \quad n \geq 0. \quad (1)$$

If  $\phi = (\phi_1, \phi_2, \dots, \phi_s)^T$  and  $\psi = (\psi_1, \psi_2, \dots, \psi_s)^T$  satisfy the spectral problem and the adjoint spectral problem

$$\phi_x = U(u, \lambda)\phi, \quad \psi_x = -U^T(u, \lambda)\psi,$$

and we set the matrix  $\bar{V} = \phi\psi^T = (\phi_k\psi_l)_{s \times s}$ , then we have the following two basic results used in binary nonlinearization<sup>5 13</sup>:

(i) the variational derivative of the spectral parameter  $\lambda$  with respect to the potential  $u$  may be expressed by

$$\frac{\delta \lambda}{\delta u} = \frac{\text{tr}(\bar{V} \frac{\partial U}{\partial u})}{-\int_{-\infty}^{\infty} \text{tr}(\bar{V} \frac{\partial U}{\partial \lambda}) dx}, \quad (2)$$

(ii) the matrix  $\bar{V}$  is a solution to the adjoint representation equation  $V_x = [U, V]$ , i.e.  $\bar{V}_x = [U, \bar{V}]$ .

Now introduce distinct eigenvalues  $\lambda_1, \dots, \lambda_N$  and let

$$\phi^{(j)} = (\phi_{1j}, \dots, \phi_{sj})^T, \quad \psi^{(j)} = (\psi_{1j}, \dots, \psi_{sj})^T \quad (1 \leq j \leq N)$$

denote the eigenvectors and the adjoint eigenvectors corresponding to  $\lambda_j$  ( $1 \leq j \leq N$ ), respectively. Make the Bargmann **symmetry constraint**

$$K_0 = JG_0 = J \sum_{j=1}^N E_j \frac{\delta \lambda_j}{\delta u} \quad \text{or} \quad G_0 = \sum_{j=1}^N E_j \frac{\delta \lambda_j}{\delta u}, \quad (3)$$

where  $E_j = -\int_{-\infty}^{\infty} \langle \bar{V}(\lambda_j), \frac{\partial U}{\partial \lambda_j} \rangle dx$ ,  $\bar{V}(\lambda_j) = \phi^{(j)} \psi^{(j)T}$ ,  $1 \leq j \leq N$ . The Bargmann constraint requires the covariant  $G_0$  to be a potential function not including any potential differential and hence from the Bargmann symmetry constraint we may find an explicit nonlinear expression for the potential

$$u = f(\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(N)}; \psi^{(1)}, \psi^{(2)}, \dots, \psi^{(N)}). \quad (4)$$

Upon instituting (4) into the Lax pairs and the adjoint Lax pairs, we get two nonlinearized Lax systems, i.e. the nonlinearized spatial system

$$\begin{cases} \phi_{jx} = U(f(\phi^{(1)}, \dots, \phi^{(N)}; \psi^{(1)}, \dots, \psi^{(N)}), \lambda_j) \phi_j, & 1 \leq j \leq N, \\ \psi_{jx} = -U^T(f(\phi^{(1)}, \dots, \phi^{(N)}; \psi^{(1)}, \dots, \psi^{(N)}), \lambda_j) \psi_j, & 1 \leq j \leq N; \end{cases} \quad (5)$$

and the nonlinearized temporal systems for  $n \geq 0$

$$\begin{cases} \phi_{jt_n} = V^{(n)}(f(\phi^{(1)}, \dots, \phi^{(N)}; \psi^{(1)}, \dots, \psi^{(N)}), \lambda_j) \phi_j, & 1 \leq j \leq N, \\ \psi_{jt_n} = -V^{(n)T}(f(\phi^{(1)}, \dots, \phi^{(N)}; \psi^{(1)}, \dots, \psi^{(N)}), \lambda_j) \psi_j, & 1 \leq j \leq N. \end{cases} \quad (6)$$

In order to discuss the integrability of (5) and (6), we choose the symplectic structure  $\omega^2$  on  $\mathbb{R}^{2sN}$

$$\omega^2 = \sum_{i=0}^s \sum_{j=0}^N d\phi_{ij} \wedge d\psi_{ij} = \sum_{i=0}^s dP_i \wedge dQ_i,$$

where  $P_i = (\phi_{i1}, \dots, \phi_{iN})^T$ ,  $Q_i = (\psi_{i1}, \dots, \psi_{iN})^T$ ,  $1 \leq i \leq s$ . We accept the following corresponding Poisson bracket for two functions  $F, G$  defined over the phase space  $\mathbb{R}^{2sN}$

$$\begin{aligned} \{F, G\} &= \omega^2(\text{Id}G, \text{Id}F) = \omega^2(X_G, X_F) \\ &= \sum_{i=1}^s \left( \left\langle \frac{\partial F}{\partial Q_i}, \frac{\partial G}{\partial P_i} \right\rangle - \left\langle \frac{\partial F}{\partial P_i}, \frac{\partial G}{\partial Q_i} \right\rangle \right), \end{aligned} \quad (7)$$

where  $IdH = X_H$  represents the Hamiltonian vector field with energy  $H$  defined by  $i_{IdH}\omega^2 = i_{X_H}\omega^2 = dH$  and  $\langle \cdot, \cdot \rangle$  represents the standard inner product of  $\mathbb{R}^N$ . Then we accept the following corresponding Hamiltonian system with the Hamiltonian function  $H$

$$\dot{P}_i = \{P_i, H\} = -\frac{\partial H}{\partial Q_i}, \quad \dot{Q}_i = \{Q_i, H\} = \frac{\partial H}{\partial P_i}, \quad 1 \leq i \leq s. \quad (8)$$

**Main Result:** *The nonlinearized spatial system (5) is a finite dimensional integrable Hamiltonian system in the Liouville sense, and the nonlinearized temporal systems (6) for  $n \geq 0$  may be transformed into a hierarchy of finite dimensional integrable Hamiltonian systems in the Liouville sense, under the control of the nonlinearized spatial system (5). Moreover the potential  $u = f$  determined by the Bargmann symmetry constraint solves the  $n$ -th soliton equation  $u_{t_n} = K_n$  in the hierarchy.*

**Idea of Proof:** Note that we have

$$(V(f, \lambda))_x = [U(f, \lambda), V(f, \lambda)], \quad (\bar{V}(\lambda_j))_x = [U(f, \lambda_j), \bar{V}(\lambda_j)]$$

and when  $u_{t_n} = K_n$ , we have

$$(V(f, \lambda))_{t_n} = [V^{(n)}(f, \lambda), V(f, \lambda)], \quad (\bar{V}(\lambda_j))_{t_n} = [V^{(n)}(f, \lambda_j), \bar{V}(\lambda_j)].$$

Therefore we may show that  $F = \frac{1}{2}\text{tr}(V(f, \lambda))^2$  is a common generating function for integrals of motion of (5) and (6) since  $F_x = \frac{1}{2}\text{tr}(V^2)_x = \frac{1}{2}\text{tr}[U, V^2] = 0$  and  $F_{t_n} = \frac{1}{2}\text{tr}(V^2)_{t_n} = \frac{1}{2}\text{tr}[V^{(n)}, V^2] = 0$ . A similar deduction may verify that  $\bar{F}_j = \frac{1}{2}\text{tr}(\bar{V}(\lambda_j))^2$ ,  $1 \leq j \leq N$ , are integrals of motion of (5) and (6), too. Noticing

$$F = \sum_{n \geq 0} F_n \lambda^{-n}, \quad \bar{F}_j = \frac{1}{2} \left( \sum_{i=1}^s \phi_{ij} \psi_{ij} \right)^2, \quad 1 \leq j \leq N, \quad (9)$$

we get a series of explicit integrals of motion:  $\bar{F}_j$ ,  $1 \leq j \leq N$ ,  $\{F_n\}_{n=0}^{\infty}$ , which may be proved to be involutive with respect to the Poisson bracket (7). Further it is not difficult to show the Liouville integrability of (5) and (6) when they can be rewritten as Hamiltonian systems with Hamiltonian functions being polynomials in  $F_m$ ,  $m \geq 1$ .

In addition, because the compatibility condition of (5) and (6) is still the  $n$ -th soliton equation  $u_{t_n} = K_n$ ,  $u = f(\phi_j; \psi_j)$  gives an involutive solution to the  $n$ -th soliton equation  $u_{t_n} = K_n$  once  $\phi_j, \psi_j$ ,  $1 \leq j \leq N$ , solve (5) and (6), simultaneously. This sort of involutive solutions also exhibits a kind of separation of independent variables  $x, t_n$  for soliton equations.

### 3 The case of AKNS Hierarchy

For AKNS hierarchy, we introduce a three-by-three matrix spectral problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -2\lambda & \sqrt{2}q & 0 \\ \sqrt{2}r & 0 & \sqrt{2}q \\ 0 & \sqrt{2}r & 2\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

In this case,  $\phi = (\phi_1, \phi_2, \phi_3)^T$  and  $u = (q, r)^T$ . A hierarchy of AKNS soliton equations<sup>13</sup>

$$u_{t_n} = K_n = \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \end{pmatrix} = JL^n \begin{pmatrix} r \\ q \end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n \geq 0 \quad (10)$$

is the compatibility conditions of the Lax pairs

$$\phi_x = U\phi, \quad \phi_t = V^{(n)}\phi, \quad V^{(n)} = (\lambda^n V)_+. \quad (11)$$

Here the operator solution  $V$  to  $V_x = [U, V]$ , the Hamiltonian operator  $J$ , the recursion operator  $L$ , and the Hamiltonian functions  $H_n$  for  $n \geq 0$  read as

$$V = \begin{pmatrix} 2a & \sqrt{2}b & 0 \\ \sqrt{2}c & 0 & \sqrt{2}b \\ 0 & \sqrt{2}c & -2a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} 2a_i & \sqrt{2}b_i & 0 \\ \sqrt{2}c_i & 0 & \sqrt{2}b_i \\ 0 & \sqrt{2}c_i & -2a_i \end{pmatrix} \lambda^{-i},$$

$$J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \frac{1}{2}\partial - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}r \end{pmatrix}, \quad H_n = \frac{2a_{n+2}}{n+1}.$$

The operators  $J$  and  $JL$  constitute a Hamiltonian pair and  $L^*$  is hereditary<sup>6</sup>.

In this AKNS case, the Bargmann symmetry constraint becomes

$$K_0 = J \frac{\delta H_0}{\delta u} = J \sum_{j=1}^N \begin{pmatrix} \sqrt{2}(\phi_{2j}\psi_{1j} + \phi_{3j}\psi_{2j}) \\ \sqrt{2}(\phi_{1j}\psi_{2j} + \phi_{2j}\psi_{3j}) \end{pmatrix}, \quad (12)$$

which engenders an explicit expression for the potential  $u$

$$u = f(\phi_{ij}; \psi_{ij}) = \sqrt{2} \begin{pmatrix} \langle P_1, Q_2 \rangle + \langle P_2, Q_3 \rangle \\ \langle P_2, Q_1 \rangle + \langle P_3, Q_2 \rangle \end{pmatrix}. \quad (13)$$

Further besides  $\bar{F}_j$ ,  $1 \leq j \leq N$ , we can directly give the following explicit integrals of motion for the nonlinearized Lax systems

$$\begin{aligned}
F &:= \frac{1}{2} \text{tr} V^2 = 4(a^2 + bc) = \sum_{m \geq 0} F_m \lambda^{-m}, \\
F_0 &= 4, \quad F_1 = -8(\langle P_1, Q_1 \rangle - \langle P_3, Q_3 \rangle), \\
F_m &= 4 \sum_{i=1}^{m-1} [(\langle A^{i-1} P_1, Q_1 \rangle - \langle A^{i-1} P_3, Q_3 \rangle) \times \\
&\quad (\langle A^{m-i-1} P_1, Q_1 \rangle - \langle A^{m-i-1} P_3, Q_3 \rangle) \\
&\quad + 2(\langle A^{i-1} P_1, Q_2 \rangle + \langle A^{i-1} P_2, Q_3 \rangle) \times \\
&\quad (\langle A^{m-i-1} P_2, Q_1 \rangle + \langle A^{m-i-1} P_3, Q_2 \rangle)] \\
&\quad - 8 \langle A^{m-1} P_1, Q_1 \rangle - \langle A^{m-1} P_3, Q_3 \rangle, \quad m \geq 2,
\end{aligned}$$

where  $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ . The nonlinearized spatial system (5) is rewritten as an integrable Hamiltonian system

$$P_{ix} = \{P_i, H\} = -\frac{\partial H}{\partial Q_i}, \quad Q_{ix} = \{Q_i, H\} = \frac{\partial H}{\partial P_i}, \quad i = 1, 2, 3 \quad (14)$$

with the Hamiltonian function

$$\begin{aligned}
H &= 2(\langle AP_1, Q_1 \rangle - \langle AP_3, Q_3 \rangle) \\
&\quad - 2(\langle P_1, Q_2 \rangle + \langle P_2, Q_3 \rangle)(\langle P_2, Q_1 \rangle + \langle P_3, Q_2 \rangle),
\end{aligned}$$

and under the control of the nonlinearized spatial system (5), the nonlinearized temporal systems (6) for  $n \geq 0$  can also be rewritten as the integrable Hamiltonian systems

$$P_{it_n} = \{P_i, H_n\} = -\frac{\partial H_n}{\partial Q_i}, \quad Q_{it_n} = \{Q_i, H_n\} = \frac{\partial H_n}{\partial P_i}, \quad i = 1, 2, 3 \quad (15)$$

with the Hamiltonian functions

$$H_n = -\frac{1}{4} \sum_{m=0}^n \frac{d_m}{m+1} \sum_{\substack{i_1 + \dots + i_{m+1} = n+1 \\ i_1, \dots, i_{m+1} \geq 1}} F_{i_1} \dots F_{i_{m+1}},$$

where the constants  $d_m$  are defined by

$$\begin{aligned}
d_0 &= 1, \quad d_1 = -\frac{1}{8}, \quad d_2 = \frac{3}{128}, \\
d_m &= -\frac{1}{2} \sum_{i=1}^{m-1} d_i d_{m-i} - \frac{1}{4} d_{m-1} - \frac{1}{8} \sum_{i=1}^{m-2} d_i d_{m-i-1}, \quad m \geq 3.
\end{aligned}$$

Moreover following the previous main result, the potential (13) with

$$P_i(x, t_n) = g_H^x g_{H_n}^{t_n} P_i(0, 0), \quad Q_i(x, t_n) = g_H^x g_{H_n}^{t_n} Q_i(0, 0), \quad i = 1, 2, 3,$$

gives rise to a sort of involutive solutions with separated variables  $x, t_n$  to the  $n$ -th AKNS soliton equation  $u_{t_n} = K_n$ . Here  $g_G^y$  denotes the Hamiltonian phase flow of  $G$  with a parameter variable  $y$  but  $P_i(0, 0), Q_i(0, 0)$  may be arbitrary initial value vectors. A finite gap property for the resulting involutive solutions may also be shown.

#### 4 Concluding remarks

We remark that the finite dimensional Hamiltonian systems generated by nonlinearization technique depend on the starting Lax pairs. Thus the same equation may be connected with different finite dimensional Hamiltonian systems once it possesses different Lax pairs. AKNS soliton equations are exactly such examples<sup>13</sup>.

We also point out that the Neumann symmetry constraint and the higher order symmetry constraints

$$K_{-1} = J \sum_{j=1}^N E_j \frac{\delta \lambda_j}{\delta u}, \quad K_m = J G_m = J \sum_{j=1}^N E_j \frac{\delta \lambda_j}{\delta u}, \quad (m \geq 1), \quad (16)$$

may be considered. These two sorts of symmetry constraints are somewhat different from the Bargmann symmetry constraints because  $K_{-1}$  is a constant vector and the conserved covariants  $G_m, m \geq 1$ , involve some differentials of the potential. This suggests that a few new tools are needed for discussing them<sup>18</sup>. Similarly, we can consider the corresponding  $\tau$ -symmetry (i.e. time first order dependent symmetry<sup>3</sup>) constraints or more generally, time polynomial dependent symmetry<sup>7</sup> constraints. Binary nonlinearization may also be well applied to discrete systems and non-Hamiltonian soliton equations such as the Toda lattice and the coupled Burgers equations<sup>11</sup>. Note that in the case of KP hierarchy, the similar Bargmann symmetry constraints have been carefully analyzed as well<sup>15</sup>, and the specific symmetries we use in constraints are sometimes called additional symmetries<sup>4</sup> and are often taken as source terms of soliton equations<sup>14</sup>. It should also be noted that the nonlinearized Lax systems are intimately related to stationary equations<sup>17</sup> and the more general nonlinearized Lax systems can be generated from the linear combination of Bargmann symmetry constraints which will be shown in a late publication.

However, in the binary nonlinearization procedure, there exist two intriguing **open problems**. The first one is why the nonlinearized spatial system (5)

and the nonlinearized temporal systems (6) for  $n \geq 0$  with the control of the nonlinearized spatial system (5) always possess Hamiltonian structures? The second one is whether or not the nonlinearized temporal systems (6) for  $n \geq 0$  are themselves integrable soliton equations without the control of the nonlinearized spatial system (5). These two problems are important and interesting but need some further investigation.

### Acknowledgments

One of the authors (W. X. Ma) would like to thank the Alexander von Humboldt Foundation for a research fellow award and the National Natural Science Foundation of China and the Shanghai Science and Technology Commission of China for their financial support. He is also grateful to Drs. W. Oevel, P. Zimmermann and G. Oevel for their helpful and stimulating discussions.

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